Canad. Math. Bull. Vol. 18 (2), 1975

NOTE ON A SUBRING OF $C^*(X)$[1]

BY
CHOO ENG-UNG

Throughout, topological spaces are assumed to be completely regular. $C(X)$ (resp. $C^*(X)$) will denote the ring of all (resp. all bounded) continuous real-valued functions. βX will denote the Stone-Cech compactification of X. In [2], Nel and Riorden defined $C^{\neq}(X)$ to be the set of all $f \in C(X)$ such that $M(f)$ is real in the residue class ring $C(X)/M$ for every maximal ideal M in $C(X)$. $C^{\neq}(X)$ is a subalgebra as well as a sublattice of $C^*(X)$. Some equivalent topological and algebraic characterizations of $C^{\neq}(X)$ are given. The main aim of this paper is to prove that X is pseudocompact iff $C^{\neq}(X)=C(X)$ iff $C^{\neq}(X)$ determines the topology of X and is uniformly closed. All notations and background information are referred to [1].

LEMMA. *If $\{x_n : n \in N\}$ is a subset of a zero set Z of X such that* $\lim_{n\to\infty} f(x_n)=r$ *and $r \notin f[Z]$, then $\{x_n : n \in N\}$ contains a C-embedded copy of N.*

Proof. Let $Z=Z(g)$ and $h=1/((f-r)^2+g^2)$. Then $h \in C(X)$. Since h is unbounded on $\{x_n : n \in N\}$, hence $\{x_n : n \in N\}$ contains a C-embeddable copy of N [1, 1.20].

The equivalence of (1) and (3) in the following theorem was given in [2].

THEOREM. *For a function $f \in C(X)$ the following are equivalent.*

(1) $f \in C^{\neq}(X)$

(2) *$f \in C^*(X)$ and $p \in \mathrm{Cl}_{\beta X}[Z(f-f^{\beta}(p))]$ for every $p \in \beta X$.*

(3) *$f \in C^*(X)$ and $f[D]$ is finite for every C-embedded copy D of N.*

(4) *$f \in C^*(X)$ and $f[Z]$ is closed for every zero set Z of X.*

Proof. We show (1)→(2)→(3)→(4)→(1).

(1)→(2). Let $f \in C^{\neq}(X)$. Then $f \in C^*(X)$ [1, 5.7]. For any $p \in \beta X$, since $M^P(f)$ is real, hence $f-r \in M^P$ for some $r \in R$. It follows that $p \in \mathrm{Cl}_{\beta X}[Z(f-r)]$ and $f^{\beta}(p)=r$.

(2)→(3). Let $p \in \mathrm{Cl}_{\beta X} D$. Then $p \in \mathrm{Cl}_{\beta X}[Z(f-f^{\beta}(p))] \cap \mathrm{Cl}_{\beta X} D$. If $Z(f-f^{\beta}(p)) \cap D=\varnothing$, then $\mathrm{Cl}_{\beta X} Z(f-f^{\beta}(p)) \cap \mathrm{Cl}_{\beta X} D=\varnothing$ [1, 1.18]. Thus $Z(f-f^{\beta}(p)) \cap D \neq \varnothing$ and hence $f^{\beta}(p) \in f[D]$. Hence $f[D]=f^{\beta}[\mathrm{Cl}_{\beta X} D]$ is closed. It follows that $f[D]$ is finite.

(3)→(4). Assume that $f[Z]$ is not closed for some zero set Z. Let $r \in \mathrm{Cl}_R[f[Z]]-f[Z]$. Choose $x_n \in Z$ $(n \in N)$ such that $\lim_{n\to\infty} f(x_n)=r$. Then, by the

Received by the editors June 6, 1973 and, in revised form, September 17, 1973.

[1] This paper is a part of the author's Ph.D. thesis at the University of British Columbia written under the supervision of J. V. Whittaker.

lemma, $\{x_n : n \in N\}$ contains a C-embedded copy D of N. It is obvious that $f[D]$ is not finite.

(4)→(1). Let M be any maximal ideal of $C(X)$. Then $M=M^P$ for some $p \in \beta X$ [1, 7.3]. Assume that $f-f^\beta(p) \notin M^P$. Then there exists a zero set Z such that $p \in \mathrm{Cl}_{\beta X} Z$ and $Z \cap Z(f-f^\beta(p))=\varnothing$. Since $f[Z]$ is closed, hence $f^\beta(p) \in f[Z]$. This is a contradiction. Therefore, $f-f^\beta(p) \in M^P$ and hence $M(f)$ is real.

There is an interesting algebraic characterization of $C^{\neq}(X)$ which depends only on the maximal ideals in $C(X)$ and $C^*(X)$.

THEOREM. *$C^{\neq}(X)$ is the largest subring of $C^*(X)$ satisfying:*
(1) *$C^{\neq}(X)$ contains all the constant functions, and*
(2) *$M^p \cap C^{\neq}(X)=M^{*p} \cap C^{\neq}(X)$ for every $p \in \beta X$.*

Proof. Suppose G is a subring of $C^*(X)$ satisfying conditions (1) and (2). Let $g \in G$. For every $p \in \beta X$, the function $g-g^\beta(p) \in M^{*p} \cap G$. By condition (2), $g-g^\beta(p) \in M^p \cap G$. Thus, $p \in \mathrm{Cl}_{\beta X}[Z(g-g^\beta(p))]$. Therefore, $g \in C^{\neq}(X)$. Hence $G \subseteq C^{\neq}(X)$. It remains to show that $C^{\neq}(X)$ satisfies the condition (2). For every $p \in \beta X$, it is obvious that $M^p \cap C^{\neq}(X) \subseteq M^{*p} \cap C^{\neq}(X)$. Let $f \in M^{*p} \cap C^{\neq}(X)$. Then $f^\beta(p)=0$ and $p \in \mathrm{Cl}_{\beta X}[Z(f-f^\beta(p))]=\mathrm{Cl}_{\beta X}[Z(f)]$. Hence, $f \in M^p$. Consequently, $M^p \cap C^{\neq}(X)=M^{*p} \cap C^{\neq}(X)$ for every $p \in \beta X$.

We are now ready to prove the main theorem.

THEOREM *The following are equivalent.*
(1) *X is a pseudocompact space.*
(2) *$C^{\neq}(X)=C(X)$.*
(3) *$C^{\neq}(X)$ determines the topology of X and $C^{\neq}(X)$ is uniformly closed.*

Proof. We show (1)→(2)→(3)→(1).

(1)→(2). Since $C(X)=C^*(X)$, hence, $M^p=M^{*p}$ for every $p \in \beta X$.

By previous theorem, it follows that $C^{\neq}(X)=C(X)$.

(2)→(3). It is obvious.

(3)→(1). Suppose X is not pseudocompact. Then X has a C-embedded copy $D=\{x_n : n \in N\}$. Let $\{O_n : n \in N\}$ be a countable collection of disjoint open sets where $x_n \in O_n$ for every $n \in N$. For every $n \in N$, there exists $f_n \in C^{\neq}(X)$ such that

(i) $f_n(x_n)=1/2^n$.
(ii) $Z(f_n)$ contains $X-O_n$.
and
(iii) $O \leq f_n \leq 1/2^n$.

Let $g=\sum_{n=1}^{\infty} f_n$. Since $C^{\neq}(X)$ is uniformly closed, hence $g \in C^{\neq}(X)$. But $g[D]$ is not closed. This is a contradiction. Consequently, X is a pseudocompact space.

REMARK. It can be proved that if X is a locally compact space, then $C^{\neq}(X)$ determines the topology of X. It is difficult to find a space X where $C^{\neq}(X)$ does not determine the topology of X.

REFERENCES

1. L. Gillman and M. Jerison, *Rings of Continuous Functions*, Van Nostrand, N.Y., 1960.

2. L. D. Nel and D. Riordan, *Note on a Subalgebra of $C(X)$*, Cand. Math. Bull. Vol. **15** **(4)**, 1972.

UNIVERSITY OF SCIENCE OF MALAYSIA,
PENANG, MALAYSIA