Compositio Mathematica **138:** 337–360, 2003.

Subregular Representations of $\mathfrak{sl}_n$ and Simple Singularities of Type A_{n-1}

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(Received: 19 March 2001; accepted in final form: 26 June 2002)

Abstract. Alexander Premet has stated the following problem: what is a relation between subregular nilpotent representations of a classical semisimple restricted Lie algebra and non-commutative deformations of the corresponding singularities? We solve this problem for type *A*.

Mathematics Subject Classifications (1991). Primary: 17B50; Secondary: 14J17, 16G10, 16S80.

1. Introduction

1.1. The last few years have seen interest grow in the representation theory of reductive Lie algebras in positive characteristic. Classical results of Kac and Weisfeiler show that the nilpotent coadjoint orbits of the Lie algebra comprise a natural parameter set for the representation theory, [25]. It is thus a basic problem to understand how the geometry of the nilpotent coadjoint orbits influences representation theory. A milestone in this direction is Premet's theorem which asserts that the dimension of any simple module associated to the orbit Ω has dimension divisible by $p^{1/2(\dim \Omega)}$, where p is the characteristic of the field, [21]. Building on Premet's theorem, Jantzen studied the subregular nilpotent coadjoint orbit in detail, obtaining a great deal of information on not only the simple modules, but also baby Verma modules, [14].

1.2. More recent work of Lusztig suggests a relationship between the representation theory associated to a nilpotent coadjoint orbit Ω and the geometry of a transverse slice to Ω, together with its desingularisation in the Springer resolution, [18]. The relationship would tie the simple and projective modules associated to Ω to certain elements in the K-theory of the transverse slice and its desingularisation. In the subregular case, the transverse slice is a Kleinian singularity and the Springer resolution provides its minimal resolution.

1.3. Quantisations of Kleinian singularities were introduced by Hodges [11] for type A and later by Crawley-Boevey and Holland [7] for all types. Premet has suggested a possible relationship between these quantisations and subregular representations of a

https://doi.org/10.1023/A:1027381710548 Published online by Cambridge University Press

simple Lie algebra. In [22] Premet examines this in characteristic zero. This paper is concerned with exploring Premet's suggestion in the modular type A case. In a subsequent paper we consider Lusztig's conjectures in the subregular case. It is the quantisations which build a bridge between the representation theory and the geometry, allowing us to connect the two sides.

1.4. Let us give some more detail. We refer the reader to later sections of the paper for unexplained definitions and notation. Fix an integer $n > 0$. Let $\mathbb{K}$ (respectively $\mathbb{L}$) be an algebraically closed field of characteristic p (respectively arbitrary characteristic). We will assume throughout that p and n are coprime. Let $\mathbb{F}$ be the prime subfield of $\mathbb{K}$. Given an integer $m \in \mathbb{Z}$ we denote its reduction modulo p by $\bar{m} \in \mathbb{F}$.

1.5. Let U be the enveloping algebra of $\mathfrak{sl}_n(\mathbb{K})$ and for $v \in \mathbb{K}[t]$ let $T(v)$ be Hodges' non-commutative deformation of a Kleinian singularity of type A over $\mathbb{K}$. Let U_χ be the reduced enveloping algebra for a subregular nilpotent functional χ and let $U_{\chi,\lambda}$ be the central reduction of U_χ determined by the weight λ. We show in Theorem 7.2 that, for $p > n$, there exists a polynomial $v_\lambda \in \mathbb{K}[t]$ such that $U_{\chi,\lambda} \cong \mathrm{Mat}(t(v_\lambda))$, where $t(v_\lambda)$ is a central reduction of $T(v_\lambda)$. Moreover, this isomorphism respects a natural $\mathbb{Z}$-grading on the algebras $U_{\chi,\lambda}$ and $t(v_\lambda)$ and so induces an equivalence between the category of $U_{\chi,\lambda}$-T_0-modules and a category of suitably graded $t(v_\lambda)$-modules.

1.6. A crucial step in the above theorem is the comparison of $U_{\chi,\lambda}$ and $t(v_\lambda)$ with a basic algebra we call the *no-cycle algebra*. This algebra is defined over any field $\mathbb{L}$, depends on a positive integer k, and is denoted $N_{\mathbb{L}}(k)$. The no-cycle algebra is a string algebra and hence its indecomposable modules can be described by a simple combinatorial procedure. As an application of this comparison we give a sufficient condition for the indecomposability of a baby Verma module belonging to the block of U_χ determined by a regular weight λ. This is formulated in terms of the geometry of the Springer fibre $\mathcal{B}_\chi$.

1.7. There are several clear directions for future work arising from this paper. Firstly, the hypothesis $p > n$ should be weakened to p and n being coprime. It would also be highly desirable to extend the results to other types. At the end of Section 7 we sketch how to extend our results to type B. It seems likely, however, that the techniques used here are not sufficient for this in general. Furthermore, we expect $N_{\mathbb{C}}(n)$-modules to correspond to a central reduction of the subregular representations of a quantum group of type A at a root of unity. We have not, however, checked this here.

1.8. The paper is organised as follows. In Section 2 we introduce the notation we will require when we deal with categories having group actions. We define the no-cycle algebra in Section 3 and describe some of its indecomposable representations. In

Section 4 we study subregular representations of $\mathfrak{sl}_n(\mathbb{K})$ and relate them to the no-cycle algebra. Section 5 takes a brief look at Gröbner–Shirshov bases for non-commutative algebras and their representations. In Section 6 we study Hodges' quantisation using, in particular, the results of the previous section. In Section 7 we prove Theorem 7.2, whilst in Section 8 we present the application to baby Verma modules.

2. $\mathbb{Z}$-Categories

2.1. A $\mathbb{Z}$-category is an Abelian $\mathbb{L}$-category equipped with exact *shift functors*, $[i]$ for each $i \in \mathbb{Z}$, together with natural isomorphisms $[i] \circ [j] \to [i+j]$. If $\mathcal{C}$ and $\mathcal{D}$ are both $\mathbb{Z}$-categories, we say that an $\mathbb{L}$-linear functor $\Phi: \mathcal{C} \to \mathcal{D}$ is a $\mathbb{Z}$-functor if the functors $\Phi \circ [i]$ and $[i] \circ \Phi$ are naturally isomorphic for every $i \in \mathbb{Z}$. A $\mathbb{Z}$-functor $\Phi : \mathcal{C} \to \mathcal{D}$ is a $\mathbb{Z}$-equivalence if there exists a $\mathbb{Z}$-functor $\Psi: \mathcal{D} \to \mathcal{C}$ such that $\Psi\Phi \cong 1_{\mathcal{C}}$ and $\Phi\Psi \cong 1_{\mathcal{D}}$. We say $\mathcal{C}$ and $\mathcal{D}$ are equivalent $\mathbb{Z}$-categories. Note that an equivalence between $\mathcal{C}$ and $\mathcal{D}$ need not be a $\mathbb{Z}$-equivalence [10, Section 5].

2.2. Let $R = \bigoplus_{i \in \mathbb{Z}} R_i$ be a $\mathbb{Z}$-graded (noetherian) algebra, that is $R_i R_j \subseteq R_{i+j}$. A $\mathbb{Z}$-graded R-module is an R-module, M, together with a $\mathbb{L}$-space decomposition $M = \bigoplus_{i \in \mathbb{Z}} M_i$ satisfying $R_i \cdot M_j \subseteq M_{i+j}$. The category of $\mathbb{Z}$-graded (finitely generated) R-modules, denoted R-Grmod (R-grmod) is an example of a $\mathbb{Z}$-category. By definition, we have $(M[i])_j = M_{j-i}$. for all $i, j \in \mathbb{Z}$.

2.3. Given $X, Y \in \mathcal{C}$ and $i \in \mathbb{Z}$, set $\mathrm{Hom}(X, Y)_i = \mathrm{Hom}_{\mathcal{C}}(X[i], Y)$. We define

$$\mathrm{Hom}(X, Y) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}(X, Y)_i,$$

a $\mathbb{Z}$-graded vector space. The identification $\mathrm{Hom}_{\mathcal{C}}(X[i], Y) \cong \mathrm{Hom}_{\mathcal{C}}(X[i+j], Y[j])$ yields a composition law for $X, Y, Z \in \mathcal{C}$: $\mathrm{Hom}(Y, Z)_j \times \mathrm{Hom}(X, Y)_i \to \mathrm{Hom}(X, Z)_{i+j}$. Then, thanks to 2.2, the space $\mathrm{End}(X) = \mathrm{Hom}(X, X)$ becomes a $\mathbb{Z}$-graded $\mathbb{L}$-algebra and $\mathrm{Hom}(X, Y)$ a $\mathbb{Z}$-graded $\mathrm{End}(X)$-module. The functor $\mathrm{Hom}(X, -)$ is a $\mathbb{Z}$-functor from $\mathcal{C}$ to $\mathrm{End}(X)$-Grmod.

3. The No-cycle Algebra

3.1. Let $k \in \mathbb{N}$ be a fixed integer. Let Q be the directed graph with k vertices and $2k$ edges labelled a_i and b_i for $i \in \mathbb{Z}/k\mathbb{Z}$, see Figure 1. Let $\mathbb{L}Q$ be the path algebra of Q. The *no-cycle algebra*, $N_{\mathbb{L}}(k)$, is the quotient of $\mathbb{L}Q$ by the two sided ideal generated by all non-trivial paths in Q which start and end at the same vertex. By inspection, $N_{\mathbb{L}}(k)$ is a $k(2k-1)$-dimensional algebra.

3.2. If $\mathbb{L}$ admits a primitive nth root of unity ζ then the no-cycle algebra can be described as a 'skew coinvariant algebra'. Let Γ be a cyclic group of order n acting on $\mathbb{L}[X, Y]$ by $g^m : (X, Y) \mapsto (\zeta^m X, \zeta^{-m} Y)$. There exists an isomorphism

$$N_{\mathbb{L}}(k) \cong \frac{\mathbb{L}[X, Y]}{(X^n, XY, Y^n)} * \Gamma = \mathbb{L}[X, Y]_{\Gamma} * \Gamma$$

which sends X to $\sum_{k=1}^{n} b_k$, Y to $\sum_{k=1}^{n} a_k$ and g to $\sum_{k=1}^{n} \zeta^{n-k} e_k$. This realization is of fundamental importance since the 'skew coinvariant algebra' controls much of the geometry surrounding the resolution of the Kleinian singularity of type A_{k-1}.

3.3. The algebra $N_{\mathbb{L}}(k)$ is $\mathbb{Z}$-graded: a vertex idempotent e_i has degree 0, a_i has degree -1 and b_i has degree 1. Following 2.2 the category of $\mathbb{Z}$-graded modules will be denoted by $N_{\mathbb{L}}(k)$-grmod.

3.4. We need some notation before describing several $N_{\mathbb{L}}(k)$-modules. For $i \in \mathbb{Z}/k\mathbb{Z}$, introduce formal inverses of the arrows a_i (respectively b_i), written a_i^* (respectively b_i^*). The head (respectively tail) of an arrow, c, is denoted $h(c)$ (respectively $t(c)$), and we define $h(c^*) = t(c)$ (respectively $t(c^*) = h(c)$). We form *formal paths* of length t, $c_1, \dots, c_t$, where each c_j is of the form c or c^* for some arrow c and $t(c_j) = h(c_{j+1})$. Given a formal path $c_1, \dots, c_t$, we define its inverse to be $c_t^*, \dots, c_1^*$, where $(c^*)^*$ equals c.

3.5. For $t \leqslant k$, let S_t be the set of formal paths $c_1 \dots c_t$ such that if $c_j = a_i$ (respectively b_i) then c_{j+1} is either a_{i-1} or b_{i-1}^* (respectively b_{i+1} or a_{i+1}^*), and similarly if $c_j = a_i^*$ (respectively b_i^*). Furthermore, if $t = k$ then exclude from S_k the formal paths consisting entirely of a's or entirely of b's and also the inverses of such formal paths. For $t < k$ (respectively $t = k$) let ρ_t be the equivalence relation on S_t which identifies a formal path with its inverse (respectively its cyclic permutations and their inverses). Let W_t be a set of equivalence class representatives in S_t for ρ_t.

3.6. For $t < k$, an element $C = c_1 \dots c_t \in W_t$ defines a $t+1$-dimensional indecomposable *string* $N_{\mathbb{L}}(k)$-module $St(C)$. A basis is given by $\{z_0, \dots, z_t\}$ where, for $1 \leqslant j \leqslant t$, the element z_j is concentrated at vertex $h(c_j)$, and z_0 is concentrated at vertex $t(c_1)$. For $1 \leqslant j \leqslant t$ if $c_j = c$, an arrow, define $c(z_j) = z_{j-1}$, whilst if $c_j = c^*$, define $c(z_{j-1}) = z_j$. All other arrows in Q act as zero.

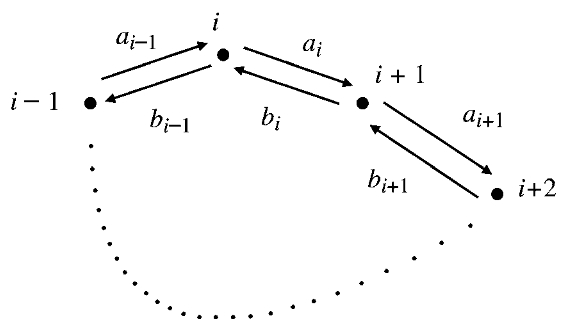

Figure 1.

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3.7. For $t = k$, and element $C = c_1, \dots, c_k \in W_k$ together with a scalar $\lambda \in \mathbb{L}^*$, define an indecomposable *band* $N_{\mathbb{L}}(k)$-module $\mathrm{Bd}_\lambda(C)$. A basis is given by $\{z_0, \dots, z_{k-1}\}$ where, for $0 \leqslant j \leqslant k-1$, the element z_j is concentrated at vertex $h(c_j)$. If $c_k = c$, an arrow, define $c(z_0) = \lambda z_{k-1}$, whilst if $c_k = c^*$, define $c(z_{k-1}) = \lambda^{-1} z_0$. For $1 \leqslant j \leqslant k-1$, if $c_j = c$, an arrow, then $c(z_j) = z_{j-1}$, whilst if $c_j = c^*$ define $c(z_{j-1}) = z_j$. All other arrows of Q act as zero.

3.8. The algebra $N_{\mathbb{L}}(k)$ belongs to a family of tame algebras called *string algebras*. Any non-zero indecomposable $N_{\mathbb{L}}(k)$-module of dimension no greater than k is isomorphic to either $\mathrm{St}(C)$ or $\mathrm{Bd}_\lambda(C)$ for some unique C and λ [6, Section 3] (see also [2]). Observe that the modules $\mathrm{St}(C)$ admit a $\mathbb{Z}$-grading compatible with the $\mathbb{Z}$-grading on $N_{\mathbb{L}}(k)$ introduced in 3.3.

4. The Reduced Enveloping Algebra

4.1. Let $\chi \in \mathfrak{sl}_n(\mathbb{L})^*$ be the functional vanishing on upper triangular matrices and defined as follows on the strictly lower triangular matrices

$$\chi(E_{i,j}) = \begin{cases} 1 & \text{if } i = j+1 \text{ and } 1 \leqslant j \leqslant n-2, \\ 0 & \text{otherwise.} \end{cases}$$

Let P (respectively P^+) be the weight lattice (respectively dominant weights) of $\mathrm{SL}_n(\mathbb{L})$ with respect to the standard choice of torus and Borel subgroup. Let $\{\varpi_1, \dots, \varpi_{n-1}\}$ be the fundamental weights and let $\rho = \varpi_1 + \cdots + \varpi_{n-1}$. The Weyl group W is the symmetric group $\mathfrak{S}_n$ acting on P. The W-orbit through $\lambda \in P$ contains a unique representative in P^+. We also need a dot action given by

$$w \bullet \lambda = w(\lambda + \rho) - \rho, \quad w \in W, \lambda \in P.$$

4.2. Let $\mathcal{B}$ be the flag variety. As a set this consists of all Borel subalgebras of $\mathfrak{sl}_n(\mathbb{L})$, that is those subalgebras which are conjugate under $\mathrm{SL}_n(\mathbb{L})$ to the upper triangular matrices. The cotangent bundle of $\mathcal{B}$ is naturally identified with the variety

$$\widetilde{\mathcal{N}} = \{(x, \mathfrak{b}) : x(\mathfrak{b}) = 0\} \subset \mathfrak{sl}_n(\mathbb{L})^* \times \mathcal{B}$$

where the first projection π_1 becomes the moment map. The Springer fibre $\mathcal{B}_\chi$ is the subvariety $\pi_1^{-1}(\chi)$ of $\mathcal{B}$.

There is a simple way to parametrise $\mathcal{B}_\chi$, [24, Section 6.3]. Given a basis u_i of $\mathbb{L}^n$, let $\mathcal{F}(u_1, \dots, u_n)$ be a flag with the span of $u_1, \dots, u_k$ as the k-dimensional space. Let v_i be an element of the standard basis of $\mathbb{L}^n$ so that $E_{i,j} v_j = v_i$. We introduce the flag

$$\mathcal{F}_{k,\alpha} = \mathcal{F}(v_1, v_2, \dots, v_{k-1}, v_k + \alpha v_n, v_n, v_{k+1}, v_{k+2}, \dots, v_{n-1})$$

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for all $(k, \alpha) \in (\{1, \dots, n-1\} \times \mathbb{L}) \cup \{(0, 0)\}$. The irreducible components of $\mathcal{B}_\chi$ are projective lines Π_k, $1 \leqslant k \leqslant n-1$ where

$$\Pi_k = \{\mathcal{F}_{n-k,\alpha} \mid \alpha \in \mathbb{L}\} \cup \{\mathcal{F}_{n-k-1,0}\}.$$

For $2 \leqslant k \leqslant n-1$ the components Π_{k-1} and Π_k intersect transversally at a point $p_{k-1,k} = \mathcal{F}_{n-k,0}$. Components Π_i and Π_j with $|i-j| > 1$ do not intersect.

Consider the following one parameter subgroup of the diagonal matrices in $\mathrm{SL}_n(\mathbb{L})$

$$T_0 = \{v(\tau) = \tau E_{1,1} + \tau E_{2,2} + \cdots + \tau E_{n-1,n-1} + \tau^{1-n} E_{n,n} : \tau \in \mathbb{L}^*\}.$$

Notice that T_0 stabilises $\mathcal{B}_\chi$, since $v(\tau) \cdot \mathcal{F}_{i,\alpha} = \mathcal{F}_{i,\tau^{-n}\alpha}$.

4.3. Let us further assume that $\mathbb{L} = \mathbb{C}$. By the Jacobson-Morozov theorem there exists an $\mathfrak{sl}_2$-triple $e, h, f \in \mathfrak{sl}_n(\mathbb{C})$ such that $\mathrm{Tr}(ex) = \chi(x)$ for each $x \in \mathfrak{sl}_n(\mathbb{C})$. Let $\mathcal{N}$ be the variety of nilpotent elements in $\mathfrak{sl}_n(\mathbb{C})$. Let

$$V_\chi = \{\mu \in \mathfrak{sl}_n(\mathbb{C})^* \mid \forall x \in \mathfrak{sl}_n(\mathbb{C})\ \mu([x, f]) = \chi([x, f])\}.$$

By [24, Theorem 6.4 and Section 7.4] V_χ is a Kleinian singularity of type A_{n-1} and $\Lambda_\chi = \pi_1^{-1}(V_\chi)$ is its minimal desingularisation with the exceptional fibre $\mathcal{B}_\chi$.

4.4. *For the rest of this section we will assume that* $p > n$. We expect, however, all results to hold under the weaker condition that p and n are coprime. The proof of Proposition 4.13 is the crucial point where we require the restriction $p > n$ and all other results of this section, up to 4.17 inclusive, will continue to hold if this proposition can be proved under the weaker hypothesis.

4.5. We are going to work with the Lie algebra $\mathfrak{sl}_n(\mathbb{K})$ now. For any element $X \in \mathfrak{sl}_n(\mathbb{K})$ let $X^{[p]} \in \mathfrak{sl}_n(\mathbb{K})$ denote the pth power of X.

Let $U(\mathfrak{sl}_n(\mathbb{K}))$ be the universal enveloping algebra of $\mathfrak{sl}_n(\mathbb{K})$. For any $X \in \mathfrak{sl}_n(\mathbb{K})$ the element $X^p - X^{[p]} \in U(\mathfrak{sl}_n(\mathbb{K}))$ is central. We will study representations of the following *subregular reduced enveloping algebra*

$$U_\chi := \frac{U(\mathfrak{sl}_n(\mathbb{K}))}{(X^p - X^{[p]} - \chi(X)^p : X \in \mathfrak{sl}_n(\mathbb{K}))}.$$

Let $\Lambda = P/pP$, an $\mathbb{F}$-vector space, and let $\psi \colon P \to \Lambda$ be the quotient map. Both the usual and the dot actions of W on P pass to Λ. Let $W(\lambda)$ be the stabiliser of $\lambda \in \Lambda$ under the dot action. Set

$$C_0 = \left\{\lambda \in P : \lambda + \rho = \sum r_i \varpi_i \text{ with } r_i \geqslant 0 \text{ and } p \geqslant (r_1 + \cdots + r_{n-1})\right\}.$$

A weight in the interior of C_0 is called *regular*.

4.6. BLOCKS

By [5, Theorem 3.18] we have a block decomposition

$$U_\chi = \bigoplus_\lambda B_{\chi,\lambda},$$

where λ runs over a set of representatives of the $W\bullet$-orbits on Λ.

Let $\mathfrak{h}$ be the diagonal Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{K})$. The Weyl group acts by algebra automorphisms on the polynomial ring $S(\mathfrak{h})$. For $\lambda \in \Lambda$ the partial coinvariants give a local, graded algebra

$$C_\lambda = \frac{S(\mathfrak{h})^{W(\lambda)}}{(S(\mathfrak{h})^W_+)}.$$

Thanks to ([19], Theorem 10) and ([22], Theorem 8.2) there is an injective algebra homomorphism $C_\lambda \longrightarrow B_{\chi,\lambda}$, whose image is central in $B_{\chi,\lambda,}$. Henceforth, we will identify C_λ with its image in $B_{\chi,\lambda}$.

4.7. We will be concerned with the category of finite-dimensional U_χ-modules, U_χ-mod, or more specifically the subcategory of $B_{\chi,\lambda}$-modules. These categories have graded analogues which we introduce now.

By construction $\chi(tXt^{-1}) = \chi(X)$ for all $X \in \mathfrak{sl}_n(\mathbb{K})$ and $t \in T_0$. As a result the action of T_0 on $U(\mathfrak{sl}_n(\mathbb{K}))$ passes to an action on the quotient U_χ.

Following Jantzen [12], a U_χ-T_0-module is a finite-dimensional vector space V over $\mathbb{K}$ that has a structure both as a U_χ-module and as a rational T_0-module such that the following compatibility conditions hold:

(1) We have $t(Xv) = (tXt^{-1})tv$ for all $X \in \mathfrak{sl}_n(\mathbb{K})$, $t \in T_0$ and $v \in V$;
(2) The restriction of the $\mathfrak{sl}_n(\mathbb{K})$-action on V to $\mathrm{Lie}(T_0)$ is equal to the derivative of the T_0-action on V.

We obtain the category U_χ-T_0-mod, whose objects are the U_χ-T_0-modules and whose morphisms are the T_0-equivariant U_χ-module homomorphisms.

For $i \in \mathbb{Z}$, there are shift functors $[i] : U_\chi\text{-}T_0\text{-mod} \longrightarrow U_\chi\text{-}T_0\text{-mod}$. These send a given U_χ-T_0-module V to the object having the same U_χ-module structure but with T_0 acting by $\nu(\tau)v = \tau^{ip}\nu(\tau)v$ for $\nu(\tau) \in T_0$ and all $v \in V$. This makes U_χ-T_0-mod a $\mathbb{Z}$-category.

By [9, 9.3] the full $\mathbb{Z}$-subcategory $B_{\chi,\lambda}$-T_0-mod of U_χ-T_0-mod is well-defined. Its objects are $B_{\chi,\lambda}$-modules with a compatible rational T_0-action. The projection functor $\mathrm{pr}_\lambda \colon U_\chi\text{-}T_0\text{-mod} \longrightarrow B_{\chi,\lambda}\text{-}T_0\text{-mod}$ is a $\mathbb{Z}$-functor.

4.8. Let $F : U_\chi\text{-}T_0\text{-mod} \longrightarrow U_\chi\text{-mod}$ denote the functor which forgets the T_0-structure. The objects of U_χ-mod which are in the image of F are called *gradable*. It follows from ([10], Corollary 3.4) and ([15], Corollary 1.4.1) that the simple U_χ-modules and their projective covers are gradable. Moreover, any lift of a simple U_χ-module is simple in U_χ-T_0-mod and any lift of a projective indecomposable U_χ-module is projective indecomposable in U_χ-T_0-mod. Suppose M is gradable, that is there exists a U_χ-T_0-module V such that $F(V) = M$. Then, by ([15], Remark 1.5), we have $F(\mathrm{soc}\ V) = \mathrm{soc}\ M$ and $F(\mathrm{rad}\ V) = \mathrm{rad}\ M$.

For any $M, N \in U_\chi$-T_0-mod, using the notation of 2.3, we have $\mathrm{Hom}_{U_\chi}(F(M), F(N)) = \oplus_i \mathrm{Hom}_{U_\chi\text{-}T_0}(M[i], N)$, ([10], Section 2).

4.9. The category U_χ-T_0-mod admits a contravariant self-equivalence, D whose square is canonically isomorphic to the identity functor, ([13], Sections 1.13 and 1.14). Moreover D fixes the simple modules in $B_{\chi,\lambda}$-T_0-mod, ([13], Proposition 2.16).

4.10 TRANSLATION FUNCTORS

Using the map $\psi\colon P \to \Lambda$, we will abuse notation by writing $B_{\chi,\lambda}$-T_0-mod and pr_λ for $\lambda \in P$ (we should really take the image of λ under ψ). Given $\lambda, \mu \in C_0$ we define a translation functor

$$T_\lambda^\mu\colon B_{\chi,\lambda}\text{-}T_0\text{-mod} \to B_{\chi,\mu}\text{-}T_0\text{-mod}$$

by $T_\lambda^\mu(V) = \mathrm{Pr}_\mu(E \otimes V)$, where E is the simple $\mathrm{SL}_n(\mathbb{K})$-module with the highest weight $w(\mu - \lambda) \in P^+$ for some $w \in W$.

Note that we get (in general) more than one functor $B_{\chi,\lambda}$-T_0-mod $\to B_{\chi,\mu}$-T_0-mod for fixed λ and μ: if μ and μ' are two distinct weights in C_0 with $\psi(\mu)$ and $\psi(\mu')$ in the same W-orbit then T_λ^μ and $T_\lambda^{\mu'}$ will be two (in general) distinct functors from $B_{\chi,\lambda}$-T_0-mod to $B_{\chi,\mu}$-T_0-mod $= B_{\chi,\mu'}$-T_0-mod.

4.11. BABY VERMA MODULES

Given $\mathfrak{b} \in \mathcal{B}_\chi$ and $\lambda \in P$ we have a one dimensional representation of $U_0(\mathfrak{b})$, the subalgebra of U_χ generated by the elements of $\mathfrak{b}$, described as follows. Let $g \in \mathrm{SL}_n(\mathbb{K})$ be such that g conjugates $\mathfrak{b}$ to the upper triangular matrices in $\mathfrak{sl}_n(\mathbb{K})$, say $\mathfrak{b}_+$. There is a one dimensional $U_0(\mathfrak{b}_+)$-module which is annihilated by strictly upper triangular matrices and on which diagonal matrices act via $\psi(\lambda)$. Conjugation by g provides an isomorphism between $U_0(\mathfrak{b})$ and $U_0(\mathfrak{b}_+)$, and so gives a one dimensional $U_0(\mathfrak{b})$-module, say $\mathbb{K}_\lambda$. It can be checked that this module is independent of the choice of $g \in \mathrm{SL}_n(\mathbb{K})$. Induction yields a *baby Verma module*

$$V(\mathfrak{b}, \lambda) = U_\chi \otimes_{U_0(b)} \mathbb{K}_\lambda$$

This is a $B_{\chi,\lambda}$-module on which C_λ acts by scalar multiplication.

4.12. The module $V(\mathfrak{b}_+, \lambda)$ can be given the structure of a $B_{\chi,\lambda}$-T_0-module where T_0 acts on $1 \otimes 1$ through λ. Set $V(\mathfrak{b}_+, \lambda)' = D(V(\mathfrak{b}_+, \lambda))$ (it follows from [12, 11.16(1)] that this is a baby Verma module with respect to the Borel subalgebra obtained by conjugating $\mathfrak{b}_+$ by the Coxeter element $s_1 s_2 \ldots s_{n-1}$).

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PROPOSITION [13, Theorem 2.6]. *Suppose* $\lambda \in C_0$ *with* $\lambda + \rho = \sum r_i \varpi_i$ *and let* $r_0 = p - (r_1 + \cdots + r_{n-1})$.

(i) *The category* $B_{\chi,\lambda}$-T_0-mod *has simple modules* $L_0, L_1, \ldots, L_{n-1}$ *(up to isomorphism and shift) where the module* L_i *has dimension* $p^{(n^2-n-2)/2} r_{n-1-i}$ *(if* $r_{n-1-i} = 0$ *then* L_i *should be omitted from the list of simple modules).*

(ii) *For each i between* 0 *and* $n-1$ *there exists a uniserial module* $V_i \in B_{\chi,\lambda}$-T_0-mod *whose Loewy layers are* $L_i, L_{i+1}[-1], \ldots, L_{i-1}[1-n]$ *(count the subscripts modulo n, and omit* V_i *and* L_i *whenever* $r_{n-1-i} = 0$*).*

(iii) *For each i between* 0 *and* $n-1$ *there exists a uniserial module* $V_i' \in B_{\chi,\lambda}$-$T_0$-mod *whose Loewy layers are* $L_i, L_{i-1}[1], \ldots, L_{i+1}[n-1]$ *(count the subscripts modulo n, and omit* V_i' *and* L_i *whenever* $r_{n-1-i} = 0$*).*

(iv) *For any* $\mu \in P$ *such that* $\psi(\mu)$ *and* $\psi(\lambda)$ *are in the same* $W\bullet$*-orbit there exists a unique i and* $j \in \mathbb{Z}$ *(respectively* i', j'*) such that* $V(\mathfrak{b}_+, \mu) \cong V_i[j]$ *(respectively* $V(\mathfrak{b}_+, \mu)' \cong V_{i'}'[j']$*).*

4.13 ENDOMORPHISMS

Let $\lambda \in P$ and let Z_λ be the centre of $B_{\chi,\lambda}$. Recall $C_\lambda \subseteq Z_\lambda$. Let M be in $B_{\chi,\lambda}$-T_0-mod. There is a homomorphism

$$\theta_M: Z_\lambda \longrightarrow \mathrm{End}_{B_{\chi,\lambda}}(F(M)),$$

sending $z \in Z_\lambda$ to the endomorphism $(m \longmapsto z \cdot m)$.

PROPOSITION. *Let* $\lambda \in C_0$ *and* Q *be a projective indecomposable module in* $B_{\chi,\lambda}$-T_0-mod. *The homomorphism*

$$\theta_Q: Z_\lambda \longrightarrow \mathrm{End}_{B_{\chi,\lambda}}(F(Q))$$

is surjective.

Proof. To make the paper as self-contained as possible, we will give a direct proof of this for regular weights λ, since the general case relies on an unpublished result of Jantzen and Sörgel, [16].

It is enough to prove this in the ungraded case. Let $P_0, \ldots, P_r$ be all distinct (up to isomorphism) projective indecomposable $B_{\chi,\lambda}$-modules. The algebra C_λ has a simple socle [8, Corollary 3.9]. Therefore if $\mathrm{Ann}_{C_\lambda}(P_i)$ is non-zero it must contain the socle of C_λ. The equality $0 = \mathrm{Ann}_{C_\lambda}(B_{\chi,\lambda}) = \cap_i \mathrm{Ann}_{C_\lambda}(P_i)$, implies that C_λ acts faithfully on at least one projective indecomposable, say P_0. By Proposition 4.12 and a result of Jantzen ([12], Proposition 10.11) the dimension of $\mathrm{End}_{B_{\chi,\lambda}}(P_0) = \dim C_\lambda$, so Z_λ generates $\mathrm{End}_{B_{\chi,\lambda}}(P_0)$.

Now assume λ is regular. It follows from ([13], Section 2.3) that we can find a translation functor T such that $T(P_0) \cong P_i$ and, by ([12], Section 11.21), that T is a self-equivalence of $B_{\chi,\lambda}$-mod. Since Z_λ can be identified with the endomorphism ring of the identity functor on $B_{\chi,\lambda}$-mod, conjugation by T induces a ring automorphism of Z_λ, say $\tilde{T}$. It follows that $\mathrm{End}_{B_{\chi,\lambda}}(P_i)$ is generated by $\tilde{T}(C_\lambda) \subseteq Z_\lambda$, as claimed.

If λ is not regular the above argument fails since it need no longer be true that we can find a translation functor T which is a self-equivalence and sends P_0 to P_i. In this situation we use the following fact, ([16], C.6 Claim 2): if the highest weight of P_i (in the graded category) belongs to C_0 and does not lie on the affine wall, then the canonical map $C_\lambda \to \mathrm{End}_{B_{\chi,\lambda}}(P_i)$ is an isomorphism. Hence, it is enough to show that we can find a representative of the isomorphism class of P_i whose highest weight belongs to C_0 and does not lie on the affine wall. A straightforward calculation shows that this follows from ([14], Section 2.3). □

4.14. It is possible to *slightly* weaken the hypothesis $p > n$ when using the results of [16], and hence the hypothesis of this whole section. The proof of Jantzen's results, however, do not apparently generalise to the best case where p and n are coprime. We leave these details to the interested reader.

4.15. Let J be the unique maximal ideal of Z_λ.

LEMMA. *Let P_i be the projective cover of L_i in $B_{\chi,\lambda}$-T_0-mod. Then* $[P_i/JP_i : L_i[j]] = \delta_{j0}$.

Proof. It is enough to prove this for ungraded $B_{\chi,\lambda}$-modules. Suppose that $F(L_i)$ is a composition factor of $F(P_i)/JF(P_i)$. Thus $F(L_i)$ appears as a direct summand of $\mathrm{rad}^m F(P_i)/\mathrm{rad}^{m+1} F(P_i)$ for some $m \in \mathbb{N}$. Hence we have a commutative diagram

$$\begin{array}{ccccccc} F(P_i) & \twoheadrightarrow & F(L_i) & \longrightarrow & \mathrm{rad}^m F(P_i)/\mathrm{rad}^{m+1} F(P_i) & \longrightarrow & F(P_i)/\mathrm{rad}^{m+1} F(P_i) \\ & \searrow & & & & & \uparrow \\ & & & - - - - - - - - & & \dashrightarrow & F(P_i) \end{array}$$

Thanks to Lemma 4.13 there exists $z \in Z_\lambda$ such that the above endomorphism of $F(P_i)$ is multiplication by z. Then $z \notin J$ since, by hypothesis, the composition factor $F(L_i)$ does not lie in $JF(P_i)$. Since Z_λ is local it follows that the endomorphism is an isomorphism and so $F(L_i)$ lies in the head of $F(P_i)$ as required. □

4.16. TWO CENTRAL REDUCTIONS

Let $\tilde{J}$ be the unique maximal ideal of C_λ and recall that J is the unique maximal ideal of Z_λ. We introduce two central reductions

$$\tilde{U}_{\chi,\lambda} = \frac{B_{\chi,\lambda}}{\tilde{J}B_{\chi,\lambda}}, \qquad U_{\chi,\lambda} = \frac{B_{\chi,\lambda}}{JB_{\chi,\lambda}}.$$

Since $\tilde{J}$ and J are homogeneous both $\tilde{U}_{\chi,\lambda}$ and $U_{\chi,\lambda}$ inherit $\mathbb{Z}$-gradings from $B_{\chi,\lambda}$. The category of graded modules $U_{\chi,\lambda}$-T_0-mod is thus a full subcategory of $B_{\chi,\lambda}$-T_0-mod and there is a $\mathbb{Z}$-action on $U_{\chi,\lambda}$-T_0-mod, inherited from $B_{\chi,\lambda}$-T_0-mod.

4.17. The main result of this section follows.

PROPOSITION. *Suppose* $\lambda \in C_0$ *with* $\lambda + \rho = \sum r_i \varpi_i$, *and let* $r_0 = p - (r_1 + \cdots + r_{n-1})$. *Let k be the number of non-zero* r_i*'s. Then* $U_{\chi,\lambda}$ *is Morita equivalent to the no-cycle algebra* $N_{\mathbb{K}}(k)$. *Moreover, if* λ *is regular there is a* $\mathbb{Z}$*-equivalence of categories*

$$U_{\chi,\lambda}\text{-}T_0\text{-mod} \longrightarrow N_{\mathbb{K}}(n)\text{-grmod}.$$

Proof. Let A be a finite dimensional algebra with simple modules $S_1, \dots, S_r$. The Gabriel quiver of A is the directed graph with vertices labelled from 1 to r and $\dim \mathrm{Ext}^1_A(S_i, S_j)$ edges from i to j. By ([3], Proposition 4.17) A is Morita equivalent to the the path algebra of its Gabriel quiver factored by some admissible ideal, that is an ideal generated by linear combinations of paths of length at least two.

Recall the notation of Proposition 4.12. Let $0 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n-1$ be such that $L_{i_t} \neq 0$, or, equivalently, $r_{n-1-i_t} \neq 0$. Set $s_t = i_{t+1} - i_t$ for $1 \leqslant t < k$ and set $s_k = n + i_1 - i_k$. Since L_i appears only once as a composition factor of V_i, $Z(B_{\chi,\lambda})$ acts by scalars on V_i, making V_i a $U_{\chi,\lambda}$-module. By [13, Proposition 2.19], for $t \neq t'$ modulo k and $j \in \mathbb{Z}$

$$\mathrm{Ext}^1_{U_{\chi,\lambda}-T_0}(L_{i_t}[j], L_{i_{t'}}) = \begin{cases} \mathbb{K}, & \text{if } t' = t+1, j = s_t \text{ or } t' = t-1, j = -s_{t-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus the Gabriel quiver of $U_{\chi,\lambda}$ is of the form 3.1, possibly with loops added at the vertices. Let B be the quotient of this quiver which is Morita equivalent to $U_{\chi,\lambda}$. We will show B is isomorphic to $N_{\mathbb{K}}(k)$.

The projective covers of the simple B-modules are spanned by the paths ending in a fixed vertex. Hence, Lemma 4.15 shows that there can be no loops at vertices and further, that B is therefore a quotient of $N_{\mathbb{K}}(k)$. In particular, its dimension is at most $k(2k-1)$.

Let T_{i_t} be the kernel of the sum of two projections $V_{i_t} \oplus V'_{i_t} \to L_{i_t}$. By Proposition 4.12

$$[F(T_{i_t}) : F(L_{i_{t'}})] = \begin{cases} 2, & \text{if } t \neq t' \\ 1, & \text{if } t = t'. \end{cases}$$

Since $F(T_{i_t})$ is a quotient module of $F(P_{i_t})$, the dimension of B can be estimated by

$$\dim B = \mathrm{End}\left(\bigoplus_{t=1}^{k} F(P_{i_t})\right) = \sum_{t,t'} [F(P_{i_t}) : F(L_{i_{t'}})] \geqslant \sum_{t,t'} [F(T_{i_t}) : F(L_{i_{t'}})] = k(2k-1).$$

We deduce that $B \cong N_{\mathbb{K}}(k)$, proving the first statement of the theorem. This also proves that T_{i_t} is the projective cover of L_{i_t} in $U_{\chi,\lambda}$-T_0-mod.

Let λ be regular, so that $k = n$, and let $T = \oplus T_{i_t}$. Thanks to 2.3 and 4.8 the algebra $E = \mathrm{End}_{U_{\chi,\lambda}}(F(T))^{\mathrm{op}}$ has a $\mathbb{Z}$-grading. By ([10], Theorem 5.4) $U_{\chi,\lambda}$-T_0-mod and $N_{\mathbb{K}}(n)$-mod are equivalent $\mathbb{Z}$-categories if $E \cong N_{\mathbb{K}}(n)$ as a $\mathbb{Z}$-graded algebra.

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But, up to a choice of scalars, b_i corresponds to a $U_{\chi,\lambda}$-T_0-module homomorphism sending $T_i[1]$ to T_{i+1}, and a_i corresponds to the a $U_{\chi,\lambda}$-T_0-module homomorphism sending $T_{i+1}[-1]$ to T_i. This proves the second claim. □

4.18. TWO CENTRAL REDUCTIONS (II)

In general the inclusion $C_\lambda \subseteq Z_\lambda$ is strict, ([8], Corollary 3.9). However, the natural homomorphism $\tilde{U}_{\chi,\lambda} \to U_{\chi,\lambda}$ is an isomorphism. This follows from the fact in the proof of Proposition 4.3 that the map

$$\theta_1 : C_\lambda \longrightarrow \mathrm{End}_{B_{\chi,\lambda}}(F(P))$$

is an isomorphism for any projective indecomposable module P. The arguments of 4.15 and 4.17 are then valid for $\tilde{U}_{\chi,\lambda}$, from which the isomorphism follows. We expect this continues to hold under the weaker hypothesis that p and n are coprime.

5. Gröbner-Shirshov Bases

5.1. We are going to use Gröbner-Shirshov bases for associative algebras (see [4] for two-sided ideals and [17] for one-sided ideals). In this section we quickly explain the technique to make our paper self-contained. Although the version for one-sided ideals [17] is sufficient for our ends, we generalise to arbitrary modules to avoid repetitions.

5.2. Let $R = \mathbb{L}\langle X_1, X_2, \ldots, X_l\rangle$ be a free associative algebra. Let F be a free R-module with generators $Y_1, \ldots, Y_k$. Although l and k are natural numbers here, one can use, with certain care, the technique for transfinite ordinals.

Let $\mathcal{R}$ be the set of monomials in R, $\mathcal{F}$ the set of monomials in F. The set $\mathcal{R} \cup \mathcal{F}$ admits a partial multiplication with a two-sided unit 1_R (one agrees that $m1 = m$ for $m \in \mathcal{F}$). We always make the assumption that a product is defined when we write the product. We start with a linear order $\succ$ on $\mathcal{R} \cup \mathcal{F}$ such that

- $\forall z \in \mathcal{R} \cup \mathcal{F}\ z \succ 1_R$;
- $z_1 \succ z_2 \Longrightarrow w z_1 v \succ w z_2 v$;
- $\forall z \in \mathcal{R} \cup \mathcal{F}$ the set $\{w \in \mathcal{R} \cup \mathcal{F} | z \succ w\}$ is finite.

A degree lexicographical order is most practical but there are different orders. For a nonzero element f of $R \cup F$ we denote the highest term of f by $\bar{f}$.

5.3. A pair of subsets $S \subseteq R$ and $T \subseteq F$ determine an algebra $A = R/RSR$ and a left A-module $M = A \otimes_R (F/RT) = F/(RT + RSF)$. The technique of Gröbner–Shirshov pairs allows to solve questions about M by producing an explicit basis of M.

Each $f \in S \cup T$ gives a rewriting rule $\bar{f} \to \bar{f} - f$. We write $a \rightsquigarrow b$ if b can be obtained from a by using rewriting rules. Note that one cannot rewrite a monomial $\bar{f}t$ unless $t = 1$ or $f \in S$.

5.4. COMPOSITION

For certain $f, g \in R \cup F$, $w \in \mathcal{R} \cup \mathcal{F}$, we can form a composition $(f, g)_w$. If $w = \bar{f}V = W\bar{g} = WZV$ for some $W, Z, V \in \mathcal{R} \cup \mathcal{F}$ with $Z \neq 1$ then the composition is

$$(f, g)_w = fV - Wg.$$

If $w = W\bar{f}V = \bar{g}$ for some $W, V \in \mathcal{R} \cup \mathcal{F}$ then the composition is

$$(f, g)_w = WfV - g.$$

These two cases are mutually exclusive.

A pair (S, T) is a Gröbner–Shirshov pair if $(f, g)_w \rightsquigarrow 0$ for all possible $f, g \in S \cup T$ and $w \in \mathcal{R} \cup \mathcal{F}$.

The following version of Shirshov's composition lemma can be proved by standard methods [4]. If $k = 1$ and $T = \emptyset$ then the statement is the standard version of Shirshov's composition lemma [4]. If $k = 1$ and T arbitrary then it is a version for left ideals [17].

5.5. SHIRSHOV'S COMPOSITION LEMMA

For S and T as above, we consider the set of monomials

$$B = \{Z \in \mathcal{F} \mid \forall f \in S \cup T\ \forall W, V\ Z \neq W\bar{f}V\}.$$

If (S, T) is a Gröbner–Shirshov pair then the image of B is a basis of M as an $\mathbb{L}$-vector space.

Moreover, for every (S, T) there exits a Gröbner–Shirshov pair (S', T') such that $RSR = RS'R$ and $RSF + RT = RS'F + RT'$.

5.6. BUCHBERGER'S ALGORITHM

A proof of existence of a Gröbner–Shirshov pair uses transfinite recursion, called Buchberger's algorithm. It proceeds as follows. One starts with $(S_0, T_0) = (S, T)$. Given (S_m, T_m) we produce the next pair (S_{m+1}, T_{m+1}) such that $RS_mR = RS_{m+1}R$ and $RS_mF + RT_m = RS_{m+1}F + RT_{m+1}$. Consider all possible compositions $(f, g)_w$ with $f, g \in S_m \cup T_m$. To each such composition, apply a sequence of rewriting rules $\bar{v} \to \bar{v} - v$ with $v \in S_m \cup T_m$ so that $(f, g)_w \rightsquigarrow [f, g]_w$ and $[f, g]_w$ cannot be rewritten any further. Note that the element $[f, g]_w$ is not canonical since we choose a sequence

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of rewriting rules to use. Another sequence can give a different answer. Define the following sets

$$I_m = \{[f,g]_w \mid f,g \in S_m \cup T_m,\ [f,g]_w \neq 0\},$$

$$J_m = \{g \mid f,g \in S_m \cup T_m,\ (f,g)_w = WfV - g,\ [f,g]_w \neq 0\}.$$

Now we can make the recursion step,

$$S_{m+1} = (S_m \cup (I_m \cap \mathcal{R})) \setminus J_m,\ T_{m+1} = (T_m \cup (I_m \cap \mathcal{F})) \setminus J_m.$$

5.7. TERMINATION OF BUCHBERGER'S ALGORITHM

If (S,T) is a Gröbner-Shirshov pair then $S_1 = S_0$, $T_1 = T_0$, and the procedure terminates immediately. If S is a Gröbner-Shirshov basis (equivalently $(S,\emptyset)$ is a Gröbner–Shirshov pair), and R/RSR is noetherian then the procedure terminates after finitely many steps.

6. Hodges' Quantisation

6.1. KLEINIAN SINGULARITIES

Let $\zeta \in \mathbb{K}$ be a primitive root of unity of degree n. Set

$$\Gamma = \left\{ g^i : g = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \right\},$$

a subgroup of $\mathrm{SL}_2(\mathbb{K})$. The natural action of Γ on $\mathbb{K}^2$ induces an action on $\mathbb{K}[X,Y] : g \cdot X = \zeta X, g \cdot Y = \zeta^{-1} Y$. The invariants of $\mathbb{K}[X,Y]$ under this action are generated by X^n, XY and Y^n. Thus, the orbit space $\mathbb{K}^2/\Gamma$ has co-ordinate ring

$$\mathcal{O}(\mathbb{K}^2/\Gamma) = \mathbb{K}[X^n, XY, Y^n] \cong \frac{\mathbb{K}[A,B,H]}{(AB - H^n)}.$$

The variety $\mathbb{K}^2/\Gamma$ has an isolated singularity at 0, a *Kleinian singularity of type* A_{n-1}.

6.2. Let $v(z) \in \mathbb{K}[z]$ be a polynomial of degree n, whose roots lie in $\mathbb{F}$. Following [11,1], we define an associative algebra, $T(v)$, over $\mathbb{K}$ with generators a, b and h satisfying the relations

$$ha = a(h+1), hb = b(h-1), ba = v(h), ab = v(h-1). \quad (1)$$

There exists a filtration on $T(v)$ such that $\mathrm{gr}(T(v)) \cong \mathbb{K}[A,B,H]/(AB - H^n)$. In other words, $T(v)$ is a deformation of a Kleinian singularity of type A_{n-1}.

Using the translation $h \mapsto h-1$ we will assume without loss of generality that 0 is a root of $v(z)$.

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6.3. A $\mathbb{Z}$-CATEGORY

There is also a $\mathbb{Z}$-grading on $T(v)$: we assign a degree n, b degree $-n$ and h degree 0. We want to study a category of graded $T(v)$-modules similar in spirit to the construction of $B_{\chi,\lambda}$-T_0-mod in Section 4.8.

Since p and n are coprime, we can find an inverse of $\bar{n}$ in $\mathbb{F}$, say $\bar{c}$. We will consider the full subcategory of finitely generated $\mathbb{Z}$-graded $T(v)$-modules consisting of objects

$$M = \bigoplus_{j\in\mathbb{Z}} M_j,$$

such that h acts on M_j through scalar multiplication by $\bar{j}\bar{c}$. Note that $a \cdot M_j \subseteq M_{j+n}$ (respectively $b \cdot M_j \subseteq M_{j-n}$) showing that this definition is compatible with the relation $ha = a(h+1)$ (respectively $hb = b(h-1)$). Denote this category by $T(v)$-grmod.

For $i \in \mathbb{Z}$, there is a shift functor $[i]: T(v)\text{-grmod} \to T(v)\text{-grmod}$: given $M \in T(v)$-grmod, set $(M[i])_j = M_{j-pi}$ for all $j \in \mathbb{Z}$. This makes $T(v)$-grmod a $\mathbb{Z}$-category. We let $F: T(v)\text{-grmod} \longrightarrow T(V)\text{-mod}$ denote the functor which forgets the $\mathbb{Z}$-structure.

6.4. We will be interested in a finite dimensional central quotient of $T(v)$.

LEMMA 6.1. *The centre of $T(v)$ is generated by a^p, b^p and $h^p - h$. It is isomorphic to the algebra of functions on a type A_{n-1} Kleinian singularity.*

Proof. It is straightforward to check that a^p, b^p and $h^p - h$ are central elements. By construction $T(v)$ is a free $\mathbb{K}[h]$-module with basis $\{a^i, b^j : i, j \geqslant 0\}$. It follows from the relations in $T(v)$ that the degrees of the homogeneous components of any nonzero central elements must be a multiple of pn. Since a^p and b^p are central we must find which polynomials in h are central. Let $q(h)$ be such a polynomial. Since $aq(h) = q(h+1)a$ we deduce that the roots of q are invariant under integer addition. It follows that $q(h)$ is a polynomial in $h^p - h$ as required.

Using the defining relations once more we have

$$a^p b^p = v(h)v(h+1)\dots v(h+p-1).$$

Since $v(h)$ has degree n in h, it follows that $a^p b^p = (h^p - h)^n$. Hence, the centre of $T(v)$ is a quotient of the ring of functions of a Kleinian singularity of type A_{n-1}. Any proper quotient of the ring of functions on a Kleinian singularity has dimension 0 or 1. Thus since $T(v)$ is finitely generated over its centre and has Gelfand–Kirillov dimension 2, the centre must be the entire ring of functions. □

6.5. Now we can introduce the protagonist of this section:

$$t(v) = \frac{T(v)}{(a^p, b^p, h^p - h)}.$$

Since the ideal $(a^p, b^p, h^p - h)$ is homogeneous, $t(v)$ inherits a $\mathbb{Z}$-grading from $T(v)$. We denote the full subcategory of $T(v)$-grmod consisting of $\mathbb{Z}$-graded $t(v)$-modules by $t(v)$-grmod.

6.6. In order to study $t(v)$ we introduce an intermediate algebra

$$\mathfrak{T}(v) = T(v)/(a^p, b^p). \tag{2}$$

Let us use a degree lexicographical order on non-commutative associative monomials in a of degree 1, b of degree $2n-1$, and h of degree 1 with $h > b > a$.

The relations of (1) already form a Gröbner–Shirshov basis. It follows that monomials not containing ha, hb, ba, or ab as submonomials form a basis of $T(v)$.

For any polynomial $f(z) \in \mathbb{K}[z]$ and a positive integer i, we denote

$$f_{(i)}(z) = \prod_{k=0}^{i-1} f(z+\bar{k}), \quad f_{(-i)}(z) = \prod_{k=0}^{i-1} f(z-\bar{k}).$$

LEMMA 6.2. *The following relations, together with those in* (1) *and* (2), *form a Gröbner-Shirshov basis of* $\mathfrak{T}(v)$,

$$a^{p-i}v_{(-i)}(h-1) \quad \textit{for } i = 1, \dots, p, \tag{3}$$

$$b^{p-i}v_{(i)}(h) \quad \textit{for } i = 1, \dots, p-1. \tag{4}$$

Proof. Let us obtain all relations in (6.6) recursively. For $i = 1, \dots, p$,

$$\begin{aligned}(ba - v(h), a^{p-i}v_{(-i)}(h-1))_{ba^{p-i}h^{ni}} &= ba^{p-i}v_{(-i)}(h-1) - (ba - v(h))a^{p-i-1}h^{ni}\\ &= ba^{p-i}(v_{(-i)}(h-1) - h^{ni}) + v(h)a^{p-i-1}h^{ni} \rightsquigarrow v(h)a^{p-i-1}(v_{(-i)}(h-1) - h^{ni}) +\\ &+ v(h)a^{p-i-1}h^{ni} = v(h)a^{p-i-1}v_{(-i)}(h-1) \rightsquigarrow av(h+1)a^{p-i-2}v_{(-i)}(h-1) \rightsquigarrow \cdots\\ &\rightsquigarrow a^{p-i-1}v(h+p-i-1)v_{(-i)}(h-1) = a^{p-i-1}v_{(-i-1)}(h-1).\end{aligned}$$

Similarly, for $i = 1, \dots, p-1$, $(b^{p-i}v_{(i)}(h), ab - v(h-1))_{ab^{p-i}h^{ni}} \rightsquigarrow b^{p-i-1}v_{(i+1)}(h)$. Now we need to show that all remaining compositions are trivial. The highest terms of defining relations are ha, hb, ba, ab, $a^{p-i}h^{ni}$, and $b^{p-i}h^{ni}$. Let us make certain that all compositions are trivial. Firstly,

$$(ba - v(h), ab - v(h-1))_{bab} = bv(h-1) - v(h)b \rightsquigarrow bv(h-1) - bv(h-1) = 0.$$

Similarly, $(ab - v(h-1), ba - v(h))_{aba} \rightsquigarrow 0$. Then

$$\begin{aligned}(hb - bh + b, b^{p-i}v_{(i)}(h))_{hb^{p-i}h^{ni}} &= hb^{p-i}(h^{ni} - v_{(i)}(h)) - bhb^{p-i-1}h^{ni} + b^{p-i}h^{ni} \rightsquigarrow\\ &\rightsquigarrow b^{p-i}((h-p+i)(h^{ni} - v_{(i)}(h)) - (h-p+i+1)h^{ni} + h^{ni} = b^{p-i}v_{(i)}(h)h \rightsquigarrow 0.\end{aligned}$$

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Analogously, the compositions $(a^{p-i}v_{(-i)}(h), ab - v(h-1))_{a^p b}$ and $(b^{p-i}v_{(i)}(h), ab- v(h-1))_{ab^p}$ are trivial. Then

$$\begin{aligned}&(ha - ah - a, ab - v(h-1))_{hab}\\&\quad = hv(h-1) - ahb - ab \rightsquigarrow hv(h-1) - a(bh-b) - ab\\&\quad = hv(h-1) - abh \rightsquigarrow 0.\end{aligned}$$

Another possible composition to consider is

$$\begin{aligned}&(ha - ah - a, a^{p-i}v_{(-i)}(h))_{ha^{p-i}h^{ni}}\\&\quad = ha^{p-i}(v_{(-i)}(h) - h^{ni}) + aha^{p-i-1}h^{ni} + a^{p-i}h^{ni} \rightsquigarrow\\&\qquad \rightsquigarrow a^{p-i}((h-i)(v_{(-i)}(h) - h^{ni}) + (h-i-1)h^{ni} + h^{ni})\\&\quad = a^{p-i}v_{(-i)}(h)(h-i) \rightsquigarrow 0.\end{aligned}$$

The remaining compositions $(ha - ah - a, a^{p-i}h^{ni})_{a^{p-i}h^{ni}a}$, $(hb - bh + b, a^{p-i}h^{ni})_{a^{p-i}h^{ni}b}$, $(ha - ah - a, b^{p-i}h^{ni})_{b^{p-i}h^{ni}a}$, and $(hb, b^{p-i}h^{ni})_{b^{p-i}h^{ni}b}$ are trivial by a similar argument. □

COROLLARY 6.7. *The dimension of* $\mathfrak{T}(v)$ *is* np^2. *Moreover, there is a direct sum decomposition*

$$\mathfrak{T}(v) = \left[\bigoplus_{i=1}^{p} a^{p-i}\mathbb{K}[h]/(v_{(i)}(h-1))\right] \oplus \left[\bigoplus_{j=1}^{p-1} b^{j}\mathbb{K}[h]/(v_{(p-j)}(h))\right]. \tag{5}$$

Proof. The direct sum decomposition (5) follows at once from the description of the Gröbner–Shirshov basis of $\mathfrak{T}(v)$. Adding dimensions of summands, we arrive at the dimension of $\mathfrak{T}(v)$, that is $2(n + 2n + \cdots (p-1)n) + pn = np^2$. □

6.8. If we write $\gcd(f(Z), g(Z))$ for the greatest common divisor of two polynomials then decomposition (5) is inherited:

$$t(v) = \bigoplus_{i=1}^{p} \frac{a^{p-i}\mathbb{K}[h]}{(\gcd(v_{(i)}(h-1), h^p - h))} \oplus \bigoplus_{j=1}^{p-1} \frac{b^{j}\mathbb{K}[h]}{(\gcd(v_{(p-j)}(h), h^p - h))}. \tag{6}$$

This decomposition allows one to compute the dimension of $t(v)$.

Let $r_1, \ldots, r_{n-1} \in \mathbb{Z}$ be such that $r_i \geqslant 0$ and $p \geqslant r_1 + \cdots + r_{n-1}$ and

$$v(z) = \prod_{i=0}^{n-1}(z - (\overline{r_1} + \cdots + \overline{r_i})).$$

Set $r_0 = p - (r_1 + \cdots + r_{n-1})$.

COROLLARY. *The dimension of* $t(v)$ *is* $2p^2 - \sum_{i=0}^{n-1} r_i^2$.

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Proof. If $f(z, i) = z - (\overline{r_1} + \cdots + \overline{r_i})$ then the roots of $f_{(j)}(z, i)$ are $\overline{r_1} + \cdots + \overline{r_i}, \ldots, \overline{r_1} + \cdots + \overline{r_i} - \bar{j}$. If $j \geqslant r_i$ then $\overline{r_1}, \ldots, \overline{r_{i-1}}, \ldots, \overline{r_1} + \cdots + \overline{r_i} - \bar{j}$ are already roots of $f_{(j)}(z, i+1)$. Thus, the dimension of the first summand in (6) is

$$p^2 - \sum_{i=0}^{n-1}(1 + 2 + \cdots + (r_i - 1) + r_i + r_i + \cdots r_i),$$

where the number of summands in the parenthesis is p. This sum equals $p^2 + (p - \sum_{i=0}^{n-1} r_i^2)/2$. Similarly, the second summand has dimension $p^2 - (p + \sum_{i=0}^{n-1} r_i^2)/2$, so that the total dimension is $2p^2 - \sum_{i=0}^{n-1} r_i^2$. □

6.9. BABY VERMA MODULES (II)

Let u (respectively u') be the subalgebra of $t(v)$ generated by a and h (respectively, b and h). Similarly let U (respectively, U') be the subalgebra of $\mathfrak{T}(v)$ generated by a and h (respectively b and h).

We introduce two sets of baby Verma modules. For $\lambda \in \mathbb{Z}$ let $\mathbb{K}_\lambda$ (respectively, $\mathbb{K}'_\lambda$) be the one-dimensional U-module (respectively, U'-module) with basis $|0\rangle$ (respectively $|1\rangle$) and

$$h|0\rangle = \bar{\lambda}|0\rangle, \qquad a|0\rangle = 0, \qquad h|1\rangle = \bar{\lambda}|1\rangle, \qquad b|1\rangle = 0.$$

The baby Verma module $V(\lambda)$ (respectively $V(\lambda)'$) is

$$V(\lambda) = \mathfrak{T}(v) \otimes_U \mathbb{K}_\lambda \quad (\text{respectively}\, V(\lambda)' = \mathfrak{T}(v) \otimes_{U'} \mathbb{K}'_\lambda).$$

LEMMA 6.10. (i) *If $\bar{\lambda}$ is not a root of $v(z)$ then $V(\lambda) = 0$, whilst if $\bar{\lambda}$ is a root of $v(z)$ then $V(\lambda)$ has a basis of p elements $|0\rangle, b|0\rangle, \ldots, b^{p-1}|0\rangle$. We have*

$$ab^k|0\rangle = v(\bar{\lambda} - \bar{k})b^{k-1}|0\rangle, \qquad hb^k|0\rangle = (\bar{\lambda} - \bar{k})b^k|0\rangle.$$

(ii) *If $\bar{\lambda} - 1$ is not a root of $v(z)$ then $V(\lambda)' = 0$, whilst if $\bar{\lambda} - 1$ is a root of $v(z)$ then $V(\lambda)'$ has a basis of p elements $|1\rangle, a|1\rangle, \ldots, a^{p-1}|1\rangle$. We have*

$$ba^k|1\rangle = v(\bar{\lambda} + \bar{k} - 1)a^{k-1}|1\rangle, \qquad ha^k|1\rangle = (\bar{\lambda} + \bar{k})a^k|1\rangle.$$

Proof. (i) For the generator $|0\rangle \in V(\lambda)$ one observes that

$$0 = ba|0\rangle - (ba - v(h))|0\rangle = v(h)|0\rangle = v(\bar{\lambda})|0\rangle.$$

Thus if $\bar{\lambda}$ is not a root of $v(z)$ then $|0\rangle = 0$ and $V(\lambda) = 0$.

Let $\bar{\lambda}$ be a root of $v(z)$. Let S be the Gröbner–Shirshov basis of $\mathfrak{T}(v)$ constructed in Lemma 6.6. The module $V(\lambda)$ is determined by the pair $(S, \{(h - \bar{\lambda})|0\rangle,\ a|0\rangle\})$ and this turns out to be a Gröbner–Shirshov pair. Indeed, there are three elements in S whose leading monomials end with a:

$$(a|0\rangle, ha - ah - a)_a = ha|0\rangle - (ha - ah - a)|0\rangle = (ah + a)|0\rangle \rightsquigarrow (\lambda a + a)|0\rangle \rightsquigarrow 0,$$

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$$(a|0\rangle, ba - v(h))_a = ba|0\rangle - (ba - v(h))|0\rangle = v(h)|0\rangle \rightsquigarrow v(\bar{\lambda})|0\rangle = 0,$$

$$(a^p, a|0\rangle)_a = a^p|0\rangle - a^{p-1}a|0\rangle = 0.$$

Elements of S whose leading monomials end with h fall into two types:

$$(a^{p-i}v_{(i)}(h-1)|0\rangle, (h-\bar{\lambda})|0\rangle)_h = (\bar{\lambda}a^{p-i}h^{ni-1} + a^{p-i}(v_{(i)}(h-1) - h^{ni}))|0\rangle \rightsquigarrow$$

$$v_{(i)}(\bar{\lambda}-1)a^{p-i}|0\rangle \rightsquigarrow 0,$$

$$(b^{p-i}v_{(i)}(h), (h-\bar{\lambda})|0\rangle)_h = (\bar{\lambda}b^{p-i}h^{ni-1} + b^{p-i}(v_{(i)}(h) - h^{ni}))|0\rangle \rightsquigarrow b^{p-i}v_{(i)}(\bar{\lambda})|0\rangle = 0.$$

Direct computation now yields the formulas for the action.

(ii) The proof is analogous. □

Thanks to the lemma we can consider $V(\lambda)$ and $V(\lambda)'$ as objects in $T(v)$-grmod. Indeed if $V(\lambda)$ (respectively $V(\lambda)'$) is nonzero we let $b^k|0\rangle$ (respectively $a^k|1\rangle$) span the $(\lambda - k)n$ (respectively $(\lambda + k)n$) homogeneous component.

6.11. Under our assumptions, we have $(\bar{\lambda} - \bar{k})^p = \bar{\lambda} - \bar{k}$. It follows that $(h^p - h)V(\lambda) = 0$ (respectively $(h^p - h)V(\lambda)' = 0$), and so $V(\lambda)$ and $V(\lambda)'$ are objects in $t(v)$ − grmod.

PROPOSITION. *Let* $v(z) = \prod_{i=0}^{n-1}(z - (\overline{r_1} + \cdots + \overline{r_i}))$ *be as in* 6.8 *and let* $r_0 = p - (r_1 + \cdots + r_{n-1})$.

(i) *The category* $t(v)$-grmod *has simple modules* $L_0, \ldots, L_{n-1}$ (*up to isomorphism and shift*) *where the dimension of* L_i *is* r_{n-1-i} (*if* $r_{n-1-i} = 0$ *then* L_i *should be omitted from the list of simple modules*).

(ii) *For each* i *lying between* 0 *and* $n-1$ *there exists a uniserial module* $V_i \in t(v)$-grmod *whose Loewy layers are* $L_i, L_{i+1}[-1], \ldots, L_{i-1}[1-n]$ (*count subscripts modulo* n, *and omit* V_i *and* L_i *whenever* $r_{n-1-i} = 0$).

(iii) *For each* i *lying between* 0 *and* $n-1$ *there exists a uniserial module* $V_i' \in t(v)$-grmod *whose Loewy layers are* $L_i, L_{i-1}[1], \ldots, L_{i+1}[n-1]$ (*count subscripts modulo* n, *and omit* V_i' *and* L_i *whenever* $r_{n-1-i} = 0$).

(iv) *For any* $\lambda \in \mathbb{Z}$ *such that* $\bar{\lambda}$ *is a root of* $v(z)$ *there exists a unique* i *and* $j \in \mathbb{Z}$ (*respectively* i', j') *such that* $V(\lambda) \cong V_i[j]$ (*respectively* $V(\lambda+1)' \cong V_{i'}'[j']$).

Proof. Set $V_0 = V(0)$ and $V_i = V(r_1 + \cdots + r_{n-1-i})[i]$. Note that if $r_{n-1-i} = 0$ then $V_i \cong V_{i+1}[-1]$. Since every $t(v)$-module has a u-fixed point we see that any simple $t(v)$-module is a quotient of $F(V_i)$ for some i (where F is the forgetful functor to ungraded modules). By Lemma 6.10(i) $F(V_i)$ is isomorphic to $\mathbb{K}[b]/(b^p)$ as a $\mathbb{K}[b]/(b^p)$-module. Since $\mathbb{K}[b]/(b^p)$ is a local algebra it follows that V_i has a simple head.

In V_i the element $b^{r_{n-1-i}}|0\rangle$ is annihilated by a and belongs to the $n(r_1+\cdots+r_{n-1-(i+1)})+pi$ component, so there is a graded $t(v)$ homomorphism

$$\theta_i : V_{i+1}[-1] \longrightarrow V_i.$$

The cokernel of θ_i, say L_i, has dimension r_{n-1-i} and is simple if $r_{n-1-i}\neq 0$ since it is generated by any basis vector $b^i|0\rangle$ it contains. If $r_{n-1-i}\neq 0$ it follows from Lemma 6.10 that L_i has a unique u-fixed point, namely $|0\rangle$. Hence, if r_{n-1-i} and r_{n-1-j} are non-zero L_i and L_j are isomorphic if and only if $i=j$. This proves (i).

Since V_i is simple-headed, dim $V_i=p$ and $\sum_{i=0}^{n-1} r_i = p$ the chain of homomorphisms

$$V_{i-1}[1-n] \xrightarrow{\theta_{i-2}} V_{i-2}[2-n] \xrightarrow{\theta_{i-3}} \cdots \xrightarrow{\theta_{i+1}} V_{i+1}[-1] \xrightarrow{\theta_i} V_i$$

proves (ii). The proof of (iii) is similar. Part (iv) is clear. □

PROPOSITION 6.12. *Let* $v(z)=\prod_{i=0}^{n-1}(z-(\overline{r_1}+\cdots+\overline{r_i}))$ *be as in* 6.8 *and set* $r_0=p-(r_1+\cdots+r_{n-1})$. *Let* k *be the number of nonzero* r_i*'s. Then* $t(v)$ *is Morita equivalent to* $N_{\mathbb{K}}(k)$. *Moreover, if* $k=n$, *there is a* $\mathbb{Z}$*-equivalence of categories*

$$t(v)\text{-grmod} \longrightarrow N_{\mathbb{K}}(n)\text{-grmod}.$$

Proof. Let $0\leqslant i_1\leqslant\cdots\leqslant i_k\leqslant n-1$ be such that $r_{n-1-i_t}\neq 0$. Let Q_{i_t} be the projective cover of L_{i_t} in $t(v)$-grmod. Recall the general formula, [3, Section 1.7]

$$\dim\, t(v) = \sum_{t=1}^{k} \dim\, Q_{i_t} \dim\, L_{i_t}.$$

Let T_{i_t} be the kernel of the sum of two projections $V_{i_t}\oplus V'_{i_t}\to L_{i_t}$. Then T_{i_t} has head isomorphic to L_{i_t} so is a quotient of Q_{i_t}. By Lemma 6.9 and Proposition 6.11 dim $T_{i_t}=2p-r_{n-1-i_t}$. Using Lemma 6.8 we find

$$\sum_{t=1}^{k}\dim\, Q_{i_t}\dim\, L_{i_t} \geqslant \sum_{t=1}^{k}\dim\, T_{i_t}\dim\, L_{i_t} = \sum_{t=1}^{k}(2p-r_{n-1-i_t})r_{n-1-i_t}$$
$$= 2p^2 - \sum_{t=1}^{k} r_{n-1-i_t}^2 = \dim\, t(v),$$

proving that $T_{i_t}\cong Q_{i_t}$.

Let $T=\oplus T_{i_t}$. The basic algebra of $t(v)$ is $\mathrm{End}_{t(v)}(F(T))^{\mathrm{op}}$. Let b_t (respectively a_t) be the homomorphism $F(T_{i_t})\to F(T_{i_{t+1}})$ (respectively $F(T_{i_{t+1}})\to F(T_{i_t})$) associated to the composition factor $F(L_{i_t})$ of $F(T_{i_{t+1}})$ (respectively $F(T_{i_t})$) lying in the second Loewy layer of $F(V_{i_{t+1}})$ (respectively $F(V'_{i_t})$). It is straightforward to check that a_t and b_t, together with the idempotents arising from the projections in $\mathrm{End}_{t(v)}(F(T))$, generate the basic algebra and satisfy the relations of the no-cycle algebra. Since

$$\dim\, \mathrm{End}_{t(v)}(F(T)) = k(2k-1)$$

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the first statement of the proposition follows. The second statement is proved in the same manner as Proposition 4.17. □

7. Proof of Premet's Conjecture

7.1. We require $p > n$ for the following theorem. Thanks to 4.18 we can replace $U_{\chi,\lambda}$ with $\tilde{U}_{\chi,\lambda}$ throughout, if we wish.

THEOREM 7.2. *Let $p > n$. Suppose $\lambda \in C_0$ with $\lambda + \rho = \sum r_i \varpi_i$ and let $v(z) = \prod_{i=0}^{n-1}(z - (\overline{r_1} + \cdots + \overline{r_i}))$. There is an isomorphism*

$$U_{\chi,\lambda} \cong \mathrm{Mat}_{p^{(n^2-n-2)/2}}(t(v)).$$

Moreover, there is a $\mathbb{Z}$-equivalence of categories

$$U_{\chi,\lambda}\text{-}T_0\text{-mod} \longrightarrow t(v)\text{-grmod}.$$

Proof. Set $r_0 = p - (r_1 + \cdots + r_{n-1})$ and let k be the number of nonzero r_i's. Let $0 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant n-1$ be such that $r_{n-1-i_t} \neq 0$. Let $L_1, \ldots, L_k$ (respectively $M_1, \ldots, M_k$) be the simple $U_{\chi,\lambda}$-T_0-modules (respectively graded $t(v)$-modules) appearing in Proposition 4.12 (respectively Proposition 6.11) and let $P_1, \ldots, P_k$ (respectively $Q_1, \ldots, Q_k$) be their projective covers. We have

$$U_{\chi,\lambda} \cong \mathrm{End}_{U_{\chi,\lambda}}(\oplus F(P_t)^{p^{(n^2-n-2)/2} r_{n-1-i_t}}) \cong \mathrm{Mat}_{p^{(n^2-n-2)/2}}(\mathrm{End}_{U_{\chi,\lambda}}(\oplus F(P_t)^{r_{n-1-i_t}}))$$

and

$$t(v) \cong \mathrm{End}_{t(v)}(\oplus F(Q_t)^{r_{n-1-i_t}}).$$

Thanks to our construction of P_t in Proposition 4.17 and Q_t in Proposition 6.12 we have a graded isomorphism

$$\mathrm{End}_{U_{\chi,\lambda}}(\oplus F(P_t)^{r_{n-1-i_t}}) \cong \mathrm{End}_{t(v)}(\oplus F(Q_t)^{r_{n-1-i_t}}),$$

proving the first statement of the theorem, together with an equivalence. The equivalence is a $\mathbb{Z}$-equivalence by ([10], Theorem 5.4). □

7.3. EXTENSION TO TYPE B

For subregular representations of Lie algebras of type B, Jantzen has proved an analogue of Proposition 4.12 and calculated several extension groups, ([13], Section 3). Then, *if we can prove an analogue of Proposition* 4.13, it follows formally from the arguments of Sections 4, and 6 and the above that the central reduction of a block of a subregular reduced enveloping algebra of type B is a matrix ring over a central reduction of Hodges' deformation of a Kleinian singularity of type A.

Unfortunately, the proof of Proposition 4.13 does not immediately generalise to type B since it is no longer true that we can find all highest weights in the fundamental alcove, C_0. For regular weights, however, this can be remedied as follows. Let $B_{\chi,\lambda}$ denote a block of a subregular reduced enveloping algebra of type B_n

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associated to a regular weight λ. Rickard's results in ([23], Section 3) remain valid in this situation. To check this requires the specific information on the wall-crossing functors, denoted γ_i for $1 \leqslant i \leqslant n$, provided by Jantzen in ([14], Section H). Thus, for $1 \leqslant i \leqslant n$, there are derived self-equivalences F_i on the bounded derived categories of $B_{\chi,\lambda}$-modules. In particular, given any $B_{\chi,\lambda}$-module M, $F_i(M)$ is the complex $\gamma_i(M) \to M$, the map being given by the counit of γ_i. Combining the known behaviour of baby Verma modules under wall-crossing, [12, 11.20], with existing results on filtrations of the projective indecomposable modules by baby Verma modules, ([12], Proposition 10.11), it can be shown that, given two projective indecomposable $B_{\chi,\lambda}$-modules Q and Q', there are integers $1 \leqslant i_1, \dots, i_t \leqslant n$ such that $F_{i_1}, \dots, F_{i_t}(Q)$ is quasi-isomorphic to Q'. Since it is known that the centre of $B_{\chi,\lambda}$ is a derived-invariant, the proof of Proposition 4.13 can be generalised, using the derived category, to deal with $B_{\chi,\lambda}$. As a result we find the central reduction of $B_{\chi,\lambda}$ is a matrix ring over the central reduction of Hodges' deformation of a Kleinian singularity of type A_{2n-1}.

8. More on Baby Verma Modules

8.1. We want to study baby Verma modules in 'general position'. To do so, we need a general lemma.

LEMMA 8.1. *Let A be a finite dimensional $\mathbb{L}$-algebra and Y a connected algebraic variety over $\mathbb{L}$. Suppose M_α, $\alpha \in Y$, is a flat family of finite-dimensional A-modules over Y. Then the Grothendieck group element $[M_\alpha] \in K_0(A)$ is independent of α.*

Proof. Let B be the basic algebra of A. There exists a (B, A)-bimodule N, flat over A, such that the functor $N \otimes_A -$ induces an equivalence between the categories of finite-dimensional A-modules and finite-dimensional B-modules. Let $K_\alpha = N \otimes_A M_\alpha$, a flat family of B-modules over Y. Given a primitive idempotent $e \in B$, it suffices to show that the dimension of eK_α is independent of $\alpha \in U$, a Zariski open neighbourhood of a point. Without loss of generality we can trivialise the family locally on U, giving $K \times U$. Then e defines an algebraic family of projection operators on the finite dimensional vector space K,

$$e_\alpha : k \mapsto \mathrm{pr}_K(e \cdot (k, \alpha)).$$

Since the dimension of eK_α is equal to the rank of e_α, and the latter is constant since Y is connected. □

8.2. Given $\lambda \in X$, it is an interesting problem to describe the isomorphism classes of all baby Verma modules $V(\mathfrak{b}, \lambda)$ as $\mathfrak{b}$ runs over $\mathcal{B}_\chi$. If λ is regular then the description of $\mathcal{B}_\chi$, Proposition 4.12 and Lemma 8.1 show that, for every $\mathfrak{b} \in \mathcal{B}_\chi$,

$$[V(\mathfrak{b}, \lambda)] = \sum_{i=0}^{n-1} [L_i] \in K_0(U_{\chi,\lambda}).$$

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Therefore, on passing to the no-cycle algebra, we see that such $V(\mathfrak{b}, \lambda)$ corresponds to a $N_{\mathbb{K}}(n)$-module $M(\mathfrak{b})$, such that $e_i M(\mathfrak{b})$ is one-dimensional for all i. Thanks to Section 2, all modules of this dimension are known.

Let $(k, \alpha) \in (\{1, \ldots, n-1\} \times \mathbb{K}\}) \cup (0, 0)$ and let $\mathfrak{b}_{k,\alpha}$ be the stabiliser of $\mathcal{F}_{k,\alpha}$. Suppose first that $\alpha = 0$. Since the torus T_0 stabilises $\mathfrak{b}_{k,0}$ for any k, twists by elements of T_0 provide a grading of $V(\mathfrak{b}_{k,0}, \lambda)$. Therefore $M(\mathfrak{b}_{k,\alpha})$ is gradable and, by Section 3, must be a direct sum of string modules if n is odd (note that some band modules are gradable for even n). When λ is regular, we expect that gradable band modules are not baby Verma modules, and that baby Verma modules are indecomposable.

The generic case is dealt with in the following proposition.

PROPOSITION 8.3. *Keep the above notation and let $\lambda \in X$ be a regular weight, $1 \leqslant k \leqslant n-1$ and $\alpha \neq 0$. Then the baby Verma module $V(\mathfrak{b}_{k,\alpha}, \lambda)$ is indecomposable.*

Proof. It follows from Section 3 that if $M(\mathfrak{b}_{k,\alpha})$ is not gradable it is necessarily a band module and, hence, indecomposable. Thus, by Proposition 4.17, it suffices to show that $V(\mathfrak{b}_{k,\alpha}, \lambda)$ does not admit a T_0-grading.

Suppose for a contradiction that $V(\mathfrak{b}_{k,\alpha}, \lambda)$ admits a T_0-grading. Let L_k be the $\mathfrak{sl}_2(\mathbb{K})$-subalgebra generated by $E_{n-k,n}, E_{n-k,n-k} - E_{n,n}$ and $E_{n,n-k}$. Any Borel subalgebra belonging to Π_k is uniquely determined by its intersection with L_k. Let $\lambda_k = \lambda(E_{n-k,n-k} - E_{n,n})$. A straightforward calculation shows that the restriction of $V(\mathfrak{b}_{k,\alpha}, \lambda)$ to L_k has a direct summand isomorphic to the baby Verma module for L_k induced from $\mathfrak{b}_{k,\alpha} \cap L_k$ with highest weight λ_k.

For $t \in T_0$ the element $t(1 \otimes 1) \in V(\mathfrak{b}_{k,\alpha}, \lambda)$ is a highest weight vector for the Borel subalgebra $t \cdot \mathfrak{b}_{k,\alpha}$, yielding an isomorphism $V(\mathfrak{b}_{k,\alpha}, \lambda) \cong V(t \cdot \mathfrak{b}_{k,\alpha}, \lambda)$. Since λ is regular $\lambda_k \neq -1$, and so, by ([20], Main Theorem), the baby Verma modules for L_k induced from different Borel subalgebras of L_k with highest weight λ_k are not isomorphic. Hence, by the last paragraph, the isomorphism $V(\mathfrak{b}_{k,\alpha}, \lambda) \cong V(t \cdot \mathfrak{b}_{k,\alpha}, \lambda)$ forces $V(\mathfrak{b}_{k,\alpha}, \lambda)$ to have infinitely many nonisomorphic direct summands, a contradiction. □

Acknowledgements

Both authors were visiting and partially supported by MSRI while this research was begun and extend their thanks to that institution. Much of the research for this paper was undertaken while the first author was supported by TMR grant ERB FMRX-CT97-0100 at the University of Bielefeld. The second author was partially supported by EPSRC grants GR/M68886 and GR/M75037.

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