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SEMIDIRECT PRODUCT COMPACTIFICATIONS

F. DANGELLO AND R. LINDAHL

1. Introduction. K. Deleeuw and I. Glicksberg [4] proved that if S and T are commutative topological semigroups with identity, then the Bochner almost periodic compactification of $S \times T$ is the direct product of the Bochner almost periodic compactifications of S and T. In Section 3 we consider the semidirect product $S \circledS T$ of two semitopological semigroups with identity and two unital C^*-subalgebras $\mathscr{A}$ and $\mathscr{B}$ of $W(S)$ and $W(T)$ respectively, where $W(S)$ is the weakly almost periodic functions on S. We obtain necessary and sufficient conditions on $\mathscr{A}$ and $\mathscr{B}$ for a semidirect product compactification of $S \circledS T$ to exist such that this compactification is a semitopological semigroup and such that this compactification is a topological semigroup. Moreover, we obtain the largest such compactifications. The largest such semitopological semigroup compactification is induced by $W^\sigma(S)$ and $W(T)$, where $W^\sigma(S)$ is a translation-invariant unital C^*-subalgebra of $W(S)$. The largest such topological semigroup compactification is induced by $A^\sigma(S)$ and $A(T)$, where $A^\sigma(S)$ is a translation-invariant unital C^*-subalgebra of $A(S)$, and $A(T)$ is the Bochner almost periodic functions on T. These results are achieved via an internal characterization of the tensor product of two algebras of bounded complex-valued functions on two sets, which we obtain in Section 2.

In Section 4 we obtain sufficient conditions for $A(S \circledS T)$ to be the tensor product of $A^\sigma(S)$ and $A(T)$ and for $W(S \circledS T)$ to be the tensor product of $W^\sigma(S)$ and $W(T)$. In these cases it follows that the Bochner and weakly almost periodic compactifications of $S \circledS T$ are semidirect product compactifications. We give an example showing that this is not generally valid and in the previous section we give examples where $A^\sigma(S) = A(S)$ and $W^\sigma(S) = W(S)$.

2. Tensor products of function algebras. For a set Z, let $B(Z)$ denote the bounded complex-valued functions on Z, and let $\mathscr{D}$ be a unital C^*-subalgebra of $B(Z)$. (We impose the uniform norm on $B(Z)$; that is, $\|f\|_u = \sup_{z \in Z} |f(z)|$.) We assume that any such $\mathscr{D}$ contains the constant functions. Let $\Delta(\mathscr{D})$ denote the structure space of $\mathscr{D}$; that is, $\Delta(\mathscr{D})$ consists of all non-zero multiplicative linear functionals on $\mathscr{D}$,

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the topology being the Gelfand (or weak-*) topology. Then $\Delta(\mathscr{D})$ is a compact Hausdorff space and by the Gelfand-Naimark theorem [**18**], the Gelfand transform $f \to \hat{f}$ given by

$$\hat{f}(\tau) = \tau(f), \quad \tau \in \Delta(\mathscr{D}), \quad f \in \mathscr{D}$$

is an isometric, conjugate-preserving algebra isomorphism from $\mathscr{D}$ onto $C(\Delta(\mathscr{D}))$. Moreover, $I(Z)$ is dense in $\Delta(\mathscr{D})$, where $I : Z \to \Delta(\mathscr{D})$ is given by

$$I(z)(f) = f(z), \quad z \in Z, \quad f \in \mathscr{D}.$$

We call $\Delta(\mathscr{D})$ the $(\mathscr{D}, I)$-*compactification* of Z. The inverse Gelfand transform will be denoted by I^*, and following the terminology in [**1**] and [**2**], we will refer to I^* as the adjoint map of I.

Until further notice our setting will be as follows. Let X and Y be sets. Let $\mathscr{A}$ [resp. $\mathscr{B}$] be a unital C^*-subalgebra of $B(X)$ [resp. $B(Y)$]. Let $\bar{X}$ be the $(\mathscr{A}, I_1)$-compactification of X and $\bar{Y}$ be the $(\mathscr{B}, I_2)$-compactification of Y. Given h in $B(X \times Y)$, x in X, y in Y, set

$${}^x h(y') = h(x, y'), \quad y' \in Y$$

and

$$h^y(x') = h(x', y), \quad x' \in X.$$

Let

$$\mathscr{C} = \{h \in B(X \times Y) : {}^x h \in \mathscr{B}, x \in X; h^y \in \mathscr{A}, y \in Y; \text{ and } \{h^y : y \in Y\} \text{ is totally bounded in } \mathscr{A}\}.$$

For f in $\mathscr{A}$, g in $\mathscr{B}$, set

$$f \otimes g(x, y) = f(x)g(y), \ (x, y) \in X \times Y.$$

Let $\mathscr{A} \otimes \mathscr{B}$ denote the unital C^*-subalgebra of $B(X \times Y)$ generated by

$$\{f \otimes g : f \in \mathscr{A}, g \in \mathscr{B}\}.$$

We will prove that $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$.

PROPOSITION 2.1. *$\mathscr{C}$ is a C^*-subalgebra of $B(X \times Y)$ containing $\mathscr{A} \otimes \mathscr{B}$.*

Proof. It follows directly that $\mathscr{C}$ is a Banach space and is self-adjoint. Moreover, given h_1 and h_2 in $\mathscr{C}$, ${}^x(h_1 h_2) = {}^x h_1 {}^x h_2$ is in $\mathscr{B}$ for all x in X and $(h_1 h_2)^y = h_1^y h_2^y$ is in $\mathscr{A}$ for all y in Y. Since $\{h_1^y : y \in Y\}$ and $\{h_2^y : y \in Y\}$ are totally bounded, so is $\{(h_1 h_2)^y : y \in Y\}$. Hence $h_1 h_2$ is in $\mathscr{C}$, which proves that $\mathscr{C}$ is a subalgebra of $B(X \times Y)$. Finally, $\mathscr{C}$ contains $\mathscr{A} \otimes \mathscr{B}$ since $f \otimes g$ is in $\mathscr{C}$ for each f in $\mathscr{A}$, g in $\mathscr{B}$.

Let $\overline{X \times Y}$ denote the $(\mathscr{C}, I)$-compactification of $X \times Y$. We will show how to identify $\bar{X} \times \bar{Y}$ with $\overline{X \times Y}$.

The following lemma will be used several times throughout the paper.

LEMMA 2.2. *Let E be a compact Hausdorff topological space and let $\mathscr{T}$ and $\mathscr{T}'$ be two Hausdorff topologies on a set Z such that $\mathscr{T}'$ is weaker than $\mathscr{T}$. Also suppose that D is a dense subset of E and that ψ is a continuous map from E into $(Z, \mathscr{T}')$. Then ψ is continuous from E into $(Z, \mathscr{T})$ if and only if $\{\psi(x) : x \in D\}$ is conditionally compact in $(Z, \mathscr{T})$.*

Proof. If ψ is continuous from E into $(Z, \mathscr{T})$, then $\psi(E)$ is compact and hence closed in $(Z, \mathscr{T})$ since E is compact and $\mathscr{T}$ is Hausdorff. Therefore, $\psi(D)$ has compact closure in $(Z, \mathscr{T})$.

Now assume that $\psi(D)$ is conditionally compact in $(Z, \mathscr{T})$. Let x be in E and let $(x_\alpha)_\alpha$ be a net in D with $x_\alpha \rightarrow x$. We show that $\psi(x_\alpha) \underset{\mathscr{T}}{\rightarrow} \psi(x)$.

Suppose $(\psi(x_\alpha))_\alpha$ does not converge to $\psi(x)$ in $(Z, \mathscr{T})$. Then there exists a $\mathscr{T}$-open neighborhood V of $\psi(x)$ and a subnet $(x_\beta)_\beta$ of $(x_\alpha)_\alpha$ such that $\psi(x_\beta)$ is in $Z \sim V$ for all β ($\sim$ denotes complement). Since $\{\psi(x) : x \in D\}$ is conditionally compact in $(Z, \mathscr{T})$ and $Z \sim V$ is $\mathscr{T}$-closed, there exists a subnet $(x_\gamma)_\gamma$ of $(x_\beta)_\beta$ and a z in $Z \sim V$ such that $\psi(x_\gamma) \underset{\mathscr{T}}{\rightarrow} z$. Since ψ is continuous from E into $(Z, \mathscr{T}')$,

$$\psi(x_\alpha) \underset{\mathscr{T}'}{\longrightarrow} \psi(x),$$

and since $\mathscr{T}'$ is weaker than $\mathscr{T}$,

$$\psi(x_\gamma) \underset{\mathscr{T}'}{\longrightarrow} z.$$

Since $\mathscr{T}'$ is Hausdorff and $(x_\gamma)_\gamma$ is a subnet of $(x_\alpha)_\alpha$, $z = \psi(x)$. Therefore, $\psi(x)$ is in $Z \sim V$, for a contradiction.

The above argument proves that $\psi(E)$ is contained in the $\mathscr{T}$-closure of $\psi(D)$, and hence, $\psi(E)$ is conditionally compact in $(Z, \mathscr{T})$. We can now repeat the above argument with D replaced by E to show that if x is in E and (x_α) is a net in E with $x_\alpha \rightarrow x$, then

$$\psi(x_\alpha) \underset{\mathscr{T}}{\rightarrow} \psi(x).$$

Hence, ψ is continuous from E into $(Z, \mathscr{T})$.

Definition 2.3. For h in $\mathscr{C}$, μ in $\bar{Y}$, set $h^\mu(x) = \mu({}^x h)$ for all x in X.

Note that $h^{I_2(y)} = h^y$ for all y in Y and h in $\mathscr{C}$.

PROPOSITION 2.4. *Given h in $\mathscr{C}$, μ in $\bar{Y}$, one has that h^μ is in $\mathscr{A}$. Moreover, $\mu \rightarrow h^\mu$ is continuous from $\bar{Y}$ into $(\mathscr{A}, \| \ \|_u)$.*

Proof. Choose a net $\{y_\alpha\}$ in Y such that $I_2(y_\alpha) \underset{w^*}{\longrightarrow} \mu$. For each x in X,

$$h^{y_\alpha}(x) = h(x, y_\alpha) = {}^x h(y_\alpha) = I_2(y_\alpha)({}^x h)$$
$$\underset{\alpha}{\to} \mu({}^x h) = h^\mu(x).$$

Hence h^{y_α} converges pointwise to h^μ. Since $\{h^y : y \in Y\}$ is totally bounded, $h^{y_\alpha} \underset{\| \|_u}{\longrightarrow} h^\mu$. Thus, h^μ is in $\mathscr{A}$.

Define ψ from $\bar{Y}$ into $\mathscr{A}$ by $\psi(\mu) = h^\mu$ for all μ in $\bar{Y}$. Then ψ is continuous in the topology of pointwise convergence on $\mathscr{A}$ and

$$\{\psi(I_2(y)) : y \in Y\} = \{h^y : y \in Y\}$$

is totally bounded. By Lemma 2.2, ψ is continuous from $\bar{Y}$ into $(\mathscr{A}, \| \quad \|_u)$.

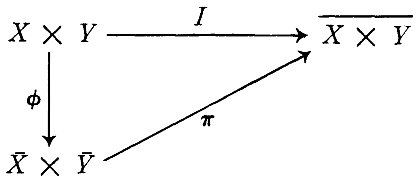
Definition 2.5. For τ in $\bar{X}$, μ in $\bar{Y}$, set $\tau \otimes \mu(h) = \tau(h^\mu)$, $h \in \mathscr{C}$.

Let $\phi(x, y) = (I_1(x), I_2(y))$ for all x in X and y in Y, and let $\pi(\tau, \mu) = \tau \otimes \mu$ for all τ in $\bar{X}$ and μ in $\bar{Y}$.

THEOREM 2.6. *The map π describes a homeomorphism from $\bar{X} \times \bar{Y}$ onto $\overline{X \times Y}$. Moreover,*

$$I_1(x) \otimes I_2(y) = I(x, y), \quad (x, y) \in X \times Y$$

from which the following diagram commutes:



Proof. First note that given $(x, y) \in X \times Y$ and h in $\mathscr{C}$, one has that

$$I_1(x) \otimes I_2(y)(h) = I_1(x)(h^{I_2(y)}) = I_1(x)(h^y)$$
$$= h(x, y) = I(x, y)(h).$$

Hence,

$$I_1(x) \otimes I_2(y) = I(x, y).$$

Let τ be in $\bar{X}$, μ in $\bar{Y}$. Then $\tau \otimes \mu$ is a linear functional on $\mathscr{C}$. For h_1, h_2 in $\mathscr{C}$,

$$\tau \otimes \mu(h_1 h_2) = \tau((h_1 h_2)^\mu) = \tau(h_1{}^\mu h_2{}^\mu) = \tau(h_1{}^\mu)\tau(h_2{}^\mu)$$
$$= \tau \otimes \mu(h_1) \cdot \tau \otimes \mu(h_2).$$

Hence, $\tau \otimes \mu$ is multiplicative. Also,

$$\tau \otimes \mu(1) = \tau(1) = 1.$$

Thus, $\tau \otimes \mu$ is in $\overline{X \times Y}$.

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For τ in $\bar{X}$, μ in $\bar{Y}$, f in $\mathscr{A}$, g in $\mathscr{B}$, we have that

$$\tau \otimes \mu(f \otimes g) = \tau((f \otimes g)^{\mu}) = \tau(\mu(g)f) = \tau(f)\mu(g).$$

It follows that π is one to one. From the first part of the proof, π maps densely into $\overline{X \times Y}$. Since $\bar{X} \times \bar{Y}$ is compact Hausdorff, it suffices to show that π is continuous. Let

$$\tau_\alpha \xrightarrow[w^*]{} \tau, \quad \mu_\alpha \xrightarrow[w^*]{} \mu,$$

and let h be in $\mathscr{C}$. From Proposition 2.4,

$$h^{\mu_\alpha} \xrightarrow[\|\ \|_u]{} h^{\mu}.$$

It follows that

$$\tau_\alpha(h^{\mu_\alpha}) \longrightarrow \tau(h^{\mu}).$$

Therefore, π is continuous and hence is a homeomorphism onto $\overline{X \times Y}$.

Recall that $\mathscr{C} = \{h \in B(X \times Y): {}^{x}h \in \mathscr{B}, x \in X; h^{y} \in \mathscr{A}, y \in Y;$ and $\{h^{y}: y \in Y\}$ is totally bounded in $\mathscr{A}\}$.

THEOREM 2.7. $\mathscr{A} \otimes \mathscr{B} = \mathscr{C}$.

Proof. Let ^ denote the Gelfand transform on $\mathscr{C}$. In showing that $\mathscr{A} \otimes \mathscr{B} = \mathscr{C}$, it suffices to prove that $(\mathscr{A} \otimes \mathscr{B})^\wedge$ separates the points of $\overline{X \times Y}$. Suppose

$$\tau \otimes \mu(h) = \tau' \otimes \mu'(h)$$

for all h in $\mathscr{A} \otimes \mathscr{B}$, where τ, τ' are in $\bar{X}$ and μ, μ' are in $\bar{Y}$. Then for f in $\mathscr{A}$,

$$\tau(f) = \tau \otimes \mu(f \otimes 1) = \tau' \otimes \mu'(f \otimes 1) = \tau'(f)$$

and so $\tau = \tau'$. Similarly, $\mu = \mu'$. Hence, $\tau \otimes \mu = \tau' \otimes \mu'$.

THEOREM 2.8.

$$\mathscr{A} \otimes \mathscr{B} = \{h \in B(X \times Y): {}^{x}h \in \mathscr{B}, x \in X; h^{y} \in \mathscr{A}, y \in Y; \text{ and } \{{}^{x}h : x \in X\} \text{ is totally bounded in } \mathscr{B}\}.$$

Proof. The proof is identical to the proof of Theorem 2.7.

A semitopological semigroup is a semigroup together with a Hausdorff topology such that the multiplication map is continuous in each variable separately. Let Z be a semitopological semigroup and let $C(Z)$ be the C^*-algebra of all bounded continuous complex-valued functions on Z. For f in $C(Z)$ and z in Z, the *left translate* of f by z is defined by

$${}_{z}f(x) = f(zx), x \in Z.$$

The right translate of f by z is defined similarly and is denoted by f_z. A function f in $C(Z)$ is called *Bochner almost periodic* on Z if $\{{}_{z}f : z \in Z\}$

has compact closure in $(C(Z), \| \ \|_u)$. Let $A(Z)$ denote all Bochner almost periodic functions on Z. Equivalently, an f in $C(Z)$ is in $A(Z)$ if and only if $\{{}_zf : z \in Z\}$ is totally bounded. Since $A(Z)$ is a translation-invariant unital C^*-subalgebra of $C(Z)$ (see [**5**]), we have the following corollary to Theorems 2.7 and 2.8.

COROLLARY 2.9. *Let S and T be semitopological semigroups and let $\mathscr{A}$ be a unital C^*-subalgebra of $A(S)$. Then*

$$\begin{aligned}\mathscr{A} \otimes A(T) &= \{h \in B(S \times T) : {}^sh \in A(T), s \in S; \\ &\quad h^t \in \mathscr{A}, t \in T; \text{ and } \{h^t : t \in T\} \text{ is totally bounded in } \mathscr{A}\} \\ &= \{h \in B(S \times T) : {}^sh \in A(T), s \in S; h^t \in \mathscr{A}, t \in T; \text{ and } \\ &\qquad \{{}^sh : s \in S\} \text{ is totally bounded in } A(T)\}.\end{aligned}$$

Remark. Berglund and Milnes [**2**] have shown that $A(S \times T) = A(S) \otimes A(T)$ whenever S and T are semitopological semigroups, where S has a right identity and T has a left identity. This result assuming S and T are commutative topological semigroups each with identity was obtained earlier by Deleeuw and Glicksberg [**4**]. We obtain Berglund and Milnes' result quite simply from the above theorems.

First let S and T be semitopological semigroups. For f in $A(S)$, g in $A(T)$, one has that

$${}_{(s,t)}(f \otimes 1) = {}_sf \otimes 1 \quad \text{and} \quad {}_{(s,t)}(1 \otimes g) = 1 \otimes {}_tg$$

for all s in S and t in T. Thus, $f \otimes 1$ and $1 \otimes g$ are in $A(S \times T)$ and so $f \otimes g = (f \otimes 1)(1 \otimes g)$ is in $A(S \times T)$.

Consequently, one has that

$$A(S) \otimes A(T) \subset A(S \times T).$$

Now assume that S has a right identity e and T has a left identity e' and consider the continuous map $I : C(S \times T) \to C(S)$ given by

$$I(h)(s) = h(s, e'), \quad h \in C(S \times T), \quad s \in S.$$

For s in S, t in T, and h in $C(S \times T)$, we have

$$h^t(s) = h(s, t) = h_{(e,t)}(s, e') = I(h_{(e,t)})(s).$$

Thus, the I image of the set of right translates of any h in $C(S \times T)$ contains $\{h^t : t \in T\}$. Therefore, if h is in $A(S \times T)$, then $\{h^t : t \in T\}$ is totally bounded. By Corollary 2.9 and the above,

$$A(S \times T) = A(S) \otimes A(T).$$

Note that one also obtains this result if he assumes that S has a left identity and T has a right identity.

Let Z be a semitopological semigroup and let $\mathscr{A}$ be a unital C^*-subalgebra of $C(Z)$. Call $\mathscr{A}$ *left M-introverted* [**17**] if $\mathscr{A}$ is translation-invariant

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and given f in $\mathscr{A}$, τ in $\Delta(\mathscr{A})$, one has that $\tau \circ f$ is in $\mathscr{A}$, where

$$\tau \circ f(z) = \tau({}_z f), \quad z \in Z.$$

A left M-introverted subalgebra $\mathscr{A}$ of $C(Z)$ is contained in $A(Z)$ if and only if $\Delta(\mathscr{A})$ is a compact Hausdorff topological semigroup (a topological semigroup is a semitopological semigroup with the additional property that the multiplication map is jointly continuous) and the embedding map of Z into $\Delta(\mathscr{A})$ is a continuous homomorphism mapping Z densely into $\Delta(\mathscr{A})$ [**1**, Corollary 9.5]. Recall that $C(\Delta(\mathscr{A}))$ is isometrically isomorphic to $\mathscr{A}$ via the adjoint of the embedding map. In fact, $\Delta(\mathscr{A})$ is the unique compact Hausdorff topological semigroup with these properties.

Let Z be a semitopological semigroup. An f in $C(Z)$ is called *weakly almost periodic* if $\{{}_z f : z \in Z\}$ has compact closure in the weak topology of $C(Z)$. Let $W(Z)$ denote all weakly almost periodic functions on Z. Since $W(Z)$ is a translation-invariant unital C^*-subalgebra of $C(Z)$ (see [**5**]), we have the following corollary to Theorems 2.7 and 2.8.

COROLLARY 2.10. *Let S and T be semitopological semigroups and let $\mathscr{A}$ be a unital C^*-subalgebra of $W(S)$. Then*

$$\begin{aligned}\mathscr{A} \otimes W(T) = \{h \in B(S \times T)&: {}^s h \in W(T), s \in S; \\ &h^t \in \mathscr{A}, t \in T; \text{ and } \{h^t : t \in T\} \text{ is totally bounded in } \mathscr{A}\} \\ = \{h \in B(S \times T)&: {}^s h \in W(T), s \in S; h^t \in \mathscr{A}, t \in T; \text{ and} \\ &\{{}^s h : s \in S\} \text{ is totally bounded in } W(T)\}.\end{aligned}$$

In general, $W(S) \otimes W(T)$ is not $W(S \times T)$. See [**2**] p. 171, [**12**] p. 590, and [**13**] p. 663, in this regard. However, one always has that $W(S) \otimes W(T) \subset W(S \times T)$; the proof is virtually the same as in showing that $A(S) \otimes A(T) \subset A(S \times T)$. The following is an indication of just how seldom these two algebras are equal.

THEOREM 2.11. *Let S be an abelian topological semigroup with* 1. *Then* $W(S) \otimes W(S) = W(S \times S)$ *if and only if* $W(S \times S) = A(S \times S)$.

Proof. (i) Sufficiency. Let f be in $W(S)$. We identify S with $S \times \{1\}$ and we let π_S denote the projection of $S \times S$ onto S. Then $f \circ \pi_S$ is in $W(S \times S) = A(S \times S)$. Since $f \circ \pi_S|_S = f$, f is in $A(S)$. Hence $W(S) = A(S)$. Thus,

$$W(S \times S) = A(S \times S) = A(S) \otimes A(S) = W(S) \otimes W(S).$$

(ii) Necessity. Let f be in $W(S)$. Define $\phi : S \times S \to S$ by

$$\phi(s, t) = st, \quad (s, t) \in S \times S.$$

Then ϕ is a continuous topological semigroup homomorphism since S is abelian. Therefore, $h = f \circ \phi$ is in $W(S \times S)$. By Corollary 2.10, $\{{}^s h : s \in S\}$ is totally bounded. Since ${}^s h = {}_s f$ for all s in S, f is in $A(S)$.

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Therefore, $W(S) = A(S)$, and so

$$W(S \times S) = W(S) \otimes W(S) = A(S) \otimes A(S) = A(S \times S).$$

Remark. It is not true in general that if $A(S) = W(S)$, then $A(S \times S) = W(S \times S)$ where S is an abelian topological semigroup with 1. Hence the condition $W(S \times S) = A(S \times S)$ cannot be replaced by $W(S) = A(S)$.

As an example, let S be an infinite null semigroup with identity adjoined; that is, $st = 0$ for $s \neq 1$, $t \neq 1$ and $s \cdot 1 = 1 \cdot s = s$ for all s in S. Equip S with the discrete topology. Given f in $B(S)$ and $s \neq 1$ in S,

$$_sf = f(0)\zeta_{S\sim\{1\}} + f(s)\zeta_{\{1\}}$$

where ζ_X denotes the characteristic function of the set X. Thus, $\{_sf : s \in S\}$ is totally bounded since $\{f(s) : s \in S\}$ is bounded. Hence $B(S) = A(S) = W(S)$. By applying Grothendieck's criterion [**8**] for weak almost periodicity, one has that

$$W(S \times S) = \{h \in B(S \times S): \{^sh: s \in S\} \text{ is weakly conditionally compact}\}.$$

Let $D = \{(s, s): s \in S\}$ and let $h = \zeta_D$. Then

$$\{^sh: s \in S\} = \{\zeta_{\{s\}}: s \in S\}$$

is not totally bounded, since S is infinite, but is weakly conditionally compact, since its weak closure is $\{\zeta_{\{s\}}: s \in S\} \cup \{0\}$. Therefore, h is in $W(S \times S)$ and h is not in $A(S \times S)$.

Let Z be a semitopological semigroup. A left M-introverted subalgebra $\mathscr{A}$ of $C(Z)$ is contained in $W(Z)$ if and only if $\Delta(\mathscr{A})$ is a compact Hausdorff semitopological semigroup and the embedding map of Z into $\Delta(\mathscr{A})$ is a continuous homomorphism mapping Z densely into $\Delta(\mathscr{A})$ [**1**, Corollary 8.5]. Also, $\Delta(\mathscr{A})$ is unique with respect to these properties and the fact that $C(\Delta(\mathscr{A}))$ is isometrically isomorphic to $\mathscr{A}$ via the adjoint of the embedding map.

LEMMA 2.12. *Let $\mathscr{A}$ be a translation-invariant unital C^*-subalgebra of $C(Z)$. If $\mathscr{A} \subset W(Z)$, then $\mathscr{A}$ is left M-introverted.*

Proof. See Lemma 8.8 of [**1**].

LEMMA 2.13. *Let Z be a semitopological semigroup and let $\bar{Z}$ be a compact semitopological [resp. topological] semigroup. Let I be a continuous homomorphism from Z onto a dense subset of $\bar{Z}$. Let $\mathscr{A} = I^*(C(\bar{Z}))$, where $I^*(F) = F \circ I$ for each F in $C(\bar{Z})$. Then $\mathscr{A}$ is a translation-invariant unital C^*-subalgebra of $W(Z)$ [resp. $A(Z)$].*

Proof. We prove the semitopological case, the topological case being similar. For F in $C(\bar{Z})$ and z in Z, $I^*(_{I(z)}F) = {}_z(I^*(F))$ and $I^*(F_{I(z)}) = (I^*(F))_z$. Since I^* is an isometric algebra isomorphism from $C(\bar{Z})$ onto

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$\mathscr{A}$, it follows that $\mathscr{A}$ is a translation-invariant unital C^*-subalgebra of $C(Z)$. For F in $C(\bar{Z})$, $\{_\tau F : \tau \in \bar{Z}\}$ is compact in the topology of pointwise convergence on $C(\bar{Z})$ since $\bar{Z}$ is a compact semitopological semigroup. From Grothendieck's theorem ([**8**], which states that weak compactness and compactness in the topology of pointwise convergence are equivalent for norm bounded subsets of $C(X)$, where X is compact Hausdorff), $\{_{I(z)}F : z \in Z\}$ is weakly conditionally compact in $C(\bar{Z})$. Since I^* is continuous from $(C(\bar{Z}), wk)$ onto $(\mathscr{A}, wk)$, $\{_z(I^*(F)) : z \in Z\}$ is weakly conditionally compact in $\mathscr{A}$. Hence, $A \subset W(Z)$.

3. Semidirect product compactifications. Our setting for the first part of this section is as follows. Let T be a semitopological semigroup, X a Hausdorff topological space, σ a semigroup homomorphism from T into the semigroup of (continuous) operators on X; that is, letting $\sigma_t = \sigma(t)$,

$$\sigma_{tt'}(x) = \sigma_t(\sigma_{t'}(x)), \quad x \in X, t, t' \in T.$$

It will be further required of σ that it be separately continuous; that is, the map $x \rightarrow \sigma_t(x)$ from X into X is continuous for each t in T and the map $t \rightarrow \sigma_t(x)$ from T into X is continuous for each x in X.

Also throughout the first part of this section, $\mathscr{A}$ will denote a unital C^*-subalgebra of $C(X)$; $\mathscr{B}$ will denote a translation-invariant unital C^*-subalgebra of $W(T)$; $\bar{X}$ will denote the $(\mathscr{A}, I_1)$-compactification of X; and $\bar{T}$ will denote the $(\mathscr{B}, I_2)$-compactification of T. By Lemma 2.12 and remarks preceding it, $\bar{T}$ is a compact semitopological semigroup.

Definition 3.1. Let $\bar{\sigma}$ be a semigroup homomorphism from $\bar{T}$ into the semigroup of continuous operators on $\bar{X}$ such that $\bar{\sigma}$ is separately continuous. Call $\bar{\sigma}$ an *extension* of σ if

$$\bar{\sigma}_{I_2(t)}(I_1(x)) = I_1(\sigma_t(x)), \quad x \in X, t \in T.$$

Note that if such a $\bar{\sigma}$ exists, then it is unique by the separate continuity of $\bar{\sigma}$.

For x in X, t in T, set $\hat{\sigma}_x(t) = \sigma_t(x)$.

THEOREM 3.2. *There exists an extension $\bar{\sigma}$ of σ if and only if the following are satisfied:*

(i) $\{f \circ \sigma_t : t \in T\}$ *is weakly conditionally compact* (w.c.c.) *in $\mathscr{A}$ for each f in $\mathscr{A}$,*

(ii) *$f \circ \hat{\sigma}_x$ is in $\mathscr{B}$ for each f in $\mathscr{A}$, x in X.*

Proof. Let $\bar{\sigma}$ be an extension of σ. Let f be in $\mathscr{A}$. For each μ in $\bar{T}$, define F_μ in $C(\bar{X})$ by

$$F_\mu(\tau) = \bar{\sigma}_\mu(\tau)(f), \quad \tau \in \bar{X}.$$

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For x in X and t in T,

$$I_1^*(F_{I_2(t)})(x) = F_{I_2(t)}(I_1(x)) = \bar{\sigma}_{I_2(t)}(I_1(x))(f)$$
$$= I_1(\sigma_t(x))(f) = f \circ \sigma_t(x),$$

where I_1^* is the adjoint map of I_1. Hence,

$$|F_{I_2(t)}(I_1(x))| = |f(\sigma_t(x))| \leqq \|f\|_u, \quad x \in X, t \in T.$$

By the separate continuity of $\bar{\sigma}$, it follows that

$$|F_{I_2(t)}(\tau)| \leqq \|f\|_u, \quad \tau \in \bar{X}, t \in T,$$

and, therefore,

$$|F_\mu(\tau)| \leqq \|f\|_u, \quad \tau \in \bar{X}, \mu \in \bar{T}.$$

Thus, $\{F_\mu: \mu \in \bar{T}\}$ is norm bounded and compact in the topology of pointwise convergence on $C(\bar{X})$, and therefore, $\{F_\mu: \mu \in \bar{T}\}$ is weakly compact in $C(\bar{X})$ by Grothendieck's theorem [**8**]. See the proof of Lemma 2.13 for a statement of this theorem. In particular, $\{F_{I_2(t)}: t \in T\}$ is w.c.c. in $C(\bar{X})$. Since $I_1^*(F_{I_2(t)}) = f \circ \sigma_t$ for each t in T and I_1^* is weakly continuous, $\{f \circ \sigma_t: t \in T\}$ is w.c.c. in $\mathscr{A}$.

For each τ in $\bar{X}$ define G_τ in $C(\bar{T})$ by

$$G_\tau(\mu) = \bar{\sigma}_\mu(\tau)(f) = F_\mu(\tau), \quad \mu \in \bar{T}.$$

That $f \circ \hat{\sigma}_x$ is in $\mathscr{B}$ for each x in X now follows by noting that

$$I_2^*(G_{I_1(x)}) = f \circ \hat{\sigma}_x,$$

where I_2^* is the adjoint map of I_2.

Now assume that $\mathscr{A}$ and $\mathscr{B}$ satisfy (i) and (ii). For f in $\mathscr{A}$ and μ in $\bar{T}$, set

$$f\sigma\mu(x) = \mu(f \circ \hat{\sigma}_x), \quad x \in X$$

and observe that $f\sigma I_2(t) = f \circ \sigma_t$ for each t in T. Let (t_α) be a net in T with $I_2(t_\alpha) \rightarrow \mu$. For x in X,

$$f\sigma\mu(x) = \mu(f \circ \hat{\sigma}_x) = \lim_\alpha I_2(t_\alpha)(f \circ \hat{\sigma}_x) = \lim_\alpha (f \circ \sigma_{t_\alpha})(x)$$

and, therefore, $f\sigma\mu$ is in the pointwise closure of $\{f \circ \sigma_t: t \in T\}$. Since the topology of pointwise convergence coincides with the weak topology on the w.c.c. set $\{f \circ \sigma_t: t \in T\}$, $f\sigma\mu$ is in the weak closure of $\{f \circ \sigma_t: t \in T\}$. Hence, $f\sigma\mu$ is in $\mathscr{A}$ since $\mathscr{A}$ is weakly closed ([**6**], p. 119). The above shows that $\mu \rightarrow f\sigma\mu$ is continuous from $\bar{T}$ into $\mathscr{A}$ with the topology of pointwise convergence. From the coincidence of the pointwise and weak topologies on the range of the map $\mu \rightarrow f\sigma\mu$, it follows that $\mu \rightarrow f\sigma\mu$ is continuous from $\bar{T}$ into $(\mathscr{A}, wk)$.

Define $\bar{\sigma}$ by

$$\bar{\sigma}_\mu(\tau)(f) = \tau(f\sigma\mu), \quad \tau \in \bar{X}, \mu \in \bar{T}, f \in \mathscr{A}.$$

It follows directly that $\bar{\sigma}_\mu(\tau)$ is in $\bar{X}$ for each μ in $\bar{T}$ and τ in $\bar{X}$ and that $\bar{\sigma}$ is separately continuous. For x in X and t in T,

$$\bar{\sigma}_{I_2(t)}(I_1(x))(f) = I_1(x)(f\sigma I_2(t)) = f(\sigma_t(x)) = I_1(\sigma_t(x))(f), f \in \mathscr{A}.$$

Hence,

$$\bar{\sigma}_{I_2(t)}(I_1(x)) = I_1(\sigma_t(x)), \quad x \in X, t \in T.$$

That $\bar{\sigma}_{\mu\mu'}(\tau) = \bar{\sigma}_\mu(\bar{\sigma}_{\mu'}(\tau))$ for μ, μ' in $\bar{T}$ and τ in $\bar{X}$ now follows from the separate continuity of $\bar{\sigma}$ and the denseness of $I_1(X)$ and $I_2(T)$ in $\bar{X}$ and $\bar{T}$ respectively.

Remark. Assuming that $\mathscr{A}$ and $\mathscr{B}$ satisfy (i) and (ii) above, one has that $\{f \circ \hat{\sigma}_x: x \in X\}$ is w.c.c. in $\mathscr{B}$. This follows by interchanging the roles of F and G in the first paragraph of the above proof and noting thereby that $\{G_{I_1(x)}: x \in X\}$ is w.c.c. in $C(\bar{T})$. Thus, for f in $\mathscr{A}$, τ in $\bar{X}$, we can define

$$f\tilde{\sigma}\tau(t) = \tau(f \circ \sigma_t), \quad t \in T.$$

Then $f\tilde{\sigma}I_1(x) = f \circ \hat{\sigma}_x$. It follows that $f\tilde{\sigma}\tau$ is in $\mathscr{B}$ and the map $\tau \to f\tilde{\sigma}\tau$ is continuous from $\bar{X}$ into $(\mathscr{B}, wk)$. Hence, $\bar{\sigma}$ also satisfies

$$\bar{\sigma}_\mu(\tau)(f) = \mu(f\tilde{\sigma}\tau), \quad \mu \in \bar{T}, \tau \in \bar{X}, f \in \mathscr{A}.$$

Definition 3.3. Call σ *jointly continuous* if the map $(x, t) \to \sigma_t(x)$ is continuous from $X \times T$ into X.

Corollary 3.4. *There exists a jointly continuous extension $\bar{\sigma}$ of σ if and only if the following are satisfied:*

(i′) *$\{f \circ \sigma_t: t \in T\}$ is totally bounded in $\mathscr{A}$ for each f in $\mathscr{A}$,*

(ii′) *$f \circ \hat{\sigma}_x$ is in $\mathscr{B}$ for each f in $\mathscr{A}$, x in X.*

Proof. Assume that $\bar{\sigma}$ is a jointly continuous extension of σ. By Theorem 3.2, (ii′) is satisfied. Fix f in $\mathscr{A}$. For μ in $\bar{T}$, define F_μ as in the proof of Theorem 3.2. Since $\bar{\sigma}$ is jointly continuous, it follows directly that $\{F_{I_2(t)}: t \in T\}$ is totally bounded in $C(\bar{X})$. Since $I_1{}^*(F_{I_2(t)}) = f \circ \sigma_t$, (i′) is satisfied.

Next assume that $\mathscr{A}$ and $\mathscr{B}$ satisfy (i′) and (ii′). Then Theorem 3.2 applies and there exists an extension $\bar{\sigma}$ of σ. For f in $\mathscr{A}$, since $\{f \circ \sigma_t: t \in T\}$ is totally bounded, it follows that $\mu \to f\sigma\mu$ (see the proof of Theorem 3.2 for the definition of this map and its weak continuity) is continuous from $\bar{T}$ into $(\mathscr{A}, \| \quad \|_u)$ by Lemma 2.2. Hence,

$$(\tau, \mu) \to \tau(f\sigma\mu) = \bar{\sigma}_\mu(\tau)(f) \text{ is in } C(\bar{X} \times \bar{T}) \text{ for each } f \text{ in } \mathscr{A}.$$

Thus, $\bar{\sigma}$ is jointly continuous.

Remark. If $\mathscr{A}$ and $\mathscr{B}$ satisfy the hypothesis of Corollary 3.4, then $\{f \circ \hat{\sigma}_x: x \in X\}$ is totally bounded in $\mathscr{B}$ for each f in $\mathscr{A}$. This follows by defining G_τ as in the proof of Theorem 3.2 and noting, as in the first paragraph of the last proof, that $\{G_{I_1(x)}: x \in X\}$ is totally bounded in $C(\bar{T})$.

COROLLARY 3.5. *Assume that T is a semitopological group and that $\mathscr{A}$ is a unital C^*-subalgebra of $C(X)$ such that given f in $\mathscr{A}$, $\{f \circ \sigma_t: t \in T\}$ is* w.c.c. *in $\mathscr{A}$, $f \circ \hat{\sigma}_x$ is in $A(T)$ for each x in X, and σ_1 is the identity map on X, where 1 is the identity of T. Then $\{f \circ \sigma_t: t \in T\}$ and $\{f \circ \hat{\sigma}_x: x \in X\}$ are totally bounded.*

Proof. Let $\mathscr{B} = A(T)$. By Theorem 3.2, there exists an extension $\bar{\sigma}$ of σ. Since $\bar{T}$ is compact, contains a dense subgroup, and has jointly continuous multiplication by the remarks preceding Corollary 2.10, $\bar{T}$ is a topological group. Since $\bar{\sigma}$ is separately continuous, $\bar{\sigma}$ is jointly continuous by Ellis' Theorem [**7**]. By Corollary 3.4 and the above remark, $\{f \circ \sigma_t: t \in T\}$ and $\{f \circ \hat{\sigma}_x: x \in X\}$ are totally bounded.

For a more recent proof of Ellis' Theorem, see [**20**].

Remark. In Corollary 3.5, one need only assume that $\Delta(A(T))$ is a topological group; for example, we could assume that T has a dense subgroup.

The setting for the remainder of this section is as follows. S and T will denote semitopological semigroups with 1; $\mathscr{E}(S)$ will denote the continuous endomorphisms of S; σ will denote a separately continuous semigroup homomorphism from T into $\mathscr{E}(S)$ such that the map

$$(s, t) \rightarrow s\sigma_t(s_0)$$

from $S \times T$ into S is continuous for each fixed s_0 in S, such that σ_1 is the identity endomorphism of S, and such that $\sigma_t(1) = 1$ for all t in T. For (s, t) and (s', t') in $S \times T$, set

$$(s, t)(s', t') = (s\sigma_t(s'), tt').$$

Then $S \times T$ with this operation and the product topology is a semitopological semigroup with identity $(1, 1)$ which we designate by $S \circledS T$. We call $S \circledS T$ the *semidirect product* of S with T induced by σ.

Remark. Notice that $f \circ \sigma_t$ is in $A(S)$ [resp. $W(S)$] for all t in T whenever f is in $A(S)$ [resp. $W(S)$]. This follows from the identity

$${}_s(f \circ \sigma_t) = {}_{\sigma_t(s)}f \circ \sigma_t,$$

which shows that the left orbit of $f \circ \sigma_t$ lies in the image of the left orbit of f under the norm [hence weakly] continuous map $F \rightarrow F \circ \sigma_t$ of $C(S)$ into $C(S)$.

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Definition 3.6. Let $\mathscr{A}$ and $\mathscr{B}$ be translation-invariant unital C^*-subalgebras of $W(S)$ and $W(T)$ respectively. Let $\bar{S}$ and $\bar{T}$ be the $(\mathscr{A}, I_1)$- and $(\mathscr{B}, I_2)$-compactifications of S and T respectively. Then $\bar{S}$ and $\bar{T}$ are both compact semitopological semigroups with identity by Lemma 2.12 and remarks preceding it. Let $\bar{\sigma} : \bar{T} \to \mathscr{E}(\bar{S})$ be such that $\bar{S} \textcircled{\bar{\sigma}} \bar{T}$ is a compact semitopological semidirect product semigroup with identity. Call $\bar{S} \textcircled{\bar{\sigma}} \bar{T}$ a *semidirect product compactification* (s.p.c.) of $S \textcircled{\sigma} T$ induced by $\mathscr{A}$ and $\mathscr{B}$ if $\bar{\sigma}$ is an extension of σ.

Landstad [**15**], Junghenn [**10, 11**], and Junghenn and Lerner [**14**] have also investigated s.p.c. of $S \textcircled{\sigma} T$ induced by subalgebras of $A(S \textcircled{\sigma} T)$ and have considered when $A(S \textcircled{\sigma} T)$ splits into a tensor product.

For the last part of the proof of the next theorem, we need to know the semigroup operation on $\bar{S}$. It is *left Arens multiplication*; that is,

$$\tau\tau'(f) = \tau(\tau' \circ f), \quad \tau, \tau' \in \bar{S}, f \in \mathscr{A}$$

where

$$\tau' \circ f(s) = \tau'({}_sf), \quad s \in S$$

as was defined in the comments preceding Corollary 2.10. Recall that $\mathscr{A}$ is left M-introverted by Lemma 2.12 and, therefore, $\tau' \circ f$ is in $\mathscr{A}$.

THEOREM 3.7. *Let $\mathscr{A}$, $\mathscr{B}$, $\bar{S}$, and $\bar{T}$ be as in Definition* 3.6. *The following are equivalent:*

1) *There exists a* s.p.c. $\bar{S} \textcircled{\bar{\sigma}} \bar{T}$ *of* $S \textcircled{\sigma} T$ *induced by* $\mathscr{A}$ *and* $\mathscr{B}$,

2) $\mathscr{A} \otimes \mathscr{B}$ *is a translation-invariant unital C^*-subalgebra of* $W(S \textcircled{\sigma} T)$,

3) $\mathscr{A}$ *and* $\mathscr{B}$ *satisfy the following for each f in* $\mathscr{A}$:

a) $\{{}_sf \circ \sigma_t : s \in S, t \in T\}$ *is* w.c.c. *in* $\mathscr{A}$,

b) $\{f_{\sigma_t(s_0)} : t \in T\}$ *is totally bounded in* $\mathscr{A}$ *for each* s_0 *in* S,

c) $f \circ \hat{\sigma}_s$ *is in* $\mathscr{B}$ *for each s in* S.

Proof. To show 1) implies 2), assume 1) and notice that since $\bar{\sigma}$ is an extension of σ, the map $\bar{P}$ defined by

$$\phi(s, t) = (I_1(s), I_2(t)) \quad \text{for } s \text{ in } S \text{ and } t \text{ in } T$$

is a continuous semigroup homomorphism from $S \textcircled{\sigma} T$ onto a dense subset of $\bar{S} \textcircled{\bar{\sigma}} \bar{T}$. Hence, letting $\mathscr{C}$ be the image of $C(\bar{S} \textcircled{\bar{\sigma}} \bar{T})$ under the adjoint map of ϕ, it follows by Lemma 2.13 that $\mathscr{C}$ is a translation-invariant unital C^*-subalgebra of $W(S \textcircled{\sigma} T)$. It remains to show that $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$. Since

$$C(\bar{S} \textcircled{\bar{\sigma}} \bar{T}) = C(\bar{S}) \otimes C(\bar{T})$$

by the Stone-Weierstrass theorem and

$$\phi^*(F \otimes G) = I_1^*(F) \otimes I_2^*(G)$$

for all F in $C(\bar{S})$ and G in $C(\bar{T})$, it follows that $\mathscr{A} \otimes \mathscr{B} \subset \mathscr{C}$. Let h be in

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$\mathscr{C}$ and let ^ denote the Gelfand transform on $\mathscr{C}$. Then $\hat{h}$ is in $C(\bar{S} \circledS \bar{T}) = C(\bar{S}) \otimes C(\bar{T})$, and so $\{(\hat{h})^{I_2(t)}: t \in T\}$ is totally bounded in $C(\bar{S})$. Since

$$I_1^*((\hat{h})^{I_2(t)}) = h^t \quad \text{for all } t \text{ in } T,$$

$\{h^t: t \in T\}$ is totally bounded in $\mathscr{A}$. Similarly, ${}^s h$ is in $\mathscr{B}$ for all s in S. Therefore, $\mathscr{C} \subset \mathscr{A} \otimes \mathscr{B}$ by Theorem 2.7. Hence, $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$.

To show 2) implies 3), we again use Theorem 2.7. Let f be in $\mathscr{A}$. For s in S and t in T,

$$_s f \circ \sigma_t = [_{(s,t)}(f \otimes 1)]^1$$

and so $_s f \circ \sigma_t$ is in $\mathscr{A}$. Since $\{_{(s,t)}(f \otimes 1): s \in S, t \in T\}$ is w.c.c., so is $\{_s f \circ \sigma_t: s \in S, t \in T\}$. Hence, a) holds. For s_0 in S and t in T,

$$f_{\sigma_t(s_0)} = [(f \otimes 1)_{(s_0,1)}]^t.$$

Hence, $\{f_{\sigma_t(s_0)}; t \in T\}$ is totally bounded. For s in S,

$$f \circ \hat{\sigma}_s = {}^1[(f \otimes 1)_{(s,1)}]$$

and so $f \circ \hat{\sigma}_s$ is in $\mathscr{B}$.

We now show that 3) implies 1). First note that conditions a) and c) imply by Theorem 3.2 that there exists an extension $\bar{\sigma}$ of σ.

We first show that for fixed s_0 in S, the map

$$(\tau, \mu) \to \tau\bar{\sigma}_\mu(I_1(s_0))$$

is continuous from $\bar{S} \times \bar{T}$ into $\bar{S}$, where $\tau\bar{\sigma}_\mu(I_1(s_0))$ is left Arens multiplication of τ and $\bar{\sigma}_\mu(I_1(s_0))$. Fix f in $\mathscr{A}$ and define γ from $\bar{T}$ into $\mathscr{A}$ by

$$\gamma(\mu) = \bar{\sigma}_\mu(I_1(s_0)) \circ f, \quad \mu \in \bar{T}.$$

Note that $\gamma(\mu)$ is in $\mathscr{A}$ since $\mathscr{A}$ is left M-introverted by Lemma 2.12. For s in S,

$$\gamma(\mu)(s) = \bar{\sigma}_\mu(I_1(s_0))(_s f).$$

Since $\bar{\sigma}$ is separately continuous, it follows that γ is continuous with the topology of pointwise convergence on $\mathscr{A}$. For t in T,

$$\gamma(I_2(t)) = \bar{\sigma}_{I_2(t)}(I_1(s_0)) \circ f = f_{\sigma_t(s_0)}$$

and so from b), $\{\gamma(I_2(t)): t \in T\}$ is totally bounded. By Lemma 2.2, γ is continuous from $\bar{T}$ into $(\mathscr{A}, \| \quad \|_u)$. Since

$$\tau\bar{\sigma}_\mu(I_1(s_0))(f) = \tau(\bar{\sigma}_\mu(I_1(s_0)) \circ f), \quad \tau \in \bar{S}, \mu \in \bar{T}$$

and γ is norm continuous, it follows that

$$(\tau, \mu) \to \tau\bar{\sigma}_\mu(I_1(s_0))(f)$$

is continuous for each f in $\mathscr{A}$, and hence

$$(\tau, \mu) \to \tau\bar{\sigma}_\mu(I_1(s_0))$$

is continuous from $\bar{S} \times \bar{T}$ into $\bar{S}$.

We now show that for fixed τ_0 in $\bar{S}$, the map $(\tau, \mu) \to \tau\bar{\sigma}_\mu(\tau_0)$ is continuous from $\bar{S} \times \bar{T}$ into $\bar{S}$. Fix f in $\mathscr{A}$ and define Γ from $\bar{S} \times \bar{T}$ into $\mathscr{A}$ by

$$\Gamma(\tau, \mu)(s') = \tau\bar{\sigma}_\mu(I_1(s'))(f), \quad \tau \in \bar{S}, \mu \in \bar{T}, s' \in S.$$

That $\Gamma(\tau, \mu)$ is in $\mathscr{A}$ follows by defining F in $C(\bar{S})$ by

$$F(\tau') = \hat{f}(\tau\bar{\sigma}_\mu(\tau')), \quad \tau' \in \bar{S}$$

where ^ is the Gelfand transform on $\mathscr{A}$, and noting that $I_1^*(F) = \Gamma(\tau, \mu)$. Since $(\tau, \mu) \to \tau\bar{\sigma}_\mu(I_1(s_0))$ is continuous for fixed s_0 in S, it follows that Γ is continuous with the topology of pointwise convergence on $\mathscr{A}$. For s in S and t in T,

$$\Gamma(I_1(s), I_2(t)) = {}_sf \circ \sigma_t.$$

By a) and Lemma 2.2, Γ is continuous from $\bar{S} \times \bar{T}$ into $(\mathscr{A}, wk)$. From the continuity of $\bar{\sigma}_\mu$ and the separate continuity of multiplication in $\bar{S}$, it follows that

$$\tau_0(\Gamma(\tau, \mu)) = \tau\bar{\sigma}_\mu(\tau_0)(f), \quad \tau, \tau_0 \in \bar{S}, \mu \in \bar{T}.$$

Consequently, for fixed τ_0 in $\bar{S}$, since Γ is weakly continuous, the map $(\tau, \mu) \to \tau\bar{\sigma}_\mu(\tau_0)(f)$ is continuous for each f in $\mathscr{A}$. Thus $(\tau, \mu) \to \tau\bar{\sigma}_\mu(\tau_0)$ is continuous from $\bar{S} \times \bar{T}$ into $\bar{S}$ for each fixed τ_0 in $\bar{S}$.

Noting that $\bar{\sigma}_{I_2(1)}$ is the identity endomorphism of $\bar{S}$ and that

$$\bar{\sigma}_\mu(I_1(1)) = I_1(1) \quad \text{for all } \mu \text{ in } \bar{T},$$

we have that $\bar{S}\circledS\bar{T}$ is a s.p.c. of $S\circledS T$.

Remark. If $\mathscr{A}$ is a translation-invariant subalgebra of $A(S)$ and $\{f \circ \sigma_t: t \in T\}$ is totally bounded for each f in $\mathscr{A}$, then

$$\{{}_sf_{s'} \circ \sigma_t: s, s' \in S, t \in T\}$$

is totally bounded for each f in $\mathscr{A}$. To see this let f be in $\mathscr{A}$ and fix $\epsilon > 0$. Since f is in $A(S)$, $\{{}_sf_{s'}: s, s' \in S\}$ is totally bounded. Thus there exists $s_1, \ldots, s_n, s_1', \ldots, s_n'$ in S such that $\{{}_{s_k}f_{s_k'}: k = 1, \ldots, n\}$ is an ϵ-net for $\{{}_sf_{s'}: s, s' \in S\}$. For each k, $\{{}_{s_k}f_{s_k'} \circ \sigma_t: t \in T\}$ is totally bounded and so there exists $t_{k,1}, \ldots, t_{k,p_k}$ in T such that

$$\{{}_{s_k}f_{s_k'} \circ \sigma_{t_{k,j}}: j = 1, \ldots, p_k\}$$

is an ϵ-net for $\{{}_{s_k}f_{s_k'} \circ \sigma_t: t \in T\}$. It follows that

$$\{{}_{s_k}f_{s_k'} \circ \sigma_{t_{k,j}}: k = 1, \ldots, n; j = 1, \ldots, p_k\}$$

is a 2ϵ-net for $\{{}_sf_{s'} \circ \sigma_t: s, s' \in S, t \in T\}$.

COROLLARY 3.8. *Let $\mathscr{A}$, $\mathscr{B}$, $\bar{S}$, and $\bar{T}$ be as in Definition* 3.6. *There exists a* s.p.c. $\bar{S}\textcircled{\bar{\sigma}}\bar{T}$ *of* $S\textcircled{\sigma}T$ *induced by* $\mathscr{A}$ *and* $\mathscr{B}$ *which is a topological semigroup if and only if* $\mathscr{A} \subset A(S)$, $\mathscr{B} \subset A(T)$, $\{f \circ \sigma_t : t \in T\}$ *is totally bounded in* $\mathscr{A}$ *and* $f \circ \hat{\sigma}_s$ *is in* $\mathscr{B}$ *for each* f *in* $\mathscr{A}$ *and* s *in* S.

Proof. If $\bar{S}\textcircled{\bar{\sigma}}\bar{T}$ is a s.p.c. of $S\textcircled{\sigma}T$ which is a topological semigroup, then $\bar{\sigma}$ is jointly continuous. By Corollary 3.4, $\{f \circ \sigma_t : t \in T\}$ is totally bounded in $\mathscr{A}$ and $f \circ \hat{\sigma}_s$ is in $\mathscr{B}$ for each f in $\mathscr{A}$ and s in S. Since $\bar{S}$ and $\bar{T}$ are topological semigroups, $\mathscr{A} \subset A(S)$ and $\mathscr{B} \subset A(T)$.

We now prove the converse. Since $\mathscr{A} \subset A(S)$, condition b) of Theorem 3.7 is satisfied and by the preceding remark, condition a) of Theorem 3.7 is satisfied. By Theorem 3.7 there exists a s.p.c. $\bar{S}\textcircled{\bar{\sigma}}\bar{T}$ of $S\textcircled{\sigma}T$ induced by $\mathscr{A}$ and $\mathscr{B}$. By Corollary 3.4, $\bar{\sigma}$ is jointly continuous. Since $\mathscr{A} \subset A(S)$ and $\mathscr{B} \subset A(T)$, $\bar{S}$ and $\bar{T}$ are topological semigroups. It follows that $\bar{S}\textcircled{\bar{\sigma}}\bar{T}$ is a topological semigroup.

Definition 3.9. Given f in $A(S)$, call f *σ-Bochner almost periodic* if for each s_1 and s_2 in S, $\{{}_{s_1}f_{s_2} \circ \sigma_t : t \in T\}$ is totally bounded. Let $A^\sigma(S)$ denote the set of all σ-Bochner almost periodic functions on S. Given f in $W(S)$, call f *σ-weakly almost periodic* if $\{{}_sf_{s_2} \circ \sigma_t : s \in S, t \in T\}$ is w.c.c. and $\{f_{\sigma_t(s_1)s_2} : t \in T\}$ is totally bounded for each s_1 and s_2 in S. Let $W^\sigma(S)$ denote the set of all σ-weakly almost periodic functions on S.

PROPOSITION 3.10. *$A^\sigma(S)$ and $W^\sigma(S)$ are translation-invariant unital C^*-subalgebras of $A(S)$ and $W(S)$ respectively. Moreover, each of these algebras is closed under composition with the family $\{\sigma_t : t \in T\}$.*

Proof. It follows directly that $A^\sigma(S)$ is a unital C^*-subalgebra of $A(S)$. Let f be in $A^\sigma(S)$ and s in S. Then for s_1, s_2 in S and t in T,

$${}_{s_1}({}_sf)_{s_2} \circ \sigma_t = {}_{ss_1}f_{s_2} \circ \sigma_t$$

and

$${}_{s_1}(f_s)_{s_2} \circ \sigma_t = {}_{s_1}f_{s_2s} \circ \sigma_t$$

from which it follows that ${}_sf$ and f_s are in $A^\sigma(S)$. Hence, $A^\sigma(S)$ is translation-invariant. From the remark preceding Definition 3.6, $A(S)$ is closed under composition with the family $\{\sigma_t : t \in T\}$. That this is true for $A^\sigma(S)$ follows from the following: for f in $A^\sigma(S)$, s_1, s_2 in S, t_0, t in T,

$${}_{s_1}(f \circ \sigma_{t_0})_{s_2} \circ \sigma_t = {}_{\sigma_{t_0}(s_1)}f_{\sigma_{t_0}(s_2)} \circ \sigma_{t_0 t}.$$

It follows directly that $W^\sigma(S)$ is a linear subspace of $W(S)$ containing the constant functions and is self-adjoint. To show that $W^\sigma(S)$ is closed, let f be in the uniform closure of $W^\sigma(S)$. For fixed s_2 in S, we must show that

$$\{{}_sf_{s_2} \circ \sigma_t : s \in S, t \in T\}$$

is w.c.c. By Grothendieck's criterion [8], it suffices to show that if (s_n') and (s_m) are sequences in S and (t_m) is a sequence in T such that

$$\lim_m \lim_n {}_{s_m}f_{s_2} \circ \sigma_{t_m}(s_n') = L_1$$

and

$$\lim_n \lim_m {}_{s_m}f_{s_2} \circ \sigma_{t_m}(s_n') = L_2,$$

then $L_1 = L_2$. Assume $L_1 \neq L_2$ and set $\epsilon = |L_1 - L_2|/2$. Choose g in $W^\sigma(S)$ such that $\|f - g\|_u < \epsilon$. Set

$$a_{m,n} = {}_{s_m}f_{s_2} \circ \sigma_{t_m}(s_n')$$

and

$$b_{m,n} = {}_{s_m}g_{s_2} \circ \sigma_{t_m}(s_n')$$

for all m and n. Then $\{b_{m,n}\}$ is bounded in the complex plane. By using a diagonalization argument, there exist $\phi(1) < \phi(2) < \ldots$ and $\psi(1) < \psi(2) < \ldots$ such that

$$\lim_m \lim_n b_{\phi(m),\psi(n)} = L_1'$$

and

$$\lim_n \lim_m b_{\phi(m),\psi(n)} = L_2'$$

for some complex numbers L_1' and L_2'. Since g is in $W^\sigma(S)$, by Grothendieck's criterion, $L_1' = L_2'$. Also,

$$\lim_m \lim_n a_{\phi(m),\psi(n)} = L_1$$

and

$$\lim_n \lim_m a_{\phi(m),\psi(n)} = L_2.$$

However,

$$|a_{m,n} - b_{m,n}| \leqq \|f - g\|_u$$

for all m and n. Hence,

$$|L_1 - L_1'| \leqq \|f - g\|_u < \epsilon$$

and

$$|L_2 - L_2'| \leqq \|f - g\|_u < \epsilon$$

and therefore

$$|L_1 - L_2| < 2\epsilon = |L_1 - L_2|$$

which is a contradiction. Thus, $L_1 = L_2$.

For s_1 and s_2 in S, we must show that $\{f_{\sigma_t(s_1)s_2}: t \in T\}$ is totally bounded. Let $\epsilon > 0$ and choose $h \in W^\sigma(S)$ such that $\|f - h\|_u < \epsilon$. There exist $t_1, \ldots, t_n$ such that

$$\{h_{\sigma_{t_k}(s_1)s_2}: k = 1, \ldots, n\}$$

is an ϵ-net for $\{h_{\sigma_t(s_1)s_2}: t \in T\}$. It follows directly that

$$\{f_{\sigma_{t_k}(s_1)s_2}: k = 1, \ldots, n\}$$

is a 3ϵ-net for $\{f_{\sigma_t(s_1)s_2}: t \in T\}$. Hence, f is in $W^\sigma(S)$ and $W^\sigma(S)$ is closed.

To see that $W^\sigma(S)$ is an algebra, let f, g be in $W^\sigma(S)$, s_1, s_2 in S, and t in T. Then

$$(fg)_{\sigma_t(s_1)s_2} = f_{\sigma_t(s_1)s_2} \cdot g_{\sigma_t(s_1)s_2}$$

from which it follows that $\{(fg)_{\sigma_t(s_1)s_2}: t \in T\}$ is totally bounded. Also, for s in S,

$${}_s(fg)_{s_2} \circ \sigma_t = [{}_sf_{s_2} \circ \sigma_t] \cdot [{}_sg_{s_2} \circ \sigma_t].$$

Either by applying Grothendieck's criterion or by applying a corollary to Grothendieck's theorem [**8**] which states that for Z a set, $\mathcal{A}$ a unital C^*-subalgebra of $B(Z)$, K a norm bounded subset of $\mathcal{A}$, then K is w.c.c. if and only if K is conditionally compact in the topology induced by the multiplicative linear functionals on $\mathcal{A}$, one obtains that

$$\{{}_s(fg)_{s_2} \circ \sigma_t: s \in S, t \in T\}$$

is w.c.c. Thus, fg is in $W^\sigma(S)$.

That $W^\sigma(S)$ is translation-invariant follows from the following: for f in $W^\sigma(S)$, s, s_1, s_2 in S and t in T one has that

$${}_s({}_{s_1}f)_{s_2} \circ \sigma_t = {}_{s_1s}f_{s_2} \circ \sigma_t,$$
$${}_s(f_{s_1})_{s_2} \circ \sigma_t = {}_sf_{s_2s_1} \circ \sigma_t,$$
$$({}_sf)_{\sigma_t(s_1)s_2} = {}_s[f_{\sigma_t(s_1)s_2}],$$

and

$$(f_s)_{\sigma_t(s_1)s_2} = f_{\sigma_t(s_1)s_2s}.$$

From the remark preceding Definition 3.6, $W(S)$ is closed under composition with the family $\{\sigma_t: t \in T\}$. That this is true for $W^\sigma(S)$ follows from the following: for f in $W^\sigma(S)$, s, s_1, s_2 in S, t_0, t in T,

$${}_s(f \circ \sigma_{t_0})_{s_2} \circ \sigma_t = {}_{\sigma_{t_0}(s)}f_{\sigma_{t_0}(s_2)} \circ \sigma_{t_0t}$$

and

$$(f \circ \sigma_{t_0})_{\sigma_t(s_1)s_2} = f_{\sigma_{t_0t}(s_1)\sigma_{t_0}(s_2)} \circ \sigma_{t_0}.$$

Let aT denote the almost periodic compactification of T (induced by $A(T)$) and wT denote the weakly almost periodic compactification of T

(induced by $W(T)$). Let aS^σ denote the compactification of S induced by $A^\sigma(S)$ and wS^σ denote the compactification of S induced by $W^\sigma(S)$. Then aS^σ is a compact topological semigroup and wS^σ is a compact semitopological semigroup by Lemma 2.12 and by remarks preceding it and Corollary 2.10.

THEOREM 3.11. *$A^\sigma(S)$ and $A(T)$ induce a* s.p.c. $aS^\sigma \circledcirc{\bar\sigma} aT$ *of* $S \circledcirc{\sigma} T$ *which is a topological semigroup. Moreover, if* $\bar S \circledcirc{\bar\sigma} \bar T$ *is a* s.p.c. *of* $S \circledcirc{\sigma} T$ *induced by* $\mathscr{A}$ *and* $\mathscr{B}$ *such that* $\bar S \circledcirc{\bar\sigma} \bar T$ *is a topological semigroup, then* $\mathscr{A} \subset A^\sigma(S)$ *and* $\mathscr{B} \subset A(T)$.

Proof. Recall from Proposition 3.10 that $f \circ \sigma_t$ is in $A^\sigma(S)$ for each f in $A^\sigma(S)$ and t in T. For s in S, t in T, f in $A^\sigma(S)$,

$$_t(f \circ \hat\sigma_s) = f \circ \sigma_t \circ \hat\sigma_s.$$

Since $\{f \circ \sigma_t: t \in T\}$ is totally bounded and the map $F \to F \circ \hat\sigma_s$ from $C(S)$ into $C(T)$ is norm continuous, $f \circ \hat\sigma_s$ is in $A(T)$. By Corollary 3.8, there is a s.p.c. $aS^\sigma \circledcirc{\bar\sigma} aT$ of $S \circledcirc{\sigma} T$ which is a topological semigroup.

Next let $\bar S \circledcirc{\bar\sigma} \bar T$ be any s.p.c. of $S \circledcirc{\sigma} T$ induced by $\mathscr{A}$ and $\mathscr{B}$ which is a topological semigroup. By Corollary 3.8, $\mathscr{A} \subset A(S)$, $\mathscr{B} \subset A(T)$, and $\{f \circ \sigma_t: t \in T\}$ is totally bounded in $\mathscr{A}$ for each f in $\mathscr{A}$. Since $\mathscr{A}$ is translation-invariant, $\mathscr{A} \subset A^\sigma(S)$.

Remark. Theorem 3.11 states that $aS^\sigma \circledcirc{\bar\sigma} aT$ is, in terms of the algebras, the largest s.p.c. of $S \circledcirc{\sigma} T$ which is a topological semigroup. The next result states that $wS^\sigma \circledcirc{\bar\sigma} wT$ is the largest s.p.c. of $S \circledcirc{\sigma} T$.

THEOREM 3.12. *$W^\sigma(S)$ and $W(T)$ induce a* s.p.c. $wS^\sigma \circledcirc{\bar\sigma} wT$ *of* $S \circledcirc{\sigma} T$. *Moreover, if* $\bar S \circledcirc{\bar\sigma} \bar T$ *is any* s.p.c. *of* $S \circledcirc{\sigma} T$ *induced by* $\mathscr{A}$ *and* $\mathscr{B}$, *then* $\mathscr{A} \subset W^\sigma(S)$ *and* $\mathscr{B} \subset W(T)$.

Proof. Recall from Proposition 3.10 that $f \circ \sigma_t$ is in $W^\sigma(S)$ for each f in $W^\sigma(S)$ and t in T. Hence, conditions a) and b) of Theorem 3.7 are satisfied. To show condition c), let f be in $W^\sigma(S)$, s in S, t in T. Then,

$$_t(f \circ \hat\sigma_s) = f \circ \sigma_t \circ \hat\sigma_s.$$

Since $\{f \circ \sigma_t: t \in T\}$ is w.c.c. and the map $F \to F \circ \sigma_s$ from $C(S)$ into $C(T)$ is weakly continuous, $f \circ \hat\sigma_s$ is in $W(T)$. By Theorem 3.7, there exists a s.p.c. $wS^\sigma \circledcirc{\bar\sigma} wT$ of $S \circledcirc{\sigma} T$.

Next let $\bar S \circledcirc{\bar\sigma} \bar T$ be a s.p.c. of $S \circledcirc{\sigma} T$ induced by $\mathscr{A}$ and $\mathscr{B}$. Let f be in $\mathscr{A}$, s_1, s_2 in S. By Theorem 3.7 a), $\{{}_sf \circ \sigma_t: s \in S, t \in T\}$ is w.c.c. in $\mathscr{A}$ and hence, $\{{}_sf_{s_2} \circ \sigma_t: s \in S, t \in T\}$ is w.c.c. in $\mathscr{A}$ by the translation invariance of $\mathscr{A}$. By Theorem 3.7 b), $\{f_{\sigma_t(s_1)}: t \in T\}$ is totally bounded in $\mathscr{A}$. Since

$$f_{\sigma_t(s_1)s_2} = (f_{s_2})_{\sigma_t(s_1)}$$

and $\mathscr{A}$ is translation-invariant, $\{f_{\sigma_t(s_1)s_2}: t \in T\}$ is totally bounded in $\mathscr{A}$.

Thus f is in $W^\sigma(S)$, and so $\mathscr{A} \subset W^\sigma(S)$. That $\mathscr{B} \subset W(T)$ follows from Definition 3.6.

THEOREM 3.13. *$A^\sigma(S)$ and $W(T)$ induce a* s.p.c. $aS^\sigma \circledS wT$ *of* $S \circledS T$ *for which $\bar{\sigma}$ is jointly continuous. Moreover, if $\bar{S} \circledS \bar{T}$ is a* s.p.c. *of* $S \circledS T$ *induced by $\mathscr{A}$ and $\mathscr{B}$ for which $\bar{S}$ is a topological semigroup and $\bar{\sigma}$ is jointly continuous, then $\mathscr{A} \subset A^\sigma(S)$.*

Proof. Clearly $A^\sigma(S)$ and $W(T)$ satisfy conditions a) and b) of Theorem 3.7. Condition c) follows as in the previous two theorems. Hence, there is a s.p.c. $aS^\sigma \circledS wT$ of $S \circledS T$ induced by $A^\sigma(S)$ and $W(T)$. By Corollary 3.4, $\bar{\sigma}$ is jointly continuous.

Next suppose that $\bar{S} \circledS \bar{T}$ is a s.p.c. of $S \circledS T$ induced by $\mathscr{A}$ and $\mathscr{B}$ such that $\bar{\sigma}$ is jointly continuous and $\bar{S}$ is a topological semigroup. By remarks preceding Corollary 2.10, $\mathscr{A} \subset A(S)$. By Corollary 3.4, $\{f \circ \sigma_t: t \in T\}$ is totally bounded for each f in $\mathscr{A}$. Since $\mathscr{A}$ is translation-invariant, $\mathscr{A} \subset A^\sigma(S)$.

COROLLARY 3.14. *If S is a semitopological group, then*

$$A^\sigma(S) = W^\sigma(S) \cap A(S).$$

Proof. From the remark preceding Corollary 3.8, it is clear that

$$A^\sigma(S) \subset W^\sigma(S) \cap A(S).$$

Let $\mathscr{A} = W^\sigma(S) \cap A(S)$ and $\mathscr{B} = W(T)$. Then $\mathscr{A}$ is a translation-invariant unital C^*-subalgebra of $A(S)$ and $\mathscr{A}$ and $\mathscr{B}$ satisfy conditions a), b), and c) of Theorem 3.7 [c) follows as in the previous three theorems].

Hence $\mathscr{A}$ and $\mathscr{B}$ induce a s.p.c. $\bar{S} \circledS wT$ of $S \circledS T$ where $\bar{S} = \Delta(\mathscr{A})$. Since $\mathscr{A} \subset A(S)$, $\bar{S}$ is a topological group (as $\bar{T}$ is in the proof of Corollary 3.5). Consider the (right) action ψ of $\bar{S}$ on $\bar{S} \circledS wT$ given by

$$(\tau', \mu')\psi_\tau = (\tau', \mu')(\tau, I_2(1)) = (\tau' \bar{\sigma}_{\mu'}(\tau), \mu'),$$

where I_2 is the embedding map of T into wT. Since ψ is separately continuous, ψ is jointly continuous by Ellis' Theorem [7]. Hence, $\bar{\sigma}$ is jointly continuous. By Theorem 3.13, $\mathscr{A} \subset A^\sigma(S)$. Consequently, $A^\sigma(S) = W^\sigma(S) \cap A(S)$.

Remark. If $\{\sigma_t: t \in T\}$ is finite, then clearly $A^\sigma(S) = A(S)$ and $W^\sigma(S) = W(S)$. The following shows that $A^\sigma(S)$ can equal $A(S)$ when $\{\sigma_t: t \in T\}$ is infinite. Let S be an infinite commutative idempotent discrete semigroup with 1. Define $\sigma: S \to \mathscr{E}(S)$ by $\sigma_t(s) = ts$ if $s \neq 1$ and $\sigma_t(1) = 1$. Let f be in $A(S)$. For s_1, s_2 in S, t in S,

$${}_{s_1}f_{s_2} \circ \sigma_t(s) = {}_{s_1 s_2 t}f(s) \quad \text{if } s \neq 1$$

and

$${}_{s_1}f_{s_2} \circ \sigma_t(1) = {}_{s_1}f_{s_2}(1).$$

It follows that f is in $A^\sigma(S)$.

The following is a more interesting example.

Example 3.15. Let S be an infinite set and let 0 and 1 be two elements of S. Define an operation on S by

$$ss' = \begin{cases} s \text{ if } s' = 1 \text{ or } s' = s \\ s' \text{ if } s = 1 \\ 0 \quad \text{otherwise.} \end{cases}$$

Equip S with the discrete topology and observe that S is a commutative idempotent semigroup with identity, 1. Define σ: $S \rightarrow \mathscr{E}(S)$ by $\sigma_t(s) = ts$ if $s \neq 1$ and $\sigma_t(1) = 1$.

From the previous remark, one has that $A^\sigma(S) = A(S)$. However, it is interesting to note that

$$A(S) = \{f \in B(S): \operatorname*{limit}_{s\to\infty} f(s) = f(0)\},$$

where $\text{limit}_{s\to\infty} f(s) = L$ means that given $\epsilon > 0$, there exists a finite subset F of S such that $|f(s) - L| < \epsilon$ for all s not in F. Also, $aS = (S, \mathscr{U})$, where $\mathscr{U}$ is the topology in which neighborhoods of $I(0)$ [I being the embedding map] are complements of finite sets and every other point is open. Also, $W^\sigma(S) = W(S)$ due to the collapsing of the sets which need to be w.c.c. or totally bounded. Finally, $wS = \beta S =$ the Stone-Čech compactification of S [that is, $W(S) = B(S)$], since βS can be made into a compact semitopological semigroup such that the embedding map I_1: $S \rightarrow \beta S$ is a homomorphism by defining

$$\tau\tau' = \begin{cases} \tau \text{ if } \tau = \tau' \text{ in } I_1(S) \text{ or } \tau' = I_1(1) \\ \tau' \text{ if } \tau = I_1(1) \\ I_1(0) \quad \text{otherwise.} \end{cases}$$

4. Almost periodic functions on semidirect products of semigroups. In this section we obtain sufficient conditions for

$$A(S \circledS T) = A^\sigma(S) \otimes A(T) \quad \text{and} \quad W(S \circledS T) = W^\sigma(S) \otimes W(T).$$

To do this we first develop some results on tensor products.

Let X and Y be sets and let $\mathscr{C}$ be a unital C^*-subalgebra of $B(X \times Y)$ such that $\{h^y: y \in Y\}$ is w.c.c. for all h in $\mathscr{C}$. By Grothendieck's criterion [**8**], $\{{}^xh: x \in X\}$ is w.c.c. for each h in $\mathscr{C}$. Let $\mathscr{A}$ be the unital C^*-subalgebra of $B(X)$ generated by $\{h^y: h \in \mathscr{C}, y \in Y\}$ and let $\mathscr{B}$ be the unital C^*-subalgebra of $B(Y)$ generated by $\{{}^xh: h \in \mathscr{C}, x \in X\}$. Let $\bar{X}$ be the

$(\mathscr{A}, I_1)$-compactification of X; $\bar{Y}$ the $(\mathscr{B}, I_2)$-compactification of Y; and $\overline{X \times Y}$ the $(\mathscr{C}, I)$-compactification of $X \times Y$.

Definition 4.1. Given h in $\mathscr{C}$, τ in $\bar{X}$, μ in $\bar{Y}$, set ${}^{\tau}h(y') = \tau(h^{y'})$ and $h^{\mu}(x') = \mu({}^{x'}h)$ for all x' in X and y' in Y.

Note that ${}^{I_1(x)}h = {}^{x}h$ and $h^{I_2(y)} = h^{y}$ for all h in $\mathscr{C}$, x in X, y in Y. Fix h in $\mathscr{C}$ and τ in $\bar{X}$. Let (x_α) be a net in X such that $I_1(x_\alpha) \to \tau$. Then $({}^{x_\alpha}h)$ converges pointwise to ${}^{\tau}h$. Since $\{{}^{x}h: x \in X\}$ is w.c.c., $({}^{x_\alpha}h)$ converges weakly to ${}^{\tau}h$. Since $\mathscr{B}$ is weakly closed, ${}^{\tau}h$ is in $\mathscr{B}$. Define ψ from $\bar{X}$ into $\mathscr{B}$ by

$$\psi(\tau) = {}^{\tau}h, \quad \tau \in \bar{X}.$$

Then ψ is continuous in the topology of pointwise convergence on $\mathscr{B}$ and $\{\psi(I_1(x)): x \in X\} = \{{}^{x}h: x \in X\}$ is w.c.c. By Lemma 2.2, the map $\tau \to {}^{\tau}h$ is continuous from $\bar{X}$ into $(\mathscr{B}, wk)$. Similarly for each h in $\mathscr{C}$ and μ in $\bar{Y}$, h^{μ} is in $\mathscr{A}$ and the map $\mu \to h^{\mu}$ is continuous from $\bar{Y}$ into $(\mathscr{A}, wk)$.

For τ in $\bar{X}$, μ in $\bar{Y}$, define

$$\tau \otimes \mu(h) = \tau(h^{\mu}), \quad h \in \mathscr{C}.$$

The following properties follow directly:

1) $\tau \otimes \mu$ is in $\overline{X \times Y}$ for all τ in $\bar{X}$, μ in $\bar{Y}$;

2) the map $(\tau, \mu) \to \tau \otimes \mu$ is separately continuous from $\bar{X} \times \bar{Y}$ into $\overline{X \times Y}$;

3) $I(x, y) = I_1(x) \otimes I_2(y)$ for all x in X, y in Y;

4) $\tau \otimes \mu(h) = \mu({}^{\tau}h)$ for all τ in $\bar{X}$, μ in $\bar{Y}$, h in $\mathscr{C}$.

Let

$$\bar{X} \otimes \bar{Y} = \{\tau \otimes \mu: \tau \in \bar{X}, \mu \in \bar{Y}\}.$$

Note that $I(X \times Y) \subset \bar{X} \otimes \bar{Y} \subset \overline{X \times Y}$ and hence $\bar{X} \otimes \bar{Y}$ is dense in $\overline{X \times Y}$. Let π be the map from $\bar{X} \times \bar{Y}$ into $\overline{X \times Y}$ such that $\pi: (\tau, \mu) \to \tau \otimes \mu$.

THEOREM 4.2. *The following are equivalent:*

i) *π is jointly continuous,*

ii) *$\{h^{y}: y \in Y\}$ is totally bounded for all h in $\mathscr{C}$,*

iii) *$\mathscr{C} \subset \mathscr{A} \otimes \mathscr{B}$.*

Proof. To show i) implies ii), assume π is jointly continuous and let h be in $\mathscr{C}$. Let (y_α) be a net in Y and assume that $\{I_2(y_\alpha)\}$ converges to some μ in $\bar{Y}$. Since $\mu \to h^{\mu}$ is continuous from $\bar{Y}$ into $(\mathscr{A}, wk)$, $\{h^{y_\alpha}\} = \{h^{I_2(y_\alpha)}\}$ converges weakly to h^{μ}. If the convergence is not uniform, by passing to subnets, we may assume that there exists a net (x_α) in X such that $\{h^{y_\alpha}(x_\alpha) - h^{\mu}(x_\alpha)\}$ does not converge to 0 and $\{I_1(x_\alpha)\}$ converges to some τ in $\bar{X}$. Since π is jointly continuous,

$$\lim_\alpha h^{y_\alpha}(x_\alpha) = \lim_\alpha I_1(x_\alpha) \otimes I_2(y_\alpha)(h) = \tau \otimes \mu(h)$$

and

$$\lim_\alpha h^\mu(x_\alpha) = \lim_\alpha I_1(x_\alpha)(h^\mu) = \tau(h^\mu) = \tau \otimes \mu(h),$$

which is a contradiction. Therefore, $\{h^{y_\alpha}\}$ converges uniformly to h^μ.

To show ii) implies iii), assume ii) and recall from Theorem 2.7 that

$$\mathscr{A} \otimes \mathscr{B} = \{h \in B(X \times Y): {}^x h \in \mathscr{B}, x \in X, h^y \in \mathscr{A}, y \in Y, \text{ and } \{h^y: y \in Y\} \text{ is totally bounded}\}.$$

Hence, $\mathscr{C} \subset \mathscr{A} \otimes \mathscr{B}$.

To show iii) implies i), assume that $\mathscr{C} \subset \mathscr{A} \otimes \mathscr{B}$. By Theorem 2.7, $\{h^y: y \in Y\}$ is totally bounded for each h in $\mathscr{C}$. By Lemma 2.2, it follows that the map $\mu \to h^\mu$ is continuous from $\bar{Y}$ into $(\mathscr{A}, \| \ \|_u)$ for each h in $\mathscr{C}$. That π is jointly continuous now follows as in the proof of Theorem 2.6.

Remark. If $\mathscr{C} \subset \mathscr{A} \otimes \mathscr{B}$, then $\bar{X} \times \bar{Y} = \overline{X \times Y}$. To see this, note that by Theorem 4.2 iii), π is jointly continuous. Hence, $\pi(\bar{X} \times \bar{Y})$ is a compact dense subset of $\overline{X \times Y}$ and, therefore, $\bar{X} \otimes \bar{Y} = \pi(\bar{X} \times \bar{Y}) = \overline{X \times Y}$.

COROLLARY 4.3. *$\mathscr{A} \otimes \mathscr{B} = \mathscr{C}$ if and only if π is a homeomorphism from $\bar{X} \times \bar{Y}$ onto $\overline{X \times Y}$.*

Proof. Assume that $\mathscr{A} \otimes \mathscr{B} = \mathscr{C}$. Let τ_1, τ_2 be in $\bar{X}$ and μ_1, μ_2 be in $\bar{Y}$ such that $\tau_1 \otimes \mu_1 = \tau_2 \otimes \mu_2$. By evaluation at $f \otimes 1$ in $\mathscr{C}$ for each f in $\mathscr{A}$, one obtains $\tau_1 = \tau_2$. Similarly, $\mu_1 = \mu_2$. Hence, π is one-to-one. By Theorem 4.2 and the above remark, π is a homeomorphism onto $\overline{X \times Y}$.

Now assume that π is a homeomorphism. Then π^* is an isometry from $C(\overline{X \times Y})$ onto $C(\bar{X} \times \bar{Y}) = C(\bar{X}) \otimes C(\bar{Y})$. Let $\phi: X \times Y \to \bar{X} \times \bar{Y}$ be given by

$$\phi(x, y) = (I_1(x), I_2(y)).$$

Then ϕ^* is an isometry from $C(\bar{X} \times \bar{Y})$ onto $\mathscr{A} \otimes \mathscr{B}$. Also, I^* is an isometry from $C(\overline{X \times Y})$ onto $\mathscr{C}$. Setting

$$\Phi = \phi^* \circ \pi^* \circ (I^*)^{-1},$$

Φ is an isometry from $\mathscr{C}$ onto $\mathscr{A} \otimes \mathscr{B}$. It follows directly that Φ^{-1} is the identity map on functions of the form $f \otimes g$ for f in $\mathscr{A}$ and g in $\mathscr{B}$. Hence, Φ is the identity map and $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$.

Our setting for the remainder of the paper is as follows. $S \circledS T$ will denote a semitopological semidirect product semigroup as in the previous section; $\mathscr{C}$ will denote a translation-invariant unital C^*-subalgebra of $W(S \circledS T)$; and $\mathscr{A}$ and $\mathscr{B}$ will be defined as earlier in this section.

PROPOSITION 4.4. *For each h in $\mathscr{C}$, $\{h^t: t \in T\}$ is* w.c.c. *Moreover, $\mathscr{A}$ is translation-invariant in $W(S)$ and $\mathscr{B}$ is translation-invariant in $W(T)$.*

Proof. Let h be in $\mathscr{C}$. For t in T and s in S,

$$h^t(s) = h_{(1,t)}(s, 1).$$

Since $\{h_{(1,t)}: t \in T\}$ is w.c.c., so is $\{h^t: t \in T\}$. Also,

$$_s(h^t) = [_{(s,1)}h]^t$$

is in $\mathscr{A}$ and

$$(h^t)_s = [h_{(s,t)}]^1$$

is in $\mathscr{A}$. Since $\mathscr{A}$ is generated by $\{h^t: h \in \mathscr{C}, t \in T\}$, it follows that $\mathscr{A}$ is translation-invariant. Also, given h in $\mathscr{C}$, t in T, since $\{_{(s,1)}h: s \in S\}$ is w.c.c., so is

$$\{[_{(s,1)}h]^t: s \in S\} = \{_s(h^t): s \in S\}.$$

Thus, h^t is in $W(S)$ and so $\mathscr{A} \subset W(S)$. That $\mathscr{B}$ is translation-invariant in $W(T)$ follows similarly.

Let $\bar{S}$, $\bar{T}$, and $\overline{S \times T}$ denote the $(\mathscr{A}, I_1)$-, $(\mathscr{B}, I_2)$-, and $(\mathscr{C}, I)$-compactifications of S, T, and $S ⓢ T$, respectively. By Lemma 2.12 and remarks preceding it, $\bar{S}$, $\bar{T}$, and $\overline{S \times T}$ are compact semitopological semigroups and the embedding maps are homomorphisms.

LEMMA 4.5. *Given τ, τ' in $\bar{S}$ and μ, μ' in $\bar{T}$, one has that*

a) $(\tau\tau') \otimes \mu = (\tau \otimes I_2(1))(\tau' \otimes \mu)$,

b) $\tau \otimes (\mu\mu') = (\tau \otimes \mu)(I_1(1) \otimes \mu')$.

In particular,

$$\tau \otimes \mu = (\tau \otimes I_2(1))(I_1(1) \otimes \mu),\ (\tau\tau') \otimes I_2(1) = (\tau \otimes I_2(1))(\tau' \otimes I_2(1)),\ \textit{and}$$

$$I_1(1) \otimes (\mu\mu') = (I_1(1) \otimes \mu)(I_1(1) \otimes \mu').$$

Proof. Since $I: S ⓢ T \to \overline{S \times T}$ is a homomorphism, one obtains that

$$(I_1(s)I_1(\sigma_t(s'))) \otimes (I_2(t)I_2(t')) = (I_1(s) \otimes I_2(t))(I_1(s') \otimes I_2(t')), \quad s, s' \in S, t, t' \in T.$$

Since π is separately continuous, it follows that

(1) $$(\tau I_1(\sigma_t(s'))) \otimes (I_2(t)\mu') = (\tau \otimes I_2(t))(I_1(s') \otimes \mu'), \quad s' \in S, t \in T, \tau \in \bar{S}, \mu' \in \bar{T}.$$

Letting $t = 1$ in (1), one has that

$$(\tau I_1(s')) \otimes \mu' = (\tau \otimes I_2(1))(I_1(s') \otimes \mu').$$

By separate continuity, a) follows. Letting $s' = 1$ in (1), one has that

$$(\tau I_1(1)) \otimes (I_2(t)\mu') = (\tau \otimes I_2(t))(I_1(1) \otimes \mu').$$

By separate continuity, b) follows.

THEOREM 4.6. *Assume that one of the following conditions is satisfied:*
P) $\bar{T}$ *is a topological group and* $1 \otimes g$ *is in* $\mathscr{C}$ *for each* g *in* $\mathscr{B}$;
Q) $\bar{S}$ *is a topological group and* $f \otimes 1$ *is in* $\mathscr{C}$ *for each* f *in* $\mathscr{A}$.
Then $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$.

Proof. Assume that condition P) is satisfied. By Corollary 4.3 it suffices to show that π is a surjective homeomorphism. Define a (right) action of $\bar{T}$ on $\overline{S \times T}$ by

$$(\nu)\psi_\mu = \nu(I_1(1) \otimes \mu), \quad \mu \in \bar{T}, \nu \in \overline{S \times T}.$$

By Lemma 4.5, $(\nu)\psi_{\mu\mu'} = (\nu)\psi_\mu\psi_{\mu'}$ for all μ, μ' in $\bar{T}$, ν in $\overline{S \times T}$. Since ψ is separately continuous, it is jointly continuous by Ellis' Theorem [**7**]. For τ in $\bar{S}$, μ in $\bar{T}$,

$$(\tau \otimes I_2(1))\psi_\mu = \tau \otimes \mu$$

by Lemma 4.5 and therefore π is jointly continuous. To show that π is one-to-one, assume that $\tau_1 \otimes \mu_1 = \tau_2 \otimes \mu_2$ where τ_1, τ_2 are in $\bar{S}$ and μ_1, μ_2 are in $\bar{T}$. By evaluation at $1 \otimes g$ in $\mathscr{C}$ for each g in $\mathscr{B}$, one obtains that $\mu_1 = \mu_2$. Choose any μ in $\bar{T}$ and note that

$$(\tau_1 \otimes \mu_1)(I_1(1) \otimes \mu_1^{-1}\mu) = \tau_1 \otimes \mu$$

and

$$(\tau_2 \otimes \mu_1)(I_1(1) \otimes \mu_1^{-1}\mu) = \tau_2 \otimes \mu$$

by b) of Lemma 4.5. Thus, $\tau_1 \otimes \mu = \tau_2 \otimes \mu$ for all μ in $\bar{T}$ and so $\tau_1(h^\mu) = \tau_2(h^\mu)$ for all h in $\mathscr{C}$ and μ in $\bar{T}$. Therefore, $\tau_1 = \tau_2$ and π is one-to-one. From the remark preceding Corollary 4.3, it follows that π is surjective. Hence $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$.

If condition Q) is assumed instead of condition P), a similar proof using a) of Lemma 4.5 will show that $\mathscr{C} = \mathscr{A} \otimes \mathscr{B}$.

PROPOSITION 4.7. $A(S \circledS T) \subset A(S \times T)$. *In particular, if* $A^\sigma(S) = A(S)$, *then* $A(S \circledS T) = A(S \times T)$.

Proof. Let h be in $A(S \circledS T)$. Since $A(S \times T) = A(S) \otimes A(T)$ (this was proved after Corollary 2.9), we apply Corollary 2.9 in showing that h is in $A(S \times T)$. For s in S, t in T,

$${}^s h(t) = {}_{(s,1)}h(1, t).$$

Thus $\{{}^s h : s \in S\}$ is totally bounded. Also,

$${}_t({}^s h)(t') = {}_{(s,t)}h(1, t'), \quad s \in S, t, t' \in T.$$

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Consequently, $\{_t({}^s h): t \in T\}$ is totally bounded and so ${}^s h$ is in $A(T)$. Finally,

$$_s(h^t)(s') = {}_{(s,1)}h(s', t), \quad s, s' \in S, t \in T.$$

Hence, $\{_s(h^t): s \in S\}$ is totally bounded and so h^t is in $A(S)$. Thus,

$$A(S \circledcirc T) \subset A(S \times T).$$

By the remark preceding Corollary 3.8 or by Theorem 3.11, it follows that

$$A^\sigma(S) \otimes A(T) \subset A(S \circledcirc T).$$

Thus, if $A^\sigma(S) = A(S)$, then $A(S \circledcirc T) = A(S \times T)$.

In **[11]** Junghenn shows that there is a s.p.c. of $S \circledcirc T$ induced by $A(S \circledcirc T)$ when T contains a dense subgroup and in such a case obtains $A(S \circledcirc T)$ as a tensor product. The following theorem together with Theorem 3.11 contain his result.

THEOREM 4.8. *Assume that aT is a topological group. Then*

$$A(S \circledcirc T) = A^\sigma(S) \otimes A(T).$$

Proof. Let $\mathscr{C} = A(S \circledcirc T)$. Let $\mathscr{A}$ and $\mathscr{B}$ be as defined earlier in this section (before Definition 4.1). Since $A(S \circledcirc T) \subset A(S \times T)$, it follows that $\mathscr{A} \subset A(S)$ and $\mathscr{B} \subset A(T)$. Given g in $A(T)$, s in S, t in T, then

$$_{(s,t)}(1 \otimes g) = 1 \otimes {}_t g$$

and thus $\{_{(s,t)}(1 \otimes g): s \in S, t \in T\}$ is totally bounded. Consequently, $1 \otimes g$ is in $\mathscr{C}$ for each g in $A(T)$. Since ${}^1(1 \otimes g) = g$ for all g in $A(T)$, $\mathscr{B} = A(T)$. Since condition P) of Theorem 4.6 is satisfied, $\mathscr{C} = \mathscr{A} \otimes A(T)$. To see that $\mathscr{A} \subset A^\sigma(S)$, let f be in $\mathscr{A}$. Then $f \otimes 1$ is in $\mathscr{C} = A(S \circledcirc T)$ and therefore

$$\{_{(s,t)}(f \otimes 1): s \in S, t \in T\}$$

is totally bounded in $\mathscr{C}$. Since

$$_{(s,t)}(f \otimes 1) = ({}_s f \circ \sigma_t) \otimes 1 \quad \text{for each } s \text{ in } S \text{ and } t \text{ in } T,$$

it follows that $\{_s f \circ \sigma_t: s \in S, t \in T\}$ is totally bounded. Moreover,

$$[_{(s,t)}(f \otimes 1)]^1 = {}_s f \circ \sigma_t$$

and so ${}_s f \circ \sigma_t$ is in $\mathscr{A}$ for each s in S and t in T by the definition of $\mathscr{A}$. Thus, f is in $A^\sigma(S)$ and $\mathscr{A} \subset A^\sigma(S)$. For f in $A^\sigma(S)$,

$$_{(s,t)}(f \otimes 1) = {}_s f \circ \sigma_t \otimes 1$$

from which it follows that $f \otimes 1$ is in $\mathscr{C}$. But $(f \otimes 1)^1 = f$ is then in $\mathscr{A}$. Therefore, $\mathscr{A} = A^\sigma(S)$.

In [2] it is shown that $w(S \times T) = wS \times T$ where S is a semitopological semigroup with right identity and T is a compact topological group. The following theorem generalizes this result to semidirect products. A similar theorem, obtained independently, appears in [16].

THEOREM 4.9. *Assume that wT is a topological group. Then*

$$W(S ⓢ T) = W^\sigma(S) \otimes W(T).$$

Proof. Let $\mathscr{C} = W(S ⓢ T)$ and $\mathscr{A}$ and $\mathscr{B}$ be as defined earlier in this section. By Proposition 4.4, $\mathscr{B} \subset W(T)$. For g in $W(T)$, it follows that $1 \otimes g$ is in $\mathscr{C}$ and hence ${}^1(1 \otimes g) = g$ is in $\mathscr{B}$. Therefore, $\mathscr{B} = W(T)$. By Theorem 4.6, $\mathscr{C} = \mathscr{A} \otimes W(T)$. To see that $\mathscr{A} \subset W^\sigma(S)$, first note that $\mathscr{A}$ is a translation-invariant subalgebra of $W(S)$ by Proposition 4.4. Since $\mathscr{A} \otimes W(T) = W(S ⓢ T)$, 2) of Theorem 3.7 holds. Therefore, a) and b) of Theorem 3.7 hold, namely, $\{{}_sf \circ \sigma_t: s \in S, t \in T\}$ is w.c.c. in $\mathscr{A}$ and $\{f_{\sigma_t(s_0)}: t \in T\}$ is totally bounded in $\mathscr{A}$ for each f in $\mathscr{A}$ and s_0 in S. Since $\mathscr{A}$ is translation-invariant and

$$f_{\sigma_t(s_1)s_2} = (f_{s_2})_{\sigma_t(s_1)},$$

$\mathscr{A} \subset W^\sigma(S)$. For f in $W^\sigma(S)$,

$${}_{(s,t)}(f \otimes 1) = ({}_sf \circ \sigma_t) \otimes 1$$

from which it follows that $f \otimes 1$ is in $\mathscr{C}$. But $(f \otimes 1)^1 = f$ is then in $\mathscr{A}$. Therefore, $\mathscr{A} = W^\sigma(S)$.

THEOREM 4.10. *Let $\mathscr{C} = \{h \in W(S ⓢ T): h^t \in A^\sigma(S)$ for all t in $T\}$. If aS^σ is a topological group, then $\mathscr{C} = A^\sigma(S) \otimes W(T)$.*

Proof. For h in $\mathscr{C}$, s_0 in S, t, t_0 in T,

$$[{}_{(s_0,t_0)}h]^t = {}_{s_0}(h^{t_0t}) \circ \sigma_{t_0}$$

is in $A^\sigma(S)$ and

$$[h_{(s_0,t_0)}]^t = (h^{tt_0})_{\sigma_t(s_0)}$$

is in $A^\sigma(S)$, since $A^\sigma(S)$ is closed under composition with the family $\{\sigma_t: t \in T\}$ by Proposition 3.10. Hence, $\mathscr{C}$ is translation-invariant. It follows directly that $\mathscr{C}$ is a unital C^*-subalgebra of $W(S ⓢ T)$. Let $\mathscr{A}$ and $\mathscr{B}$ be as defined earlier in this section. From the definition of $\mathscr{C}$, $\mathscr{A} \subset A^\sigma(S)$. For f in $A^\sigma(S)$, $f = (f \otimes 1)^t$ for all t in T and so $f \otimes 1$ is in $\mathscr{C}$ and hence f is in $\mathscr{A}$ and $\mathscr{A} = A^\sigma(S)$. Hence, condition Q) of Theorem 4.6 is satisfied and so $\mathscr{C} = A^\sigma(S) \otimes \mathscr{B}$. By Proposition 4.4, $\mathscr{B} \subset W(T)$. If g is in $W(T)$, then $1 \otimes g$ is in $\mathscr{C}$ since $(1 \otimes g)^t$ is a constant function on S for each t in T. Since ${}^1(1 \otimes g) = g$ is in $\mathscr{B}$, $\mathscr{B} = W(T)$.

The following example shows that in general

$$A(S \circledS T) \neq A^\sigma(S) \otimes A(T) \quad \text{and} \quad W(S \circledS T) \neq W^\sigma(S) \otimes W(T).$$

Example 4.11. *Let* $S = (\mathbf{R}, +)$ where $\mathbf{R}$ denotes the real numbers with the usual topology and let $T = \{2^{-n}: n = 0, 1, 2, \ldots\}$ under multiplication with the discrete topology. Define $\sigma_t(s) = ts$ for all t in T, s in S. We first show that $A^\sigma(S)$ consists of just the constant functions. Choose any net $\{t_\alpha\}$ in T such that

$$\lim_\alpha t_\alpha = 0 \quad \text{and} \quad \lim_\alpha I_2(t_\alpha) = \mu,$$

where μ is in aT. Fix s in S and let $s_\alpha = s/t_\alpha$. By passing to subnets, we may assume that $\lim_\alpha I_1(s_\alpha) = \tau_s$ where τ_s is in aS^σ. Then

$$I_1(s) = \lim_\alpha I_1(\sigma_{t_\alpha}(s_\alpha)) = \lim_\alpha \bar{\sigma}_{I_2(t_\alpha)}(I_1(s_\alpha)) = \bar{\sigma}_\mu(\tau_s),$$

where $\bar{\sigma}$ is the extension of σ induced by $A^\sigma(S)$ and $A(T)$. By Theorem 3.11, such a $\bar{\sigma}$ exists and is jointly continuous. For s' in S and f in $A^\sigma(S)$,

$$\begin{aligned}\bar{\sigma}_\mu(I_1(s'))(f) &= \lim_\alpha \bar{\sigma}_{I_2(t_\alpha)}(I_1(s'))(f) = \lim_\alpha I_1(t_\alpha s')(f) \\ &= \lim_\alpha f(t_\alpha s') = f(0) = I_1(0)(f).\end{aligned}$$

Therefore, $\bar{\sigma}_\mu(\tau) = I_1(0)$ for all τ in aS^σ and so $I_1(s) = \bar{\sigma}_\mu(\tau_s) = I_1(0)$. Hence, $f(s) = f(0)$ for all f in $A^\sigma(S)$.

We next show that $A(S) \otimes C_0(T) \subset A(S \circledS T)$, where $C_0(T)$ consists of those functions in $B(T)$ which vanish at infinity. Let f be in $A(S)$ and g in $C_0(T)$. Then

$$_{(s,t)}(f \otimes g)(s', t') = f(s + ts')g(tt'), \quad s, s' \in S, t, t' \in T.$$

Given $\epsilon > 0$, there exists a $\delta > 0$ such that if $t < \delta$, then

$$|_{(s,t)}(f \otimes g)(s', t')| < \epsilon \quad \text{for all } s, s' \text{ in } S \text{ and } t' \text{ in } T,$$

since g is in $C_0(T)$. Since $\{t \in T: t \geqq \delta\}$ is finite, it suffices to show that $\{_{(s,t_0)}(f \otimes g): s \in S\}$ is totally bounded for fixed t_0 in T. Since

$$_{(s,t_0)}(f \otimes g) = {}_sf \circ \sigma_{t_0} \otimes {}_{t_0}g$$

and f is in $A(S)$, $\{_{(s,t_0)}(f \otimes g): s \in S\}$ is totally bounded. Hence,

$$A(S) \otimes C_0(T) \subset A(S \circledS T)$$

and so $A(S \circledS T) \neq A^\sigma(S) \otimes A(T)$.

To see that $W(S \circledS T) \neq W^\sigma(S) \otimes W(T)$, one can argue as follows. Suppose that $W(S \circledS T) = W^\sigma(S) \otimes W(T)$. Then from the previous paragraph,

$$A(S) \otimes C_0(T) \subset W^\sigma(S) \otimes W(T).$$

Let f be any non-constant function in $A(S)$, and let g be the identity function on T. Then $f \otimes g$ is in $W^\sigma(S) \otimes W(T)$ and hence, $(f \otimes g)^1 = f$ is in $W^\sigma(S)$. Thus, f is in $A(S) \cap W^\sigma(S) = A^\sigma(S)$ by Corollary 3.14. Since $A^\sigma(S)$ consists of just the constant functions, this is a contradiction.

It is possible to show more. We will show that $W^\sigma(S)$ consists of just the constant functions; from which it also follows that

$$W(S \circledS T) \neq W^\sigma(S) \otimes W(T),$$

since as before, $A(S) \otimes C_0(T) \subset A(S \circledS T) \subset W(S \circledS T)$.

Since wS^σ is a compact commutative semitopological semigroup (the commutativity follows from the fact that wS^σ has separately continuous multiplication and contains a dense commutative subsemigroup), the kernel, K, of wS^σ is a compact topological group ([**5**], Corollary 2.5). Let e be the identity of K. We first show that $e \circ f$ is a constant function for all f in $W^\sigma(S)$. Fix an f in $W^\sigma(S)$ and set $f_0 = f - (e \circ f)$. Recall that $e \circ f$ is in $W^\sigma(S)$ since $W^\sigma(S)$ is left M-introverted. Then,

$$e \circ f_0 = (e \circ f) - e \circ (e \circ f) = (e \circ f) - (e \circ f) = 0$$

since for s in S,

$$(e \circ (e \circ f))(s) = e({}_s(e \circ f)) = e(e \circ {}_sf) = ee({}_sf) = e({}_sf) = (e \circ f)(s).$$

Note that the third equality involves left Arens multiplication, as defined before Theorem 3.7. Also, $(e \circ f)^\wedge = (\hat{f})_e$, where $\wedge$ denotes the Gelfand transform on $W^\sigma(S)$.

Let $F = (\hat{f})_e$ and note that K is an ideal in wS^σ [**5**]. Define ρ from wS^σ into K by $\rho(\tau') = \tau' e$ for all τ' in wS^σ, and note that ρ is a continuous homomorphism. Since K is a compact topological group, by Lemma 5.2 of [**5**],

$$\rho^*(C(K)) \subset A(wS^\sigma)$$

where $\rho^*(G) = G \circ \rho$ for all G in $C(K)$. Since $\hat{f}|_K$ is in $C(K)$, where $\hat{f}|_K$ denotes the restriction of $\hat{f}$ to K, $\rho^*(\hat{f}|_K) = F$ is in $A(wS^\sigma)$. Thus, $\{F_\tau : \tau \in wS^\sigma\}$ is norm compact and, therefore, $\{F_{I_1(s)} : s \in S\}$ is totally bounded in $C(wS^\sigma)$, where I_1 is the embedding map of S into wS^σ. Hence, $\{(e \circ f)_s : s \in S\}$ is totally bounded in $W^\sigma(S)$, and so $e \circ f$ is in $A(S) \cap W^\sigma(S) = A^\sigma(S)$ by Corollary 3.14. Hence, $e \circ f$ is a constant function for all f in $W^\sigma(S)$.

Let τ be in wS^σ. We now show that $\tau e = e$. Let f be in $W^\sigma(S)$. Then $f = c + f_0$ where c is a constant and f_0 is as defined above. Since $e \circ f_0 = 0$, $e(f_0) = 0$ and so

$$\tau e(f) = \tau e(c) + \tau e(f_0) = c = e(f).$$

Let I_2 be the embedding map of T into wT; and let $\bar{\sigma}$ be the extension of σ induced by $W^\sigma(S)$ and $W(T)$. By Theorem 3.12, such a $\bar{\sigma}$ exists and is separately continuous. Note that $wT \sim I_2(T)$ is non-empty for, since

$A(T)$ separates the points of T, I_2 is one-to-one and hence I_2 cannot map a discrete T onto a compact wT. Let μ be in $wT \sim I_2(T)$. We now show that $\bar{\sigma}_\mu(\tau) = I_1(0)$ for all τ in wS^σ. Let (t_α) be a net in T with $I_2(t_\alpha) \to \mu$ and note that (t_α) converges to 0. For s in S,

$$\bar{\sigma}_{I_2(t_\alpha)}(I_1(s)) = I_1(t_\alpha s) \to I_1(0)$$

and, by the separate continuity of $\bar{\sigma}$,

$$\bar{\sigma}_{I_2(t_\alpha)}(I_1(s)) \to \bar{\sigma}_\mu(I_1(s)).$$

Therefore, $\bar{\sigma}_\mu(I_1(s)) = I_1(0)$ for all s in S. By the separate continuity of $\bar{\sigma}$ again and the fact that $I_1(S)$ is dense in wS^σ,

$$\bar{\sigma}_\mu(\tau) = I_1(0) \quad \text{for all } \tau \text{ in } wS^\sigma.$$

We next show that $\bar{\sigma}_{I_2(t)}(e) = e$ for all t in T. Fix t in T. Since

$$\bar{\sigma}_{I_2(t)}(I_1(s)) = I_1(ts), \quad s \in S,$$

$\bar{\sigma}_{I_2(t)}$ maps $I_1(S)$ onto $I_1(S)$ and, therefore, $\bar{\sigma}_{I_2(t)}$ maps wS^σ onto wS^σ. Hence, there exists a τ' in wS^σ such that

$$\bar{\sigma}_{I_2(t)}(\tau') = e.$$

Since $\tau e = e$ for all τ in wS^σ,

$$\bar{\sigma}_{I_2(t)}(e) = \bar{\sigma}_{I_2(t)}(\tau' e) = \bar{\sigma}_{I_2(t)}(\tau')\bar{\sigma}_{I_2(t)}(e) = e\bar{\sigma}_{I_2(t)}(e) = e.$$

We now show that wS^σ consists of a single point, and hence $W^\sigma(S)$ consists of just the constant functions. Since $\bar{\sigma}_{I_2(t)}(e) = e$ for all t in T and $\bar{\sigma}$ is separately continuous, $\bar{\sigma}_\mu(e) = e$ for all μ in wT. Since $\bar{\sigma}_\mu(e) = I_1(0)$ for all μ in $wT \sim I_2(T)$, $e = I_1(0)$. Then, for all τ in wS^σ,

$$\tau = \tau I_1(0) = \tau e = e.$$

Therefore, wS^σ is a single point.

We complete this paper by obtaining certain conditions which force a semidirect product to be a direct product.

A character of a topological group G is a continuous homomorphism from G into the circle group (= the group of complex numbers of modulus one). Let $\hat{G}$ denote all characters of G.

Given any χ_1, χ_2 in $\hat{G}$ with $\chi_1 \neq \chi_2$, one has that

$$\|\chi_1 - \chi_2\|_u \geqq \sqrt{3}.$$

THEOREM 4.12. *Let S be a topological group such that $\hat{S} \cap A^\sigma(S)$ separates the points of S, and let T be connected. Then $S \circledS T = S \times T$.*

Proof. For each χ in $\hat{S} \cap A^\sigma(S)$, $\{\chi \circ \sigma_t : t \in T\}$ is totally bounded and contained in $\hat{S}$. Hence, $\{\chi \circ \sigma_t : t \in T\}$ is finite. Fix a χ in $\hat{S} \cap A^\sigma(S)$. Let

$$U = \{t \in T : \chi \circ \sigma_t = \chi\}.$$

Then U is both closed and open in T and 1 is in U. Thus, $U = T$ and $\chi \circ \sigma_t = \chi$ for all t in T, for all χ in $\hat{S} \cap A^\sigma(S)$. Since $\hat{S} \cap A^\sigma(S)$ separates the points of S, σ_t = the identity endomorphism for all t in T. Hence, $S ⓢ T = S \times T$.

COROLLARY 4.13. *Let S be a locally compact, Hausdorff, abelian topological group and let T be a connected semitopological group. If $S ⓢ T$ is maximally almost periodic, then $S ⓢ T = S \times T$.*

Proof. From [**9**], p. 345, $\hat{S}$ separates the points of S. As in Corollary 3.5, aT and aS^σ are topological groups. Hence, by Theorem 4.8, $A^\sigma(S) \otimes A(T)$ separates the points of $S ⓢ T$. Thus, $A^\sigma(S)$ separates the points of S and so the embedding map I_1: $S \rightarrow aS^\sigma$ is one-to-one. Since aS^σ is a compact abelian topological group, $(aS^\sigma)^\wedge$ separates the points of aS^σ. Since $(aS^\sigma)^\wedge$ is isometrically isomorphic to $\hat{S} \cap A^\sigma(S)$ via the adjoint map I_1^* restricted to $(aS^\sigma)^\wedge$, $\hat{S} \cap A^\sigma(S)$ separates the points of S.

COROLLARY 4.14. *Let G be any semidirect product of $(\mathbf{R}^n, +)$ with $(\mathbf{R}^m, +)$ induced by some σ such that σ_t is not the identity endomorphism for some t in $\mathbf{R}^m$. Then $A^\sigma(\mathbf{R}^n) \neq A(\mathbf{R}^n)$.*

Proof. If $A^\sigma(\mathbf{R}^n) = A(\mathbf{R}^n)$, then $\hat{\mathbf{R}}^n \cap A^\sigma(\mathbf{R}^n) = \hat{\mathbf{R}}^n$ separates the points of $\mathbf{R}^n$. By Theorem 4.12, $G = \mathbf{R}^n \times \mathbf{R}^m$, which is a contradiction.

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Shippensburg State College,
Shippensburg, Pennsylvania;
Morehead State University,
Morehead, Kentucky