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ISHIKAWA'S ITERATIONS OF REAL LIPSCHITZ FUNCTIONS

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In this paper, we consider Ishikawa's iteration scheme to compute fixed points of real Lipschitz functions. Two general convergence theorems are obtained. Our results generalise the result of Hillam.

1. INTRODUCTION

Bailey [1] gave a proof of Krasnoselski's Theorem [4] restricted to the real line. In [2], using the fact that the real line is totally ordered, Hillam established the following more general result.

THEOREM A. *Let $f\colon [a, b] \to [a, b]$ be a function that satisfies a Lipschitz condition with constant L. Let x_1 in $[a, b]$ be arbitrary and define $x_{n+1} = (1-\lambda)x_n + \lambda f(x_n)$ where $\lambda = (1+L)^{-1}$. If $\{x_n\}$ denotes the resulting sequence, then $\{x_n\}$ converges monotonically to a point z in $[a, b]$ where $f(z) = z$.*

In this paper, using a somewhat more sophisticated argument, we consider a Lipschitz function f which maps the closed interval $[a, b]$ into itself, and under very general conditions, prove that Ishikawa's iteraiton scheme [3] always converges to a fixed point of f. These results improve and generalise Theorem A.

2. PRELIMINARIES

Recall that $f\colon [a, b] \to [a, b]$ is L-Lipschitz if there exists a constant $L \geqslant 0$ such that $|f(x) - f(y)| \leqslant L\,|x - y|$ for all $x, y \in [a, b]$. Clearly, each L-Lipschitz function is continuous.

LEMMA 1. [5, Theorem 1]. *Let f be a continuous self-mapping of $[a, b]$. If the iteration scheme $\{x_n\}$ converges to z, then z is a fixed point of f, where $\{x_n\}$ is defined by*

$$(1) \qquad \begin{aligned} x_1 \in [a, b], \quad x_{n+1} &= \alpha_n f(y_n) + (1-\alpha_n)x_n, \\ y_n &= \beta_n f(x_n) + (1-\beta_n)x_n, \quad n \geqslant 1 \end{aligned}$$

and $\{\alpha_n\}$, $\{\beta_n\}$ satisfy

(i) $0 \leqslant \alpha_n, \beta_n \leqslant 1$ *for all* n,

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(ii) $\sum_{n=1}^{\infty} \alpha_n$ *diverges, and*

(iii) $\lim_{n\to\infty} \beta_n = 0$.

Lemma 2. [5, Theorem 10]. *Let $f\colon [a,b] \to [a,b]$ be a L-Lipschitz function. Let $\{\alpha_n\}$, $\{\beta_n\}$ satisfy*

(i) $0 \le \alpha_n,\ \beta_n \leqslant 1$ *for all* n,

(ii) $\lim_{n\to\infty} \sup \alpha_n > 0$, *and*

(iii) $\lim_{n\to\infty} \sup \beta_n < L^{-1}$.

If the sequence $\{x_n\}$ defined by (1) converges to z, then z is a fixed point of f.

Proof: From the triangle inequality it follows that

$$\begin{aligned}\alpha_n |z - f(z)| &\leqslant \alpha_n |z - x_n| + \alpha_n |x_n - f(y_n)| + \alpha_n |f(y_n) - f(z)| \\ &\leqslant \alpha_n |z - x_n| + |x_{n+1} - x_n| + \alpha_n L |y_n - z| \\ &\leqslant \alpha_n |z - x_n| + |x_{n+1} - x_n| + \alpha_n\beta_n L |f(x_n) - z| \\ &\qquad + \alpha_n(1-\beta_n) L |x_n - z|.\end{aligned}$$

It follows that $\lim_{n\to\infty} \sup \alpha_n \left(1 - L \lim_{n\to\infty} \sup \beta_n\right) |z - f(z)| \leqslant 0$. Since $\lim_{n\to\infty} \sup \alpha_n \left(1 - L \lim_{n\to\infty} \sup \beta_n\right) > 0$, then $|z - f(z)| = 0$. Hence $f(z) = z$. ∎

Lemma 3. *Let $f\colon [a,b] \to [a,b]$ be a L-Lipschitz function. The sequence $\{x_n\}$ is given by (1) with $0 \leqslant \alpha_n,\ \beta_n \leqslant 1$ for all n. If there is a fixed point z of f in the interval between x_n and x_{n+1}, and $\gamma_n = \alpha_n[1 + L + L(L-1)\beta_n] - 1$, then*

$$|x_{n+1} - z| \leqslant \gamma_n |x_n - z|. \tag{2}$$

Proof: If $x_n \leqslant z \leqslant x_{n+1}$, we have

$$\begin{aligned}x_{n+1} - z &= \alpha_n(f(y_n) - f(z)) + (1-\alpha_n)(x_n - z) \\ &\leqslant \alpha_n L |y_n - z| + (1-\alpha_n)(x_n - z) \\ &\leqslant \alpha_n L[1 + (L-1)\beta_n](z - x_n) + (\alpha_n - 1)(z - x_n) \\ &= \gamma_n |x_n - z|.\end{aligned}$$

If $x_n \geqslant z \geqslant x_{n+1}$, by a similar argument, we have $z - x_{n+1} \leqslant \gamma_n |x_n - z|$. Hence (2) holds. The proof is complete. ∎

Lemma 4. *Let $f\colon [a,b] \to [a,b]$ be a L-Lipschitz function. Suppose that the sequence $\{x_n\}$ is given by (1) with $0 < \alpha_n \leqslant 1$, $0 \leqslant \beta_n < (1+L)^{-1}$ and $x_n \neq x_{n+1}$*

for all n. Then x_n, x_{n+1}, x_{n+2} is monotone (that is, $x_n < x_{n+1} < x_{n+2}$ or $x_n > x_{n+1} > x_{n+2}$) if and only if $f(x_n) - x_n$ and $f(x_{n+1}) - x_{n+1}$ have the same signs.

PROOF: We first state some properties which will be useful in later developments. If $f(x_n) > x_n$, then

$$f(y_n) > y_n \geqslant x_n; \tag{3}$$

if $f(x_n) < x_n$, then

$$f(y_n) < y_n \leqslant x_n. \tag{4}$$

Indeed, if $f(x_n) > x_n$, then $y_n = \beta_n f(x_n) + (1 - \beta_n)x_n \geqslant x_n$. If there is a fixed point z of f in $[x_n, y_n]$, then

$$\begin{aligned} 0 < |z - x_n| \leqslant |y_n - x_n| &= \beta_n \, |f(x_n) - x_n| \\ &\leqslant \beta_n \, |f(x_n) - f(z)| + \beta_n \, |z - x_n| \\ &\leqslant \beta_n(L+1) \, |z - x_n| < |z - x_n| \end{aligned}$$

which is a contradiction. Thus there is no fixed point of f in $[x_n, y_n]$. It follows from $f(x_n) > x_n$ that $f(y_n) > y_n$. Hence (3) holds. If $f(x_n) < x_n$, by a similar argument, (4) holds.

Now we show that the stated condition is sufficient. Suppose that $f(x_n) - x_n$ and $f(x_{n+1}) - x_{n+1}$ have the same signs; then we must have that x_n, x_{n+1}, x_{n+2} is monotone. If $f(x_n) > x_n$ and $f(x_{n+1}) > x_{n+1}$, by (3), we have

$$f(y_n) > y_n \geqslant x_n \quad \text{and} \quad f(y_{n+1}) > y_{n+1} \geqslant x_{n+1}.$$

Hence

$$x_{n+1} - x_n = \alpha_n(f(y_n) - x_n) > 0,$$
$$x_{n+2} - x_{n+1} = \alpha_{n+1}(f(y_{n+1}) - x_{n+1}) > 0,$$

that is, $x_n < x_{n+1} < x_{n+2}$. If $f(x_n) < x_n$ and $f(x_{n+1}) < x_{n+1}$, by a similar argument, we have $x_n > x_{n+1} > x_{n+2}$.

To see that the condition is necessary, without loss of generality, we may assume that $x_n < x_{n+1} < x_{n+2}$. Clearly, since $x_n \neq x_{n+1}$ for all n, we have $f(x_n) \neq x_n$ and $f(x_{n+1}) \neq x_{n+1}$. If $f(x_n) < x_n$ and $f(x_{n+1}) > x_{n+1}$, by (4), $f(y_n) < y_n \leqslant x_n$. Hence $x_{n+1} - x_n = \alpha_n(f(y_n) - x_n) < 0$; this contradicts the assumption that $x_n < x_{n+1}$. If $f(x_n) > x_n$ and $f(x_{n+1}) < x_{n+1}$, similar reasoning contradicts the assumption that $x_{n+1} < x_{n+2}$. Therefore $f(x_n) - x_n$ and $f(x_{n+1}) - x_{n+1}$ have the same signs. $\square$

DEFINITION: We say that $\{x_n\}$ switches directions at x_{n+1} if either

$$x_n < x_{n+1} > x_{n+2} \text{ or } x_n > x_{n+1} < x_{n+2}.$$

By Lemma 4, we see that x_{n+1} switches direction if and only if $f(x_n)-x_n$ and $f(x_{n+1})-x_{n+1}$ have opposite signs, where $\{x_n\}$ is defined by (1) with $0 < \alpha_n \leqslant 1$, $0 \leqslant \beta_n < (1+L)^{-1}$ and $x_n \neq x_{n+1}$ for all n.

LEMMA 5. *Suppose $\{x_n\}$ defined by (1) has successive switches of direction at x_{n_1+1} and x_{n_2+1}, and $0 < \alpha_n \leqslant 2(1+\varepsilon_n)^{-1}[1+L+L(L-1)\beta_n]^{-1}$, $0 < \varepsilon_n < 1$, $0 \leqslant \beta_n < (1+L)^{-1}$ and $x_n \neq x_{n+1}$ for all n. Then*

(a) *x_n lies between x_{n_1} and x_{n_1+1} for $n_1+1 \leqslant n \leqslant n_2+1$;*
(b) $|x_{n_2} - x_{n_2+1}| \leqslant (1-\varepsilon_{n_1})\,|x_{n_1} - x_{n_1+1}|$.

PROOF: We may suppose that $x_{n_1} < x_{n_1+1} > x_{n_1+2}$ and $x_{n_2} > x_{n_2+1} < x_{n_2+2}$. Let $m = \inf\{x : f(x) = x, x_{n_1} \leqslant x \leqslant x_{n_1+1}\}$. By Lemma 3, we have that $|x_{n_1+1} - m| \leqslant \gamma_{n_1}|x_{n_1} - m|$ and $m \geqslant (1+\gamma_{n_1})^{-1}(\gamma_{n_1}x_{n_1} + x_{n_1+1})$. Hence

$$x_{n_1+1} - m \leqslant \gamma_{n_1}(1+\gamma_{n_1})^{-1}(x_{n_1+1} - x_{n_1}). \tag{5}$$

Since x_n decreases for $n_1+1 \leqslant n \leqslant n_2+1$, we have that $x_{n_2} > m$. Indeed, if $m > x_{n_2}$, then there exists $n_1 < n' < n_2$ such that $x_{n'} > m > x_{n'+1}$. Lemma 4 implies that $f(x_{n_1+1}) - x_{n_1+1}$ and $f(x_{n_1}) - x_{n_1}$ have opposite signs; $f(x_{n'+1}) - x_{n'+1}$ and $f(x_{n_1+1}) - x_{n_1+1}$ have same signs. Thus $f(x_{n'+1}) - x_{x'+1}$ and $f(x_{n_1}) - x_{n_1}$ have opposite signs. Hence f has a fixed point in the interval between $x_{n'+1}$ and x_{n_1}. On the other hand, by Lemma 3 and $x_{n_1+1} \geqslant x_{n'}$, we have

$$\begin{aligned} m - x_{n'+1} &\leqslant \gamma_{n'}(x_{n'} - m) \\ &\leqslant \gamma_{n'}(x_{n_1+1} - m) \\ &\leqslant \gamma_{n'}\gamma_{n_1}(m - x_{n_1}). \end{aligned}$$

Hence $m > x_{n'+1} \geqslant (1-\gamma_{n'}\gamma_{n_1})m + \gamma_{n'}\gamma_{n_1}x_{n_1} \geqslant x_{n_1}$. The minimality of m implies that f has no fixed point in the interval between $x_{n'+1}$ and x_{n_1}. This is a contradiction. Therefore, we have either

$$\text{(i)} \quad x_{n_2+1} > m > x_{n_1} \quad \text{or} \quad \text{(ii)} \quad x_{n_2} > m > x_{n_2+1}.$$

By $x_{x_1+1} \geqslant x_{n_2}$, $0 \leqslant \gamma_{n_2} < 1$ and Lemma 3, in either case, we have

$$x_{n_2+1} \geqslant \gamma_{n_1}x_{n_1} + (1-\gamma_{n_1})m. \tag{6}$$

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By (6), we have $x_{n_1+1} \geqslant x_n \geqslant x_{n_2+1} \geqslant \gamma_{n_1} x_{n_1} + (1-\gamma_{n_1})m \geqslant x_{n_1}$. The conclusion (a) holds.

Since $x_{n_1+1} \geqslant x_{n_2}$, (5) and (6) show that

$$\begin{aligned}0 < x_{n_2} - x_{x_2+1} &\leqslant x_{n_1+1} - [\gamma_{n_1} x_{n_1} + (1-\gamma_{n_1})m] \\ &= \gamma_{n_1}(x_{n_1+1} - x_{n_1}) + (1-\gamma_{n_1})(x_{n_1+1} - m) \\ &\leqslant \gamma_{n_1}(x_{n_1+1} - x_{n_1}) + (1-\gamma_{n_1})\gamma_{n_1}(1+\gamma_{n_1})^{-1}(x_{n_1+1} - x_{n_1}) \\ &= [2 - 2(1+\gamma_{n_1})^{-1}](x_{n_1+1} - x_{n_1}) \\ &\leqslant (1-\varepsilon_{n_1})(x_{n_1+1} - x_{n_1}).\end{aligned}$$

The conclusion (b) is proved. □

3. Main results

Theorem 1. *Let $f\colon [a,b] \to [a,b]$ be a L-Lipschitz function. If $0 < \varepsilon < 1$, $0 \leqslant \alpha_n < 2(1+\varepsilon)^{-1}[1 + L + L(L-1)\beta_n]^{-1}$, and $0 \leqslant \beta_n < (1+L)^{-1}$ for all n, then the sequence $\{x_n\}$ defined by (1) converges to some point z which lies between x_n and x_{n+1} whenever $\{x_n\}$ switches directions at x_{n+1}. Moreover, if for all n,*

$$0 \leqslant \alpha_n \leqslant [1 + L + L(L-1)\beta_n]^{-1},$$

then the convergence is monotone.

Proof: (1) Suppose that $x_n \neq x_{n+1}$ for all n. Clearly, we have $\alpha_n > 0$ for all n.

If $\{x_n\}$ switches direction only finitely often then convergence follows since the sequence is eventually monotone. Suppose therefore that the sequence switches directions infinitely often at $x_{n_1+1}, x_{n_2+1}, \ldots, x_{n_k+1} \ldots$.

Lemma 5 shows that for $n_k + 1 \leqslant n \leqslant n_{(k+1)} + 1$, x_n lies between x_{n_k} and x_{n_k+1} and that

$$\left|x_{n_{(k+1)}} - x_{n_{(k+1)}+1}\right| \leqslant (1-\varepsilon)\left|x_{n_k} - x_{n_k+1}\right|.$$

Inductively, we see that $|x_n - x_m| \leqslant (1-\varepsilon)^k(b-a)$ for n and $m > n_{(k+1)}$. So $\{x_n\}$ is a Cauchy sequence and so has limit z. Moreover, z lies between x_{n_k} and x_{n_k+1}.

(2) Suppose $x_{n_0} = x_{n_0+1}$ for some n_0. Without loss of generality, we can assume that $\{x_n\}$ denotes

$$x_1, \ldots, x_{n_1}, x_{n_1+1}, \ldots, x_{n_2}, x_{n_2+1}, \ldots, x_{n_3}, \ldots, x_{n_k+1}, \ldots, x_{n_{(k+1)}}, \ldots,$$

where $x_1 = x_2 = \ldots = x_{n_1}$, $x_{n_k+1} = x_{n_k+2} = \ldots = x_{n_{(k+1)}}$, $x_{n_k} \neq x_{n_{(k+1)}}$, $(k = 1, 2, \ldots)$. Clearly, $\{x_n\}$ converges if and only if the subsequence $\{x_{n_k}\}$ of $\{x_n\}$

converges. On the other hand, if $\{\alpha_n\}$ and $\{\beta_n\}$ of (1) are replaced by $\{\alpha_{n_k}\}$ and $\{\beta_{n_k}\}$, respectively, then the sequence of Ishikawa iteration defined by (1) is exactly the subsequence $\{x_{n_k}\}$ of $\{x_n\}$. It follows from (1) that $\{x_{n_k}\}$ converges. Hence $\{x_n\}$ converges.

Finally, if $0 \leqslant \alpha_n \leqslant [1 + L + L(L-1)\beta_n]^{-1}$ and $0 \leqslant \beta_n < (1+L)^{-1}$, then it follows from Lemma 3 and Lemma 4 that no change of direction is possible. The proof is complete. $\square$

Note that, to establish convergence, it is only necessary to assume that

$$\limsup_{n\to\infty} \alpha_n < 2[1 + L + L(L-1)\limsup_{n\to\infty} \beta_n]^{-1},$$
$$\limsup_{n\to\infty} \beta_n < (1+L)^{-1}.$$

By Theorem 1 and Lemma 1, we have now proved:

THEOREM 2. *Let $f\colon [a, b] \to [a, b]$ be a L-Lipschitz function. The sequence $\{x_n\}$ defined by (1) satisfies: (i) $0 \leqslant \alpha_n, \beta_n \leqslant 1$ for all n, (ii) $\sum_{n=1}^{\infty} \alpha_n$ diverges, and (iii) $\lim_{n\to\infty} \beta_n = 0$. Then*

(1) *if $\limsup_{n\to\infty} \alpha_n < 2(1+L)^{-1}$, then $\{x_n\}$ converges to a fixed point of f;*
(2) *f $\alpha_n \leqslant [1 + L + L(L-1)\beta_n]^{-1}$ for all n, then $\{x_n\}$ converges monotonically to a fixed point of f.*

REMARK 1. The special case of Theorem 2.2 for $\beta_n = 0$ generalises Theorem A. Therefore Theorem 2 improves and generalises Theorem A.

By Theorem 1 and Lemma 2, we have also proved:

THEOREM 3. *Let $f\colon [a, b] \to [a, b]$ be a L-Lipschitz function. The sequence $\{x_n\}$ defined by (1) satisfies: (i) $0 \leqslant \alpha_n, \beta_n \leqslant 1$ for all n, (ii) $\limsup_{n\to\infty} \alpha_n > 0$, and (iii) $\limsup_{n\to\infty} \beta_n < (1+L)^{-1}$. Then*

(1) *if $\limsup_{n\to\infty} \alpha_n < 2[1 + L + L(L-1)\limsup_{n\to\infty} \beta_n]^{-1}$, then $\{x_n\}$ converges to a fixed point of f;*
(2) *if $\alpha_n \leqslant [1 + L + L(L-1)\beta_n]^{-1}$ for all n, then $\{x_n\}$ converges monotonically to a fixed point of f.*

REMARK 2. Theorem 3 improves Theorem 2 when $\limsup_{n\to\infty} \alpha_n > 0$.

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