

## WEIGHTED PSEUDO ALMOST PERIODIC SOLUTIONS FOR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

LI-LI ZHANG<sup>✉</sup> and HONG-XU LI

(Received 24 December 2014; accepted 23 April 2015; first published online 16 June 2015)

### Abstract

By applying the theory of exponential dichotomies and contraction mapping, we establish some existence and uniqueness results for weighted pseudo almost periodic solutions of some differential equations with piecewise constant arguments. For this purpose, we also describe some basic properties of weighted pseudo almost periodic sequences.

2010 Mathematics subject classification: primary 34K14.

Keywords and phrases: exponential dichotomy, weighted pseudo almost periodic sequence, weighted pseudo almost periodic solution, piecewise constant argument.

### 1. Introduction

The papers [1, 6, 10] discuss the pseudo almost periodicity, quasi-periodicity and periodicity of differential equations with piecewise constant arguments,

$$x'(t) = A(t)x(t) + \sum_{j=0}^r A_j(t)x([t - j]) + f(t), \quad (1.1)$$

where  $A, A_j : \mathbb{R} \rightarrow \mathbb{R}^{q \times q}$  ( $j = 0, \dots, r$ ),  $f : \mathbb{R} \rightarrow \mathbb{R}^q$  and  $[\cdot]$  denotes the greatest integer function.

It is interesting and worthwhile to study (1.1) for the case of weighted pseudo almost periodicity. The main purpose of this paper is to consider this problem. To be precise, we establish some existence and uniqueness results for weighted pseudo almost periodic solutions of (1.1). To facilitate this, we use the concept of *weighted pseudo almost periodic sequence* which is introduced in [13]. Moreover, we give a unique decomposition property for weighted pseudo almost periodic sequences and present the relationship between weighted pseudo almost periodic functions and weighted pseudo almost periodic sequences, which is quite different from the almost

---

This work is supported by the National Natural Science Foundation of China (No. 11372196) and by the Natural Science Foundation of Hebei Province, China (No. A2014210104).

© 2015 Australian Mathematical Publishing Association Inc. 0004-9727/2015 \$16.00

periodic case but similar to the pseudo almost periodic case. We believe these basic properties will be useful in related studies. Dichotomy theory for difference equations also plays a central role in this work.

The outline of the paper is as follows. Some notation and preliminary results, including some basic properties of weighted pseudo almost periodic sequences, are given in Section 2. Our main result is stated in Section 3.1. Some lemmas on the corresponding difference equations are given in Section 3.2. Finally, the main result is proved in Section 3.3.

## 2. Preliminaries

Throughout this paper, we always denote by  $|\cdot|$  the Euclidean norm when the argument is a vector and the corresponding operator norm when the argument is a matrix. Let  $BC(\mathbb{R}, \mathbb{R}^q)$  be the space of bounded continuous functions  $u : \mathbb{R} \rightarrow \mathbb{R}^q$ ;  $BC(\mathbb{R}, \mathbb{R}^q)$  equipped with the sup norm defined by  $\|u\| = \sup_{t \in \mathbb{R}} |u(t)|$  is a Banach space. Furthermore,  $C(\mathbb{R}, \mathbb{R}^q)$  denotes the space of continuous functions from  $\mathbb{R}$  to  $\mathbb{R}^q$ .

**2.1. Weighted pseudo almost periodic functions.** Let  $U$  be the collection of functions (weights)  $\rho : \mathbb{R} \rightarrow (0, +\infty)$  which are locally integrable over  $\mathbb{R}$ . If  $\rho \in U$ , we set

$$\mu(T, \rho) := \int_{-T}^T \rho(t) dt \quad \text{for } T > 0.$$

Define

$$U_\infty := \left\{ \rho \in U : \lim_{T \rightarrow \infty} \mu(T, \rho) = \infty \right\}$$

and

$$U_B := \left\{ \rho \in U_\infty : \rho \text{ is bounded with } \inf_{t \in \mathbb{R}} \rho(t) > 0 \right\}.$$

For  $\rho_1, \rho_2 \in U_\infty$ ,  $\rho_1$  is said to be equivalent to  $\rho_2$ , denoted by  $\rho_1 \sim \rho_2$ , if  $\rho_1/\rho_2 \in U_B$ . Then ' $\sim$ ' is a binary equivalence relation on  $U_\infty$  (see [3]). For  $\rho \in U_\infty$ ,  $c \in \mathbb{R}$ , define  $\rho_c$  by  $\rho_c(t) = \rho(t + c)$  for  $t \in \mathbb{R}$ . We define

$$U_T := \{ \rho \in U_\infty : \rho \sim \rho_c \text{ for each } c \in \mathbb{R} \}.$$

It is easy to see that  $U_T$  contains many weights, for example,  $1, e^t, 1 + 1/(1 + t^2), 1 + |t|^n$  with  $n \in \mathbb{N}$ , and so on.

**DEFINITION 2.1 [5].** A set  $S \subset \mathbb{R}$  is said to be relatively dense if there exists  $L > 0$  such that  $[a, a + L] \cap S \neq \emptyset$  for all  $a \in \mathbb{R}$ . A function  $f \in C(\mathbb{R}, \mathbb{R}^q)$  is said to be almost periodic if the  $\varepsilon$ -translation set of  $f$ ,

$$T(f, \varepsilon) = \{ \tau \in \mathbb{R} : |f(t + \tau) - f(t)| < \varepsilon \forall t \in \mathbb{R} \}, \quad (2.1)$$

is relatively dense for each  $\varepsilon > 0$ . Denote by  $AP(\mathbb{R}^q)$  the set of all such functions.

For  $\rho \in U_\infty$ , the *weighted ergodic spaces* are defined by

$$PAP_0(\mathbb{R}^q, \rho) := \left\{ f \in BC(\mathbb{R}, \mathbb{R}^q) : \lim_{T \rightarrow \infty} \frac{1}{\mu(T, \rho)} \int_{-T}^T |f(t)| \rho(t) dt = 0 \right\}.$$

**DEFINITION 2.2 [3].** Let  $\rho \in U_\infty$ . A function  $f \in BC(\mathbb{R}, \mathbb{R}^q)$  is called weighted pseudo almost periodic (or  $\rho$ -pseudo almost periodic) if it can be expressed as  $f = f^{ap} + f^e$ , where  $f^{ap} \in AP(\mathbb{R}^q)$  and  $f^e \in PAP_0(\mathbb{R}^q, \rho)$ . Denote by  $PAP(\mathbb{R}^q, \rho)$  the set of all such functions.

The functions  $f^{ap}$  and  $f^e$  in Definition 2.2 are called the *almost periodic* and the *weighted ergodic perturbation* components of  $f$ , respectively. Moreover, if  $PAP_0(\mathbb{R}^q, \rho)$  is translation invariant, then the decomposition  $f^{ap} + f^e$  of  $f$  is unique (see [8]), and  $PAP_0(\mathbb{R}^q, \rho)$  and  $PAP(\mathbb{R}^q, \rho)$  are Banach spaces with the norm inherited from  $BC(\mathbb{R}, \mathbb{R}^q)$  (see [4]).

**2.2. Discontinuous weighted pseudo almost periodic function.** To deal with the discontinuity of the function  $t \rightarrow \lfloor t \rfloor$ , we use the concept of discontinuous weighted pseudo almost periodic functions. We denote

$$C_m = \left\{ f : \mathbb{R} \rightarrow \mathbb{R}^q : \begin{array}{l} f \text{ is continuous on } \mathbb{R} \setminus \mathbb{Z} \text{ and has finite limits} \\ \text{on the left and on the right at any point in } \mathbb{Z} \end{array} \right\}$$

and

$$BC_m = \{f \in C_m : f \text{ is bounded}\}.$$

It is clear that  $BC_m$  is a Banach space with the sup norm  $\|x\| = \sup_{t \in \mathbb{R}} |x(t)|$  (see [1]).

**DEFINITION 2.3.** Let  $\rho \in U_\infty$  and let  $T(f, \varepsilon)$  be as in Definition (2.1). Set

$$\begin{aligned} \mathcal{F}_0 &= \left\{ f \in BC_m : \lim_{T \rightarrow \infty} \frac{1}{\mu(T, \rho)} \int_{-T}^T |f(s)|\rho(s) ds = 0 \right\}, \\ \mathcal{F}_1 &= \{f \in BC_m : \forall \varepsilon > 0, T(f, \varepsilon) \cap \mathbb{Z} \text{ is relatively dense}\}, \\ \mathcal{F} &= \mathcal{F}_0 + \mathcal{F}_1. \end{aligned}$$

It is easy to see that

$$PAP_0(\mathbb{R}^q, \rho) \subset \mathcal{F}_0, \quad AP(\mathbb{R}^q) \subset \mathcal{F}_1 \quad \text{and} \quad PAP(\mathbb{R}^q, \rho) \subset \mathcal{F}.$$

**2.3. Weighted pseudo almost periodic sequences.**

**DEFINITION 2.4 [5].** A sequence  $x : \mathbb{Z} \rightarrow \mathbb{R}^q$  is called an almost periodic sequence if the  $\varepsilon$ -translation set of  $x$ ,

$$T(x, \varepsilon) = \{\tau \in \mathbb{Z} : |x(n + \tau) - x(n)| \leq \varepsilon \forall n \in \mathbb{Z}\},$$

is a relatively dense set for all  $\varepsilon > 0$ .  $\tau$  is called the  $\varepsilon$ -period for  $x$ . Denote the set of all these sequences  $x$  by  $APS(\mathbb{R}^q)$ .

Let  $U_s$  denote the collection of sequences (weights)  $\varrho : \mathbb{Z} \rightarrow (0, +\infty)$ . For  $\varrho \in U_s$  and  $T \in \mathbb{Z}^+$ , where  $\mathbb{Z}^+ = \{n \in \mathbb{Z} : n \geq 0\}$ , set

$$\mu_s(T, \varrho) = \sum_{n=-T}^T \varrho(n).$$

Define

$$U_{s\infty} := \left\{ \varrho \in U_s : \lim_{T \rightarrow \infty} \mu_s(T, \varrho) = \infty \right\},$$

and

$$U_{sB} := \left\{ \varrho \in U_{s\infty} : \varrho \text{ is bounded with } \inf_{n \in \mathbb{Z}} \varrho(n) > 0 \right\}.$$

For  $\varrho_1, \varrho_2 \in U_{s\infty}$ ,  $\varrho_1$  is said to be equivalent to  $\varrho_2$ , denoted by  $\varrho_1 < \varrho_2$ , if  $\{\varrho_1(n)/\varrho_2(n)\}_{n \in \mathbb{Z}} \in U_{sB}$ . Then it is easy to see that ‘<’ is a binary equivalence relation on  $U_{s\infty}$ . For  $\varrho \in U_{s\infty}$ ,  $k \in \mathbb{Z}$ , define  $\varrho_k$  by  $\varrho_k(n) = \varrho(n + k)$  for  $n \in \mathbb{Z}$ . We set

$$U_{sT} = \{ \varrho \in U_{s\infty} : \varrho < \varrho_k \text{ for each } k \in \mathbb{Z} \}.$$

**DEFINITION 2.5 [13].**

- (i) Let  $\varrho \in U_{s\infty}$ . A sequence  $x : \mathbb{Z} \rightarrow \mathbb{R}^q$  is said to be a  $\varrho$ - $PAP_0$  sequence if it is bounded and satisfies

$$\lim_{T \rightarrow \infty} \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T |x(n)|_{\varrho(n)} = 0.$$

Denote the set of all such sequences  $x$  by  $PAP_0S(\mathbb{R}^q, \varrho)$ .

- (ii) Let  $\varrho \in U_{s\infty}$ . A sequence  $x : \mathbb{Z} \rightarrow \mathbb{R}^q$  is said to be a weighted pseudo almost periodic sequence (or a  $\varrho$ -pseudo almost periodic sequence) if  $x$  can be written as a sum  $x = x^{ap} + x^e$  with  $x^{ap} \in APS(\mathbb{R}^q)$  and  $x^e \in PAP_0S(\mathbb{R}^q, \varrho)$ . Here,  $x^{ap}$  and  $x^e$  are called the almost periodic component and weighted ergodic perturbation, respectively, of the sequence  $x$ . Denote the set of all such sequences  $x$  by  $PAPS(\mathbb{R}^q, \varrho)$ .

The following two results are given in [13, 14].

**LEMMA 2.6.** Let  $\rho \in U_T$  and set

$$\varrho(n) = \int_n^{n+1} \rho(t) dt \quad \text{for } n \in \mathbb{Z}. \tag{2.2}$$

Then  $\varrho \in U_{sT}$ , and for  $c \in \mathbb{R}$  there exist positive constants  $C_1, C_2$  such that, for  $T$  large enough,

$$C_1 \mu(T + c, \rho) \leq \mu_s(\lfloor T \rfloor, \varrho) \leq C_2 \mu(T + c, \rho).$$

**PROPOSITION 2.7.**  $PAP_0S(\mathbb{R}^q, \varrho)$  with  $\varrho \in U_{sT}$  is translation invariant. That is, for  $x \in PAP_0S(\mathbb{R}^q, \varrho)$  and  $k \in \mathbb{Z}$ , we have  $x(\cdot - k) \in PAP_0S(\mathbb{R}^q, \varrho)$ .

We now present some more propositions on weighted pseudo almost periodic sequences.

**PROPOSITION 2.8.** Let  $g \in C(\mathbb{R}, \mathbb{R}^q)$ ,  $\rho \in U_T$ ,  $\varrho(n)$  be given by (2.2) and  $x \in PAP_0S(\mathbb{R}^q, \varrho)$ . Suppose that there exist  $L > 0$  and a finite subset  $Z_0 \subset \mathbb{Z}$  such that

$$|g(t)| \leq L \max_{k \in Z_0} |x(n + k)|, \quad t \in [n, n + 1), \quad n \in \mathbb{Z}.$$

Then  $g \in PAP_0(\mathbb{R}^q, \rho)$ .

**PROOF.** It is easy to see that  $g$  is bounded on  $\mathbb{R}$ . By Lemma 2.6 and Proposition 2.7,

$$\begin{aligned} & \lim_{T \rightarrow \infty} \frac{1}{\mu(T, \rho)} \int_{-T}^T |g(t)| \rho(t) dt \\ & \leq \lim_{T \rightarrow \infty} \frac{1}{\mu(T, \rho)} \sum_{n=-\lfloor T \rfloor - 1}^{\lfloor T \rfloor + 1} \int_n^{n+1} |g(t)| \rho(t) dt \\ & \leq \lim_{T \rightarrow \infty} \frac{1}{\mu(T, \rho)} \sum_{n=-\lfloor T \rfloor - 1}^{\lfloor T \rfloor + 1} \int_n^{n+1} L \max_{k \in \mathbb{Z}_0} |x(n+k)| \rho(t) dt \\ & \leq L \sum_{k \in \mathbb{Z}_0} \lim_{T \rightarrow \infty} \frac{1}{\mu(T, \rho)} \sum_{n=-\lfloor T \rfloor - 1}^{\lfloor T \rfloor + 1} |x(n+k)| \int_n^{n+1} \rho(t) dt \\ & = L \sum_{k \in \mathbb{Z}_0} \lim_{T \rightarrow \infty} \frac{\mu_s(\lfloor T \rfloor + 1, \varrho)}{\mu(T, \rho)} \cdot \frac{1}{\mu_s(\lfloor T \rfloor + 1, \varrho)} \sum_{n=-\lfloor T \rfloor - 1}^{\lfloor T \rfloor + 1} |x(n+k)| \varrho(n) \\ & = 0, \end{aligned}$$

which means that  $g \in PAP_0(\mathbb{R}^q, \rho)$ . This completes the proof. □

**PROPOSITION 2.9.** Let  $x = x^{ap} + x^e \in PAPS(\mathbb{R}^q, \varrho)$  with  $x^{ap} \in APS(\mathbb{R}^q)$ ,  $x^e \in PAP_0S(\mathbb{R}^q, \varrho)$  and  $\varrho \in U_{sT}$ . Then  $x^{ap}(\mathbb{Z}) \subset \overline{\text{co } x(\mathbb{Z})}$ .

**PROOF.** Let  $\rho(t) = \varrho(n)$  for  $t \in [n, n + 1), n \in \mathbb{Z}$ . It is easy to see that  $\rho \in U_T$  and  $\varrho(n) = \int_n^{n+1} \rho(t) dt$  for  $n \in \mathbb{Z}$ . Let

$$\begin{aligned} f(t) &= (x(n+1) - x(n))(t - n) + x(n), \\ f^{ap}(t) &= (x^{ap}(n+1) - x^{ap}(n))(t - n) + x^{ap}(n), \\ f^e(t) &= (x^e(n+1) - x^e(n))(t - n) + x^e(n), \end{aligned}$$

for  $t \in [n, n + 1), n \in \mathbb{Z}$ . Then  $f = f^{ap} + f^e$  and  $f, f^{ap}$  and  $f^e$  are continuous with  $f(n) = x(n), n \in \mathbb{Z}$ , and

$$|f^e(t)| \leq 2 \max\{|x^e(n)|, |x^e(n+1)|\} \quad \text{for } t \in [n, n + 1), n \in \mathbb{Z}.$$

Thus by Proposition 2.8 we have  $f^e \in PAP_0(\mathbb{R}^q, \rho)$ . It is well known that  $f^{ap} \in AP(\mathbb{R}^q)$  (see, for example, [5]). So  $f \in PAP(\mathbb{R}^q, \rho)$ . Following the same lines as in the proof of [12, Lemma 1.3], we can easily see that  $\overline{f^{ap}(\mathbb{R})} \subset \overline{f(\mathbb{R})} = \overline{\text{co } x(\mathbb{Z})}$  (see also the proof of [4, Theorem 3.1]). Thus  $x^{ap}(\mathbb{Z}) \subset \overline{\text{co } x(\mathbb{Z})}$  since  $x^{ap}(n) = f^{ap}(n)$  for  $n \in \mathbb{Z}$ . This completes the proof. □

By the proof of Proposition 2.9, we can also see that  $x \in PAPS(\mathbb{R}^q, \varrho)$  implies that there exists  $f \in PAP(\mathbb{R}^q, \rho)$  with  $f(n) = x(n), n \in \mathbb{Z}$ . Conversely, we consider the following question: does  $f \in PAP(\mathbb{R}^q, \rho)$  imply  $\{f(n)\}_{n \in \mathbb{Z}} \in PAPS(\mathbb{R}^q, \varrho)$ ?

Here  $\rho$  and  $\varrho$  are as in Proposition 2.8. It is well known that  $f \in AP(\mathbb{R}^q)$  implies  $\{f(n)\}_{n \in \mathbb{Z}} \in APS(\mathbb{R}^q)$ . But this is not true for a weighted pseudo almost periodic

function. For example, let  $\rho(t) = 1 + t^2$  and define the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(t) = \begin{cases} e^{-n^2|t-n|} - e^{-1}, & t \in \left[ n - \frac{1}{n^2}, n + \frac{1}{n^2} \right], n = 2, 3, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that  $\rho \in U_T, f \in PAP_0(\mathbb{R}, \rho)$  and

$$f(n) = \begin{cases} 1 - e^{-1}, & n = 2, 3, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly,  $\{f(n)\}_{n \in \mathbb{Z}}$  is not a  $\varrho$ - $PAP_0$  sequence.

However, by [7, Proposition 2.3], if  $x$  is a pseudo almost periodic function then  $\{x(n)\}_{n \in \mathbb{Z}}$  is a pseudo almost periodic sequence provided that  $x$  is uniformly continuous. The same conclusion also holds for weighted pseudo almost periodic functions. In fact, we have the following result.

**PROPOSITION 2.10.** *Let  $\rho \in U_T, \varrho$  be given by (2.2) and  $x \in PAP(\mathbb{R}^q, \rho)$ . If  $x$  is uniformly continuous, then  $\{x(n)\}_{n \in \mathbb{Z}} \in PAPS(\mathbb{R}^q, \varrho)$ .*

**PROOF.** It is well known that  $\{x^{ap}(n)\}_{n \in \mathbb{Z}} \in APS(\mathbb{R}^q)$ . By the uniform continuity of  $x(t)$  and  $x^{ap}(t)$ , we see that  $x^\varepsilon(t)$  is uniformly continuous. Thus given  $\varepsilon > 0$ , there exists  $\delta \in (0, 1)$  such that  $|t_1 - t_2| < \delta$  implies  $|x^\varepsilon(t_1) - x^\varepsilon(t_2)| < \varepsilon$ . Consequently,

$$\begin{aligned} & \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T |x^\varepsilon(n)| \varrho(n) \\ & \leq \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T \left( \frac{1}{\delta} \int_n^{n+\delta} (|x^\varepsilon(n) - x^\varepsilon(t)| + |x^\varepsilon(t)|) dt \right) \varrho(n) \\ & \leq \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T \left( \frac{1}{\delta} \int_n^{n+\delta} (\varepsilon + |x^\varepsilon(t)|) dt \right) \varrho(n) \\ & \leq \frac{1}{\delta} \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T \int_n^{n+1} |x^\varepsilon(t)| dt \varrho(n) + \varepsilon. \end{aligned}$$

As in the similar argument of [13, Lemma 3.2],

$$\left\{ \int_n^{n+1} |x^\varepsilon(t)| dt \right\}_{n \in \mathbb{Z}} \in PAP_0S(\mathbb{R}, \varrho),$$

and so

$$\lim_{T \rightarrow \infty} \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T |x^\varepsilon(n)| \varrho(n) = 0.$$

That is,  $\{x^\varepsilon(n)\}_{n \in \mathbb{Z}} \in PAP_0S(\mathbb{R}^q, \varrho)$  and hence  $\{x(n)\}_{n \in \mathbb{Z}} \in PAPS(\mathbb{R}^q, \varrho)$ . The proof is complete. □

**PROPOSITION 2.11.** *Let  $\varrho \in U_{sT}$ . Then*

$$PAPS(\mathbb{R}^q, \varrho) = APS(\mathbb{R}^q) \oplus PAP_0S(\mathbb{R}^q, \varrho).$$

**PROOF.** If  $x = y_i + z_i$  with  $y_i \in APS(\mathbb{R}^q)$ ,  $z_i \in PAP_0S(\mathbb{R}^q, \varrho)$ ,  $i = 1, 2$ , then

$$(y_1 - y_2) + (z_1 - z_2) = 0.$$

It is clear that  $APS(\mathbb{R}^q)$  and  $PAP_0S(\mathbb{R}^q, \varrho)$  are linear spaces, whence  $y_1 - y_2 \in APS(\mathbb{R}^q)$  and  $z_1 - z_2 \in PAP_0S(\mathbb{R}^q, \varrho)$ . Hence, by Proposition 2.9 we have  $y_1(n) - y_2(n) \in \{0\}$  for  $n \in \mathbb{Z}$ . Therefore,  $y_1 = y_2$  and  $z_1 = z_2$ . This completes the proof.  $\square$

We note that, for  $x \in PAPS(\mathbb{R}^q, \varrho)$  with  $\varrho \in U_{sT}$ , Proposition 2.11 implies that the decomposition  $x = x^{ap} + x^e$  is unique, where  $x^{ap} \in APS(\mathbb{R}^q)$  and  $x^e \in PAP_0S(\mathbb{R}^q, \varrho)$ .

### 3. The main result

In the sequel, we always assume that  $f \in \mathcal{F}$ , where  $\rho \in U_T$ , and  $\varrho$  is given by (2.2).

#### 3.1. Statement of our main result.

**DEFINITION 3.1.** We say that a function  $x : \mathbb{R} \rightarrow \mathbb{R}^q$  is a  $\rho$ -pseudo almost periodic solution of (1.1) if  $x \in PAP(\mathbb{R}^q, \rho)$  and the following conditions are satisfied:

- (i) the derivative  $x'$  of  $x$  exists on  $\mathbb{R}$  except possibly at the points  $t = n, n \in \mathbb{Z}$ , where one-sided derivatives exist;
- (ii)  $x$  satisfies (1.1) in the intervals  $(n, n + 1), n \in \mathbb{Z}$ .

By the variation constant formula, the solution of (1.1) can be given by

$$x(t) = X(t)X^{-1}(n)x(n) + \int_n^t X(t)X^{-1}(s) \left[ \sum_{j=0}^r A_j(s)x(n-j) + f(s) \right] ds, \tag{3.1}$$

for  $t \in (n, n + 1]$ , where  $X(t)$  denotes the fundamental matrix associated to the equation

$$x'(t) = A(t)x(t). \tag{3.2}$$

Then by the continuity of  $x(t)$  we get

$$x(n + 1) = X(n + 1)X^{-1}(n)x(n) + \int_n^{n+1} X(n + 1)X^{-1}(s) \left[ \sum_{j=0}^r A_j(s)x(n-j) + f(s) \right] ds,$$

which can be rewritten as

$$x(n + 1) = \sum_{j=0}^r D_j(n)x(n-j) + h(n), \tag{3.3}$$

where

$$\begin{cases} D_0(n) = X(n+1)X^{-1}(n) + \int_n^{n+1} X(n+1)X^{-1}(s)A_0(s) ds, \\ D_j(n) = \int_n^{n+1} X(n+1)X^{-1}(s)A_j(s) ds, \quad j = 1, \dots, r, \\ h(n) = \int_n^{n+1} X(n+1)X^{-1}(s)f(s) ds. \end{cases}$$

By putting

$$z(n) = \begin{pmatrix} x(n) \\ x(n-1) \\ \vdots \\ x(n-r) \end{pmatrix}, \quad H(n) = \begin{pmatrix} h(n) \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$D(n) = \begin{pmatrix} D_0(n) & D_1(n) & \dots & D_{r-1}(n) & D_r(n) \\ I_q & 0 & \dots & 0 & 0 \\ 0 & I_q & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I_q & 0 \end{pmatrix},$$

(3.3) takes the form

$$z(n+1) = D(n)z(n) + H(n), \tag{3.4}$$

and its homogeneous form is

$$z(n+1) = D(n)z(n). \tag{3.5}$$

In the following, we assume that for each  $n \in \mathbb{Z}$ ,  $D_r(n)$  is invertible. Then for each  $n \in \mathbb{Z}$ ,  $D(n)$  is invertible.

The homogeneous equation (3.5) is said to admit an exponential dichotomy on  $\mathbb{Z}$  if there exist positive constants  $K, \alpha$  and a projection  $P$  ( $P^2 = P$ ) such that

$$\begin{cases} |Z(n)PZ^{-1}(m)| \leq Ke^{-\alpha(n-m)}, & n \geq m, \\ |Z(n)(I - P)Z^{-1}(m)| \leq Ke^{-\alpha(m-n)}, & m \geq n, \end{cases} \tag{3.6}$$

where  $Z(n)$  is the fundamental matrix solution of (3.5) with  $Z(0) = I$ .

We now state the main result in this paper.

**THEOREM 3.2.** *Assume that system (3.5) admits an exponential dichotomy on  $\mathbb{Z}$  with parameters  $(P, K, \alpha)$ . Then there exists a unique  $\rho$ -pseudo almost periodic solution for (1.1).*



**REMARK 3.3.** We claim that (3.5) admits an exponential dichotomy on  $\mathbb{Z}$  provided that

$$\sup_{n \in \mathbb{Z}} \sum_{j=0}^r |D_j(n)| < \frac{1}{\sqrt{r+1}}. \quad (3.7)$$

In fact, by (3.7) and a standard calculation (here we omit the details), we can obtain that

$$\theta = \left| \prod_{i=1}^T D(n-i) \right| < 1 \quad \text{for some } T > 1,$$

and by (3.5) we have

$$|Z(n)| \leq \theta \sup\{|Z(m)| : |m-n| \leq T, m \in \mathbb{Z}^+\}, \quad n \geq T.$$

Now by [2, Propositions 3.1 and 3.2], (3.5) has an exponential dichotomy on  $\mathbb{Z}$  since  $\{D(n)\}_{n \in \mathbb{Z}}$  is almost periodic.

### 3.2. Weighted pseudo almost periodic difference equation.

**LEMMA 3.4** [9]. *Suppose that (3.5) admits an exponential dichotomy on  $\mathbb{Z}$  with parameters  $(P, K, \alpha)$  and  $\{H(n)\}_{n \in \mathbb{Z}}$  is a bounded sequence. Then (3.4) has a unique solution  $z(n)$  bounded on  $\mathbb{Z}$ . Moreover,*

$$|z(n)| \leq K(1 + e^{-\alpha})(1 - e^{-\alpha})^{-1} \sup_{m \in \mathbb{Z}} |H(m)|, \quad n \in \mathbb{Z}.$$

We now show the weighted pseudo almost periodicity of  $\{H(n)\}_{n \in \mathbb{Z}}$ .

**LEMMA 3.5** [11]. *Assume that  $|A(t)| \leq M$  for  $t \in \mathbb{R}$ . Then there exists  $K_0 > 0$  such that the following statements hold:*

- (i)  $|X(t)X^{-1}(s)| \leq K_0$  for  $0 < t - s \leq 1$ .
- (ii) If  $\tau \in T(A, \varepsilon)$ , then  $|X(t + \tau)X^{-1}(s + \tau) - X(t)X^{-1}(s)| \leq K_0 \varepsilon e^M$  for  $0 < t - s \leq 1$ , where  $X(t)$  is the fundamental matrix solution of (3.2) with  $X(0) = I$ .

By a similar proof to [13, Lemma 3.2], we can easily get the following lemma.

**LEMMA 3.6.** *Let  $f \in \mathcal{F}_0$ . Then*

$$\left\{ \int_n^{n+1} |f(t)| dt \right\}_{n \in \mathbb{Z}} \in PAP_0S(\mathbb{R}, \varrho).$$

**LEMMA 3.7.**  $\{h(n)\}_{n \in \mathbb{Z}} \in PAP(\mathbb{R}^q, \varrho)$ .

**PROOF.** Let  $f = f_1 + f_0$ , where  $f_1 \in \mathcal{F}_1$ ,  $f_0 \in \mathcal{F}_0$ . Let

$$h^{ap}(n) = \int_n^{n+1} X(n+1)X^{-1}(s)f_1(s) ds.$$

From [1, Lemma 22],  $\{h^{ap}(n)\}_{n \in \mathbb{Z}} \in APS(\mathbb{R}^q)$ . By Lemma 3.5(i),

$$\begin{aligned} |h^e(n)| &= |h(n) - h^{ap}(n)| \\ &= \left| \int_n^{n+1} X(n+1)X^{-1}(s)f_0(s) ds \right| \leq K_0 \int_n^{n+1} |f_0(s)| ds. \end{aligned}$$

It follows from Lemma 3.6 that

$$\left\{ \int_n^{n+1} |f_0(s)| ds \right\}_{n \in \mathbb{Z}} \in PAP_0S(\mathbb{R}, \varrho).$$

Hence,  $\{h^e(n)\}_{n \in \mathbb{Z}} \in PAP_0S(\mathbb{R}^q, \varrho)$ . This completes the proof. □

We note that  $\{H(n)\}_{n \in \mathbb{Z}}$  is  $\varrho$ -pseudo almost periodic by Lemma 3.7. Moreover, since  $\{D_j(n)\}_{n \in \mathbb{Z}}, j = 0, \dots, r$ , are almost periodic sequences by [11, Proposition 9],  $\{D(n)\}_{n \in \mathbb{Z}}$  is almost periodic.

We now establish the weighted pseudo almost periodicity for the difference equation (3.4).

**LEMMA 3.8.** *Suppose that (3.5) admits an exponential dichotomy on  $\mathbb{Z}$  with parameters  $(P, K, \alpha)$ . Then (3.4) has a unique  $\varrho$ -pseudo almost periodic solution  $z(n)$ .*

**PROOF.** Let  $H = H^{ap} + H^e$ . By [11, Proposition 11],

$$z(n+1) = C(n)z(n) + H^{ap}(n)$$

has a unique almost periodic solution  $\{z^{ap}(n)\}_{n \in \mathbb{Z}}$ . Let

$$z^e(n) = \sum_{m \leq n-1} Z(n)PZ^{-1}(m+1)H^e(m) - \sum_{m \geq n} Z(n)(I-P)Z^{-1}(m+1)H^e(m)$$

for  $n \in \mathbb{Z}$ , where  $Z(n)$  is the fundamental matrix solution of (3.5) with  $Z(0) = I$ . Then by (3.6),

$$|z^e(n)| \leq \left( \sum_{m \leq n-1} Ke^{-\alpha(n-m-1)} + \sum_{m \geq n} Ke^{-\alpha(m+1-n)} \right) \sup_{n \in \mathbb{Z}} |H^e(n)|.$$

That is,  $\{z^e(n)\}_{n \in \mathbb{Z}}$  is bounded. So by Lemma 3.4,  $z^e(n)$  is the unique bounded solution of the equation

$$z(n+1) = C(n)z(n) + H^e(n).$$

Thus, again by Lemma 3.4,  $z = z^{ap} + z^e$  is the unique bounded solution of (3.4) satisfying

$$|z(n)| \leq K(1 + e^{-\alpha})(1 - e^{-\alpha})^{-1} \sup_{m \in \mathbb{Z}} |H(m)|, \quad n \in \mathbb{Z}.$$

Now it is sufficient to prove that  $z^e \in PAP_0S(\mathbb{R}^{q(r+1)}, \varrho)$ . From (3.6),

$$\begin{aligned} &\frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T |z^e(n)| \varrho(n) \\ &\leq \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T \left( \sum_{m \leq n-1} Ke^{-\alpha(n-m-1)} |H^e(m)| + \sum_{m \geq n} Ke^{-\alpha(m+1-n)} |H^e(m)| \right) \varrho(n) \\ &\triangleq \sigma_1(T) + \sigma_2(T), \end{aligned}$$

where

$$\begin{aligned} \sigma_1(T) &= \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T \sum_{m \leq n-1} K e^{-\alpha(n-m-1)} |H^e(m)| \varrho(n), \\ \sigma_2(T) &= \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T \sum_{m \geq n} K e^{-\alpha(m+1-n)} |H^e(m)| \varrho(n). \end{aligned}$$

For  $T, k \in \mathbb{Z}^+$ , let

$$\Phi_T(k) = \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T |H^e(n-k-1)| \varrho(n).$$

From Proposition 2.7,

$$\lim_{T \rightarrow \infty} \Phi_T(k) = 0 \quad \text{and} \quad |\Phi_T(k)| \leq \|H^e\| \quad \text{for } k \in \mathbb{Z}^+. \tag{3.8}$$

Given  $\varepsilon > 0$ , it is clear that there exists an integer  $N > 0$  such that

$$\sum_{k=N+1}^{\infty} e^{-\alpha k} < \varepsilon. \tag{3.9}$$

Then by (3.8), there exists  $T_0 > 0$  such that for  $T > T_0$ ,

$$\Phi_T(k) < \frac{\varepsilon}{N+1} \quad \text{for } k \in [0, N]. \tag{3.10}$$

Now by (3.8)–(3.10), for  $T > T_0$ ,

$$\begin{aligned} \sigma_1(T) &= \frac{K}{\mu_s(T, \varrho)} \sum_{n=-T}^T \sum_{m=-\infty}^{n-1} e^{-\alpha(n-m-1)} |H^e(m)| \varrho(n) \\ &= \frac{K}{\mu_s(T, \varrho)} \sum_{n=-T}^T \sum_{k=0}^{+\infty} e^{-\alpha k} |H^e(n-k-1)| \varrho(n) \\ &= K \sum_{k=0}^{+\infty} e^{-\alpha k} \frac{1}{\mu_s(T, \varrho)} \sum_{n=-T}^T |H^e(n-k-1)| \varrho(n) \\ &= K \sum_{k=0}^{+\infty} e^{-\alpha k} \Phi_T(k) \\ &= K \sum_{k=0}^N e^{-\alpha k} \Phi_T(k) + K \sum_{k=N+1}^{+\infty} e^{-\alpha k} \Phi_T(k) \\ &\leq K(N+1) \frac{\varepsilon}{N+1} + K \|H^e\| \varepsilon = K(1 + \|H^e\|) \varepsilon. \end{aligned}$$

This implies that  $\lim_{T \rightarrow \infty} \sigma_1(T) = 0$ . Similarly, we can prove that

$$\lim_{T \rightarrow \infty} \sigma_2(T) = 0.$$

Therefore  $z^e \in PAP_0S(\mathbb{R}^{q(r+1)}, \varrho)$ , and the proof is complete. □

**3.3. Proof of Theorem 3.2.**

**PROOF.** From Lemma 3.8, we know that the Equation (3.3) has a unique  $\varrho$ -pseudo almost periodic solution  $x(n)$  satisfying

$$|x(n)| \leq c \sup_{n \in \mathbb{Z}} |h(n)| \leq cK_0 \|f\|, \quad n \in \mathbb{Z}. \tag{3.11}$$

For  $t \in (n, n + 1], n \in \mathbb{Z}$ , by (3.1), (3.11) and Lemma 3.5,

$$\begin{aligned} |x(t)| &\leq |X(t)X^{-1}(n)|x(n) + \int_n^t |X(t)X^{-1}(s)| \left( \sum_{j=0}^r |A_j(s)|x(n-j) + |f(s)| \right) ds \\ &\leq K_0|x(n)| + \int_n^t K_0 \left( \sum_{j=0}^r |A_j(s)|x(n-j) + |f(s)| \right) ds \\ &\leq cK_0^2 \|f\| + K_0 \left( \sum_{j=0}^r \|A_j\|cK_0 \|f\| + \|f\| \right) = c_1 \|f\|, \end{aligned}$$

where  $c_1 = cK_0^2(1 + \sum_{j=0}^r \|A_j\|) + K_0$ . That is,  $x(t)$  is bounded on  $\mathbb{R}$ . Let  $f = f_1 + f_0$ , where  $f_1 \in \mathcal{F}_1, f_0 \in \mathcal{F}_0$ . Let

$$x^{ap}(t) = X(t)X^{-1}(n)x^{ap}(n) + \int_n^t X(t)X^{-1}(s) \left[ \sum_{j=0}^r A_j(s)x^{ap}(n-j) + f_1(s) \right] ds$$

for  $t \in (n, n + 1], n \in \mathbb{Z}$ . Then by a standard method we can prove that  $x^{ap} \in AP(\mathbb{R}^q)$  (see the corresponding part of the proof of [1, Theorem 23]).

Let

$$\begin{aligned} x^e(t) &= x(t) - x^{ap}(t) \\ &= X(t)X^{-1}(n)x^e(n) + \int_n^t X(t)X^{-1}(s) \sum_{j=0}^r A_j(s)x^e(n-j) ds + \int_n^t X(t)X^{-1}(s)f_0(s) ds \\ &\triangleq \sigma_1(t) + \sigma_2(t) \end{aligned}$$

where

$$\begin{aligned} \sigma_1(t) &= X(t)X^{-1}(n)x^e(n) + \int_n^t X(t)X^{-1}(s) \sum_{j=0}^r A_j(s)x^e(n-j) ds, \\ \sigma_2(t) &= \int_n^t X(t)X^{-1}(s)f_0(s) ds, \end{aligned}$$

for  $t \in (n, n + 1], n \in \mathbb{Z}$ . By Lemma 3.5, we have

$$\begin{aligned} |\sigma_1(t)| &\leq K_0|x^e(n)| + K_0 \sum_{j=0}^r \|A_j\| |x^e(n-j)| \\ &\leq K_0 \left( 1 + \sum_{j=0}^r \|A_j\| \right) \max_{0 \leq j \leq r} |x^e(n-j)| \end{aligned}$$

and

$$|\sigma_2(t)| \leq K_0 \int_n^{n+1} |f_0(s)| ds$$

for  $t \in (n, n + 1]$ ,  $n \in \mathbb{Z}$ . It follows from Proposition 2.8 and Lemma 3.6 that  $\sigma_1, \sigma_2 \in PAP_0(\mathbb{R}^q, \rho)$ , and consequently  $x^e \in PAP_0(\mathbb{R}^q, \rho)$ . This completes the proof.  $\square$

## References

- [1] E. Ait Dads and L. Lhachimi, 'Pseudo almost periodic solutions for equation with piecewise constant argument', *J. Math. Anal. Appl.* **371** (2010), 842–854.
- [2] A. I. Alonso, J. L. Hong and R. Obaya, 'Exponential dichotomy and trichotomy for difference equations', *Comput. Math. Appl.* **38** (1999), 41–49.
- [3] T. Diagana, 'Weighted pseudo almost periodic functions and applications', *C. R. Acad. Sci. Paris I* **343** (2006), 643–646.
- [4] T. Diagana, 'Weighted pseudo-almost periodic solutions to some differential equations', *Nonlinear Anal.* **68** (2008), 2250–2260.
- [5] A. M. Fink, *Almost Periodic Differential Equations*, Lecture Notes in Mathematics, 377 (Springer, Berlin, 1974).
- [6] T. Küpper and R. Yuan, 'On quasi-periodic solutions of differential equations with piecewise constant argument', *J. Math. Anal. Appl.* **267** (2002), 173–193.
- [7] H. X. Li, 'Pseudo almost periodic sequences and some nonlinear differential equations with piecewise constant argument', *Nonlinear Funct. Anal. Appl.* **10** (2005), 479–493.
- [8] J. Liang, T. J. Xiao and J. Zhang, 'Decomposition of weighted pseudo-almost periodic functions', *Nonlinear Anal.* **73** (2010), 3456–3461.
- [9] K. J. Palmer, 'Exponential dichotomies, the shadowing lemma and transversal homoclinic points', *Dynam. Rep.* **1** (1988), 265–306.
- [10] R. Yuan, 'The existence of almost periodic solutions of retarded differential equations with piecewise constant argument', *Nonlinear Anal.* **48** (2002), 1013–1032.
- [11] R. Yuan and J. L. Hong, 'The existence of almost periodic solutions for a class of differential equations with piecewise constant argument', *Nonlinear Anal.* **28** (1997), 1439–1450.
- [12] C. Y. Zhang, 'Pseudo almost periodic solutions of some differential equations', *J. Math. Anal. Appl.* **181** (1994), 62–76.
- [13] L. L. Zhang and H. X. Li, 'Weighted pseudo almost periodic solutions of second order neutral differential equations with piecewise constant argument', *Nonlinear Anal.* **74** (2011), 6770–6780.
- [14] L. L. Zhang and H. X. Li, 'Weighted pseudo almost periodic solutions of second-order neutral-delay differential equations with piecewise constant argument', *Comput. Math. Appl.* **62** (2011), 4362–4376.

LI-LI ZHANG, Department of Mathematics and Physics,  
Shijiazhuang Tiedao University, Shijiazhuang, Hebei 050043, PR China  
e-mail: [zlyou0@gmail.com](mailto:zlyou0@gmail.com)

HONG-XU LI, Department of Mathematics, Sichuan University,  
Chengdu, Sichuan 610064, PR China  
e-mail: [hoxuli@sohu.com](mailto:hoxuli@sohu.com)