CORRIGENDUM
to the paper

COMPLETIONS OF SEMILATTICES OF CANCELLATIVE SEMIGROUPS

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K. Shoji has pointed out to me that construction [**1**] does not always yield a completion. In the notation of [**1**], the homomorphism from the strong semilattice of cancellative semigroups S to its purported completion T in Abian's order is not always a monomorphism. The difficulty arises when there is $e \in E$, $e = \sup\{e' \in E \mid e' < e\}$ but $\{\phi_{e,e'}\}_{e'<e}$ is not faithful, i.e. there are x, y with $x \neq y$ in S_e such that $\phi_{e,e'}(x) = \phi_{e,e'}(y)$ for all $e' < e$. A modification of the construction saves all parts of Theorem 1 except the fact that the new embedding $S \subseteq T$ need not preserve suprema existing in S; it does if S is a semilattice of groups. The sequel [**2**] also needs a modification in the form of an additional hypothesis.

THEOREM 1 (cf. [**1**, Theorem 1]). *Let $S = \bigcup_E S_e$ be a strong semilattice of cancellative semigroups. Then S has a completion T in Abian's order where T is also a strong semilattice of cancellative semigroups. If the S_e are groups the completion is supremum preserving.*

The remaining results of [**1**] need not be changed except that the phrase "supremum preserving" must be dropped from Theorems 5 and 6.

The modified construction is in two stages. The first is to eliminate the problems which hinder the construction in [**1**], and then the latter is applied to the result. The example suggested by K. Shoji is a very simple one, namely that shown as A in Fig. 1, where $\{1, g\}$ is a group. The original construction yields B, while what is wanted is something like C where the boundable set $\{e, f\}$ now has a supremum, h.

Given $S = \bigcup_E S_e$, a chain of extensions is built transfinitely as follows. Suppose for an ordinal α, $S^\alpha = \bigcup_{E^\alpha} S_e^\alpha$ has already been constructed and that for some $e \in E^\alpha$, $e = \sup\{e' \in E^\alpha \mid e' < e\}$, but $\{\phi^\alpha_{e,e'}\}$ is not faithful. Then a new element $\bar{e}$ is added to E^α with $e' < \bar{e} < e$, for all $e' < e$, and multiplication is defined by

$$g\bar{e} = \begin{cases} \bar{e} & \text{if } ge = e, \\ ge & \text{if } ge < e. \end{cases} \quad (g \in E^\alpha)$$

Then $E^{\alpha+1} = E^\alpha \cup \{\bar{e}\}$. Also $S^{\alpha+1}$ is formed as $S_f^{\alpha+1} = S_f^\alpha$ if $f \neq \bar{e}$ and $S_{\bar{e}}^{\alpha+1}$ is the inverse limit of the system $\{\phi_{e,e'} \mid e' < e\}$.

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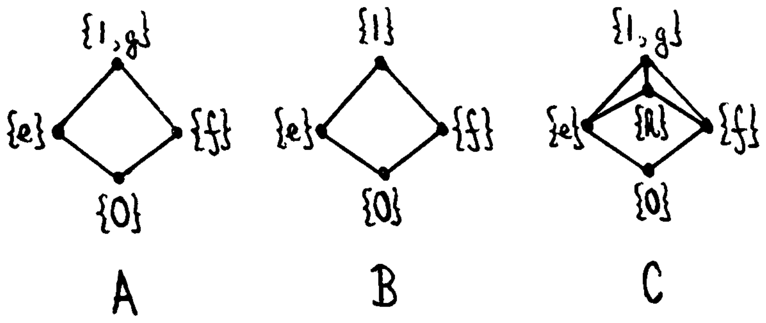

Figure 1

If β is a limit ordinal then $S^\beta = \bigcup_{\alpha<\beta} S^\alpha$.

The next lemma shows that what has been done at one stage in the process is not destroyed later.

LEMMA 1. (1) *If $e \in E^\zeta$ and $e \neq \sup_{E^\zeta}\{e' < e\}$ then, for $\gamma > \zeta$, $S_e^\gamma = S_e^\zeta$ and $e \neq \sup_{E^\gamma}\{e' < e\}$.*

(2) *If $e \in E^\zeta$, $e = \sup_{E^\zeta}\{e' < e\}$ and $\{\phi_{e,e'}\}$ is faithful, then, for $\gamma > \zeta$, $S_e^\gamma = S_e^\zeta$.*

Proof. (1) The construction does not change any existing S_e^ζ in subsequent stages. Further if $e = \sup_{E^\gamma}\{e' < e\}$, let α be the least ordinal with $e = \sup_{E^\alpha}\{e' < e\}$. For any $\zeta \leq \beta < \alpha$, there is some $u \in E^\beta$ with u an upper bound of $\{e' < e \mid e' \in E^\beta\}$, but $u \nleq e$. Hence for some least σ, $\beta < \sigma \leq \alpha$, there is $v \in E^\sigma$, $v < e$ but $v \nleq u$. It follows that $v = \bar{w}$ for some $w \in E^{\sigma-1}$, for σ is clearly not a limit ordinal. Then $u\bar{w} < \bar{w}$ so that $u\bar{w} = uw$ and $e\bar{w} = \bar{w}$ so that $ew = w$. Thus in $E^{\sigma-1}$, $w < e$ and $w \nleq u$, contradicting the choice of σ.

(2) is obvious.

By the lemma, for some ordinal γ, the construction stops with no $e \in E^\gamma$ with $e = \sup_{E^\gamma}\{e' < e\}$ and $\{\phi_{e,e'}\}$ not faithful. Let $\bar{S} = S^\gamma$, $\bar{E} = E^\gamma$.

LEMMA 2. *Every element of $\bar{S}$ is the supremum of a boundable subset of S.*

Proof. It is first noted that if at some stage in the construction $E^{\alpha+1} = E^\alpha \cup \{\bar{e}\}$, then $e \in E$. If not, then e was added at some stage, let us say in going from E^β to $E^{\beta+1}$. Then in $E^{\beta+1}$, $e = \sup\{e' < e\}$ and $\{\phi_{e,e'}\}$ is faithful. In all subsequent steps the corresponding family $\{\phi_{e,e'}\}$ is faithful, so that e is not used again in the construction. This contradiction shows that $e \in E$.

This shows that if $E^{\alpha+1} = E^\alpha \cup \{\bar{e}\}$ for some α, then $S_e^\beta = S_e$ for all β.

Next, suppose that every element of S^α is the supremum of a subset of S. Let $E^{\alpha+1} = E^\alpha \cup \{\bar{e}\}$. By construction, every element of $S_{\bar{e}}^{\alpha+1}$ is the supremum of all the elements below it, and these are, by the induction hypothesis, suprema of subsets of S. As already seen, if $t \in S_{\bar{e}}^{\alpha+1}$, $t \in S$. Finally if $t \in S_f^{\alpha+1}$, $f \neq e$, $f \neq \bar{e}$, then $S_f^{\alpha+1} = S_f^{\alpha}$. If t is not the supremum of a subset of S in $S^{\alpha+1}$, then there is $u \in S_{\bar{e}}^{\alpha+1}$ which is an upper bound for $X = \{s \in S \mid s \leq t\}$, but $t \nleq u$. If $f\bar{e} = \bar{e}$ then $fe = e$ and $\phi_{f,e}(t)$ is an upper bound of X below t, since $X \cap S_{\bar{e}}^{\alpha+1} = \phi$. Hence $f\bar{e} < \bar{e}$ and $f\bar{e} = fe$. For $x \in X$, $x \in S_g^{\alpha+1}$, $g < f$ and $g < \bar{e}$. Hence $g < f\bar{e}$ and it follows that $\phi_{f,fe}(t)$ and $\phi_{\bar{e},fe}(u)$ would be upper bounds for X in $S_{fe}^{\alpha+1}$, and hence they coincide, say $\phi_{f,fe}(t) = v$. Then v would be greater than or equal to the supremum of X in S^{α}, contrary to the induction hypothesis.

If α is a limit ordinal and every element of S^{β} is a supremum in S^{β} of a subset of S, for all $\beta < \alpha$, then for $t \in S^{\alpha}$, if t is not the supremum of $X = \{s \in S \mid s \leq t\}$, then there is $\beta < \alpha$ such that there is an element u with $u \in S^{\beta}$, u an upper bound for X but $t \nleq u$. This contradicts the induction hypothesis.

Now the completion may be constructed using the techniques of [**1**]. To do so, S is first embedded in $\bar{S}$, as above, and then $\bar{S}$ may be completed.

If S is a semilattice of groups, suprema which exist in S are preserved in the passage to $\bar{S}$. One sees that if $s = \sup_S X$ we may take $X = \{x \in S \mid x < s\}$ and then if $s \neq \sup_{\bar{S}} X$ it is because at some stage in the construction of $\bar{S}$, $\bar{e}$ is added and $\phi_{e,\bar{e}}(s) = u$ is a new upper bound for X. But $\phi_{e,\bar{e}}$ is not a monomorphism, so that any preimage of u in S_e must be an upper bound for X; this is impossible since two elements of S_e are incomparable.

In [**2**] it was claimed that the above construction may be used to construct the injective hull of certain S-sets where S is a semilattice of groups. The claim is false as stated since if $\bar{S} \neq S$ the extension $S \subseteq \bar{S}$ is not essential, although the completion T is indeed S-injective. However we shall show that $S = \bar{S}$ in the important case where the semilattice of groups S is non-singular.

Johnson and McMorris [**3**, Theorem 2] characterize semilattices of groups, with 0, $S = \bigcup_E S_e$ and which are non-singular. Necessary and sufficient conditions are that (i) E be disjunctive and (ii) for any large ideal L and $e = e^2 \notin L$, $\bigcap \{\ker \phi_{e,e'} \mid e' < e\} = \{e\}$. In this case it will be seen that $S = \bar{S}$. The following weaker theorem replaces the theorem of [**2**].

THEOREM. *Let $S = \bigcup_E S_e$ be a non-singular semilattice of groups, with 0. If F is the BL completion of E, then the completion T constructed over F is the injective hull of S, as an S-set. T is the complete semigroup of quotients of S.*

Proof. It suffices to show that, in the notation established earlier, $\bar{S} = S$. Hence it must be shown that if $e \in E$ is such that $e = \sup_E\{e' < e\}$ then $\{\phi_{e,e'}\}$ is faithful. Let $J = \bigcup_{e'<e} S_{e'} \cup \bigcup_{e \wedge f = 0} S_f$. It is seen that J is an ideal. To show that J is large it suffices to show that for any $0 \neq g \in E$, some non-zero multiple of g is in J. If $g < e$ then $g \in J$; if $g = e$ then there is $0 \neq e' < e$ and $ge' = e' \in J$. Finally, if $g \nleq e$ then there exists $0 < f \leq g$ such that $f \wedge e = 0$ since E is disjunctive, and then $f = fg \in J$. But now the condition (ii) above says precisely that $\{\phi_{e,e'}\}$ is faithful.

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