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THE BOTT SUSPENSION AND THE INTRINSIC JOIN

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Introduction. If $(G; U, V)$ is a triad with G a group we define

$$C_G(U, V) = \{g \in G | [g, u] \in V \text{ for all } u \in U\}$$

where $[g, u] = gug^{-1}u^{-1}$ is the commutator. $C_G(U, V)$ will be called the (left) center of U in G modulo V or in brief a (left) C-space. If G is a topological group it will be understood that the topology on $C_G(U, V)$ is the relative topology of G. For $(G; H, K, L)$ a tetrad of topological groups with $L \subset K$ and $(R, S) \subset (C_G(L, H), C_G(K, H))$ we define the (left) C-pairing as the map $(R, S) \times (K, L) \to (G, H)$ induced by the commutator map in G, where

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

This paper then shows that in the classical groups the Bott suspension, the James join, the Samelson product, and the Jacobi identity are all part of the same phenomenon, C-pairings. By treating these topics in a unified manner we will show how the James join can be used to prove the existence of Lundell's factorization [**6**] of the Bott suspension for the unitary groups. The unified treatment then makes it possible to obtain some new results concerning the action of this factorization on the Samelson ring.

1. C-spaces and pairings. In the sequel we will make use of the following Hall identities valid for any elements a, b, c of a group G:

$H(1)$ $[a, b]\,[b, a] = e$

$H(2)$ $[a, bc] = [a, b]\,[a, c]\,[[c, a], b]$

$H(2')$ $[ab, c] = [a, [b, c]]\,[b, c]\,[a, c]$

$H(3)$ $[[a, b], c^b]\,[[b, c], a^c]\,[[c, a], b^a] = e$

$H(3')$ $[a^c, [b, c]]\,[c^b, [a, b]]\,[b^a, [c, a]] = e$

where $a^b = bab^{-1}$ and e denotes the identity of G.

To help acquaint the reader with C-spaces we list a few elementary properties for a triple of groups, (G, H, K).

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Lattice of inclusions.

$$\begin{array}{ccccc}
 & & G & & \\
 & & \cup & & \\
 & & C_G(K, H) & & \\
 & \cup & & \cup & \\
C_G(K, K) & & & & C_G(H, H) \\
\cup \quad & \cup & & \cup & \quad \cup \\
K & & C_G(H, K) & & H \\
 & & \cup & & \\
 & & C_G(H) & &
\end{array}$$

where $C_G(H) = C_G(H, e)$ is the center of H in G.

Algebraic properties.

$A(1)$ $C_G(K, H) = \{g \in G | gKg^{-1} \subset H\}$

$A(2)$ $a \in C_G(H, H)$, $b \in C_G(K, H) \Rightarrow ab \in C_G(K, H)$

$A(3)$ $a \in C_G(K, K)$, $b \in C_G(K, H) \Rightarrow ba \in C_G(K, H)$

$A(4)$ $a, b \in C_G(H, K) \Rightarrow ab \in C_G(H, K)$

$A(5)$ $a \in C_G(H, K)$, $h \in H \Rightarrow [a, h], [h, a], hah^{-1} \in C_G(H, K)$.

We prove $A(4)$ to indicate the use of the Hall identities.

Proof. Let $a, b \in C_G(H, K)$ and $h \in H$. From the definition of $C_G(H, K)$ and the assumption that K is a group it easily follows that $[h, a]$, $[h, b]$ and $[[b, h], a]$ are in K and consequently so is $[h, ab] = [h, a]\,[h, b]\,[[b, h], a]$. Hence $ab \in C_G{}^*(H, K) = C_G(H, K)$.

In particular we see from $A(4)$ that if (G, H, K) is a triple of topological groups then $C_G(H, K)$ as a submonoid of G is an H-space.

For $(G; H, K, L)$ a tetrad of groups with $L \subset K$ we adopt the following notation

$$T = T(G, H; K, L) = (C_G(L, H)/C_G(K), C_G(K, H)/C_G(K))$$
$$F = F(G, H; K, L) = (C_G(L, H)/C_G(K), *)$$

and these will be called the T and F pairs for the tetrad.

If $(G; H, K, L)$ is a tetrad of topological groups with $L \subset K$, the C-pairings of interest are then the following:

(I) the Samelson pairing
$(G, e) \times (G, e) \to (G, e)$

(II) the relative Samelson pairings
$(H, e) \times (G, H) \to (G, H)$ and $(G, H) \times (H, e) \to (G, H)$

(III) the transfer pairings (definition)
$(R, S) \times (K, L) \to (G, H)$ and $(K, L) \times (R, S) \to (G, H)$
where $(R, S) \subset T(G; H, K, L)$.

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These pairings induce products in homotopy in the usual manner [**1**], which we denote by $\langle a, b \rangle$.

PROPOSITION 1.1. *If a and b are given by one of the following sets of conditions*
(i) $a \in \pi_r(G)$, $b \in \pi_s(G)$
(ii) $a \in \pi_r(H)$, $b \in \pi_s(G, H)$
(iii) $a \in \pi_r(R, S)$, $b \in \pi_s(K, L)$
then $\langle a, b \rangle = (-1)^{rs-1} \langle b, a \rangle$.

Proof. Parts (i) and (ii) are well known [**5**; **8**], and the proof of (iii) is similar.

In certain cases the transfer pairing can be improved.

PROPOSITION 1.2. *For (G, H, K, L) a 4-tuple of groups and P any pair contained in $T(G, H; K, L)$ there is a factorization (left cosets) of the transfer pairing*

$$\begin{array}{ccc} P \times (K, L) & \longrightarrow & (G, H) \\ \downarrow & & \downarrow p \\ P \times (K/L, *) & \dashrightarrow & (G/H, *). \end{array}$$

Furthermore this factorization induces a pairing

$$J : (\Omega P, *) \times (K/L, *) \to (\Omega(G/H), *)$$

for which there is the commutative square

$$\begin{array}{ccc} \pi_r(P) \otimes \pi_s(K/L) & \xrightarrow{\langle\, ,\rangle} & \pi_{r+s}(G/H) \\ \Big\downarrow \partial \times 1 & J & \Big\downarrow (-1)^s \partial \\ \pi_{r-1}(\Omega P) \otimes \pi_s(K/L) & \longrightarrow & \pi_{r+s-1}(\Omega G/H) \end{array}$$

where $\Omega(X, A)$ denotes the space of paths $(I, 1, 0) \to (X, A, *)$ and ∂ is the transgression of the path-loop fibration.

Proof. Let $k \in K$, $t \in L$, $g \in C_G(L, H)$ and $u \in C_G(K)$. Then

$$[gu, kt] = [g, kt] = [g, k]\,[g, t]\,[[t, g], k].$$

By definition of g, $[[g, t]$ is in H and consequently so is $[g, t]\,[[t, g], k]$. Thus $p[gu, kt] = p[g, k]$ and the factorization exists. The pairing, J, is then obvious and the commutativity of the square is an exercise in the use of representatives of homotopy classes and their adjoints. The sign $(-1)^s$ occurs since the order in which the adjoints are taken do not line up.

In particular if we set $P = F(G, H; K, L)$ in Proposition 1.2, we see that diagram (2), which is a commutative diagram of C-pairings, can be factored. Thus elements, $b \in \pi_k(F)$, can be interpreted as maps of "fibrations" (diagram (3)) – i.e., the map $K/L \to \Omega^k(G/H)$ is the adjoint of a map

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$S^k \wedge K/L \to G/H$ which is defined via a representative of b. Such maps of fibrations can, in turn, be used to define spectra and maps of spectra (see Lundell [**7**]).

$$\begin{tikzcd}
(H, e) \times (H, e) \arrow[r] \arrow[d] & (H, e) \arrow[d] \\
(H, e) \times (G, e) \arrow[r] \arrow[d] & (G, e) \arrow[d] \\
(H, e) \times (G, H) \arrow[r] & (G, H)
\end{tikzcd}
\qquad
\begin{tikzcd}
F \times (L, e) \arrow[r] \arrow[d] & (H, e) \arrow[d] \\
F \times (K, e) \arrow[r] \arrow[d] & (G, e) \arrow[d] \\
F \times (K, L) \arrow[r] & (G, H)
\end{tikzcd}
\qquad
\begin{tikzcd}
L \arrow[r] \arrow[d] & \Omega^k H \arrow[d] \\
K \arrow[r] \arrow[d] & \Omega^k G \arrow[d] \\
K/L \arrow[r] & \Omega^k G/H
\end{tikzcd}$$

diagram (1) diagram (2) diagram (3)

We finally observe from the commutative diagrams, diagram (1) and diagram (2), together with the following proposition that elements in $\pi_*(H)$ or $\Omega_*(F)$ will induce maps of long exact homotopy sequences of pairs.

PROPOSITION 1.3. *If a and b are given by one of the following sets of conditions*
(i) $a \in \pi_r(H)$, $b \in \pi_s(G, H)$
(ii) $a \in \pi_r(F)$, $b \in \pi_s(K, L)$,
then $\partial\langle a, b\rangle = \langle a, \partial b\rangle$ *and* $\partial\langle b, a\rangle = (-1)^r\langle \partial b, a\rangle$ *where ∂ denotes the transgression homomorphism for the appropriate pair.*

Proof. Again (i) is a standard result with a proof that can be adapted to prove (ii).

2. Jacobi identities. In the proof of Proposition 2.1 below we show that Jacobi identities can be verified in a natural way using the Hall identity, $H(3')$.

PROPOSITION 2.1. *Suppose that $(K, L) \subset (G, H)$ are two pairs of topological groups and that a, b, c are given by one of the following sets of conditions:*
(i) $a \in \pi_p(G)$, $b \in \pi_q(G)$, $c \in \pi_r(G)$
(ii) $a \in \pi_p(H)$, $b \in \pi_q(H)$, $c \in \pi_r(G, H)$
(iii) $a \in \pi_p(L)$, $b \in \pi_q(L)$, $c \in \pi_r(R, S)$; $(R, S) \subset T(G, H; H, L)$
(iv) $a \in \pi_p(L)$, $b \in \pi_q(K, L)$, $c \in \pi_r(R, S)$; $(R, S) \subset F(G, H; K, L)$.
Then a, b, c satisfy the following Jacobi identities:

$$(-1)^{pr}\langle a, \langle b, c\rangle\rangle + (-1)^{rq}\langle c, \langle a, b\rangle\rangle + (-1)^{pq}\langle b, \langle c, a\rangle\rangle = 0$$
$$(-1)^{rp}\langle\langle c, b\rangle, a\rangle + (-1)^{qr}\langle\langle b, a\rangle, c\rangle + (-1)^{pq}\langle\langle a, c\rangle, b\rangle = 0$$

where the products in these identities must be interpreted as absolute Samelson products, relative Samelson products or C-products as needed. For example in case (iv) *all elements of the Jacobi identity lie in $\pi_{p+q+r}(G, H)$. Furthermore, $\langle a, b\rangle$, $\langle c, a\rangle$ and $\langle b, c\rangle$ lie in $\pi_{p+q}(K, L)$, $\pi_{p+r}(H)$, and $\pi_{q+r}(G, H)$ respectively. Thus in the product $\langle b, \langle c, a\rangle\rangle$ we must interpret b as an element of $\pi_q(G, H)$ via the inclusion $(K, L) \subset (G, H)$.*

Proof. Since the format of the proof is the same in all cases we prove (iv) only. Let $J^{n-1} = \dot{I}^{n-1} \times I \cup I^{n-1} \times 1$

$$\begin{aligned} f &: (I^p, \dot{I}^p) \to (L, e) \\ g &: (I^q, \dot{I}^q, J^{q-1}) \to (K, L, e) \\ h &: (I^r, \dot{I}^r, J^{r-1}) \to (R, S, e) \end{aligned}$$

represent a, b, and c respectively where the basepoint of the unit n-cube, I^n, is taken to be the origin. We then define basepoint preserving homotopies $H_i : (I^{p+q+r}, \dot{I}^{p+q+r}, *) \times I \to (G, H, e)$ for $i = 1, 2, 3$ as follows

$$\begin{aligned} H_1(x, y, z; t) &= [h(tz)f(x)h^{-1}(tz), [g(y), h(z)]] \\ H_2(x, y, z; t) &= [g(ty)h(z)g^{-1}(ty), [f(x), g(y)]] \\ H_3(x, y, z; t) &= [f(tx)g(y)f^{-1}(tx), [h(z), f(x)]] \end{aligned}$$

where tu denotes the scalar multiplication of a vector.

Now when $t = 0$, H_1, H_2, and H_3 are seen to represent $\langle a, \langle b, c\rangle\rangle$, $(-1)^{(p+q)r}\langle c, \langle a, b\rangle\rangle$ and $(-1)^{p(r+s)}\langle b, \langle c, a\rangle\rangle$ respectively. On the other hand we see from the Hall identity $H(3')$ that at $t = 1$, $H_1H_2H_3 = e$ and the proof is complete.

We remark that the above list of Jacobi identities is not exhaustive and that whenever the various iterated C-products of three elements are all defined then there is probably a Jacobi identity.

3. The C-space structure of the classical groups. It O_n denotes one of the classical groups O_n, U_n, or Sp_n, let $O_{n,k} = O_n/O_{n-k}$ be the Stiefel manifold of k-frames in n-space and $G_{n,k} = O_{n,k}/O_k$ the Grassmann manifold of k-planes in n-space. Our choice of commutator then requires that $O_{n,k}$ be a left coset space which in turn fixes the matrix interpretation of $O_{n,k}$ as the set of matrices consisting of n rows and k orthonormal column vectors.

We next adopt the following conventions regarding imbeddings of these groups. With $p \leqq q$ let $i, i' : O_p \to O_q$ be the imbeddings

$$i(A_p) = \begin{bmatrix} A_p & 0 \\ 0 & I_{q-p} \end{bmatrix}, \quad i'(A_p) = \begin{bmatrix} I_{q-p} & 0 \\ 0 & A_p \end{bmatrix}.$$

If O_p and O_p' denote the image of O_p under i and i' respectively then the pairs (O_q, O_p) and (O_q, O_p') have a precise meaning and we define projections

$$p : (O_q, O_p) \to (O_{q,p-q}, *) \text{ and } p' : (O_q, O_p') \to (O_{q,q-p}', *).$$

Consequently with $m \leqq n$ and $(G, H, K) = (O_{n+k}, O_n, O_{n-m})$ the C-spaces $C_{n+k}(n - m, n) = C_G(K, H)$, $C_{n+k}(n) = C_G(H)$ have a fixed interpretation. We then obtain the following structure theorem using standard techniques of linear algebra.

THEOREM 3.1. *For $m \leqq n$, $C_{n+k}(n - m, n)$ is the set of matrices of the form $i(A)i'(B)$ where $A \in O_n$ and $B \in O_{k+m}$.*

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We shall refer to O_n, O_{k+m}' and $O_m'' = O_n \cap O_{k+m}'$ as the left, right, and overlap subspaces of $C_{n+k}(n - m, n)$. Furthermore, an inclusion

$$C_{n+k}(n - m, n) \subset C_{n'+k'}(n' - m', n')$$

will be called aligned if it preserves this subspace structure and this can happen if and only if $n \leqq n'$, $m \leqq m'$, $m + k \leqq m' + k'$, and $n + k \leqq n' + k'$. Moreover, if equality occurs in any of these inequalities the aligned inclusion will be unique.

With G a topological group suppose that X and Y are right and left G-spaces respectively. Then $X \times Y$ is a right G-space with the action of G defined

$$(X \times Y) \times G \to X \times Y: ((x, y), g) \to (x, y)g = (xg, g^{-1}y).$$

The orbit space will then be denoted $(X \times Y)/G$ and with this notation we therefore see from Theorem 3.1 that for $n > m$, $C_{n+k}(n - m, n) = (O_n \times O_{k+m}')/O_m''$. One should take care, however, not to confuse this notation with the various left and right coset spaces which will be used in the sequel.

We abbreviate the T and F pairs for the tetrad $(O_{n+r}; O_{n+r-s}, O_n, O_{n-s})$ to

$$T_{n,s}{}^r = (C_{n+r}(n - s, n + r - s)/C_{n+r}(n), C_{n+r}(n, n + r - s)/C_{n+r}(n))$$
$$F_{n,s}{}^r = (C_{n+r}(n - s, n + r - s)/C_{n+r}(n), *)$$

so that the transfer pairing now has the following (factored) form.

$$T_{n,s}{}^r \times (O_{n,s}, *) \to (O_{n+r,s}, *)$$

Finally, for $O_r' \subset C_{n+r}(n)$ there are pairs

$$\dot{T}_{n,s}{}^r = (C_{n+r}(n - s, n + r - s)/O_r', C_{n+r}(n, n + r - s)/O_r')$$

which in addition to having properties similar to the pairs $T_{n,s}{}^r$ also give inclusions $\dot{T}_{n,s}{}^r \to \dot{T}_{n+p,s}{}^r$ which are induced from aligned inclusions. The main result of this section is then the following.

THEOREM 3.2. (i) *For* $r, n \geqq s$, $k \geqq 0$; $\pi_k(\dot{T}_{n,s}{}^r) = \pi_k(G_{r+s,s}, G_{r,s})$.

(ii) *For* $p \geqq 0$ *the natural inclusion* $\dot{T}_{n,s}{}^r \to \dot{T}_{n+p,s}{}^r$ *induces isomorphism in homotopy.*

(iii) *For* $k \geqq 0$, $\pi_k(T_{n,s}{}^r) = \pi_k(\dot{T}_{n,s}{}^r)$.

Proof. Let $(X, A) = (C_{n+r}(n - s, n + r - s), C_{n+r}(n, n + r - s))$. Since O_r' acts freely from the right on X and A we see that $\pi_*(X, A) = \pi_*(\dot{T}_{n,s}{}^r)$. For $n \geqq s$, O_{n+r-s} is the left subspace of both X and A and thus acts freely from the left on both spaces. Recalling that $X = (O_{n+r-s} \times O_{r+s}')/O_r''$ and identifying O_{n+r-s} with $(O_{n+r-s} \times O_r'')/O_r''$ we see that there is a fibration

$$O_{n+r-s} \to X \to O_r'/O_{r-s}''$$

and we consequently have

$$\pi_*(X, A) = \pi_*(O_{r+s,s}, O_{r,s}) = \pi_*(G_{r+s,s}, G_{r,s}).$$

For (ii) let (X', A') be the corresponding pair for $\dot{T}_{n+p,s}{}^r$ then the above left group actions on the spaces of the pairs (X, A) and (X', A') commute with inclusion $(X, A) \subset (X', A')$ and the induced map on the quotient spaces is an isomorphism. The proof of (iii) parallels (i).

4. Transfer products in the classical groups. We shall now confront the technical difficulties in relating the transfer product to the intrinsic map of James.

For A and B pointed spaces the join, $A * B$ of A and B is the pointed space obtained from $A \times I \times B$ by identifying $a \times O \times B$ with $a \times O \times (*)$, $a \in A$ and $A \times 1 \times b$ with $(*) \times 1 \times b$, $b \in B$. The basepoint of $A * B$ is the class $(*) \times 1 \times (*)$.

If we now consider an element of $O_{n,k}$ as a matrix of k rows and n columns then James [**4**] defines the intrinsic map $h : O_{m,k} * O_{n,k} \to O_{m+n,k}$ by $h(U, V, t) = (U \cos \theta, V \sin \theta)$ where $\theta = \pi t/2$. The transpose of this map defines the intrinsic map when elements of $O_{m,k}$ have m rows and k columns. In homotopy we get the James join, denoted $a * b$.

With the notation of the preceding section we proceed to define maps

$$\lambda : O_{m,k}' \to \Omega \dot{T}_{n,k}{}^m.$$

First with $\theta = \pi t/2$ and $A \in O_m$ set

$$r(t) = \begin{bmatrix} \sin \theta I_k & 0 & \cos \theta I_k \\ 0 & I_{m-k} & 0 \\ -\cos \theta I_k & 0 & \sin \theta I_k \end{bmatrix}, \quad R(A, t) = r(t) \begin{bmatrix} A & \\ & I_k \end{bmatrix} r^{-1}(t).$$

If $(X, Y) = (C_{n+m}(n - k, n + m - k), C_{n+m}(n, n + m - k))$ then R is a map

$$(O_m, O_{m-k}') \times I \to (X, Y).$$

For $t = 0$ or $A \in O_{m-k}'$ we furthermore have $R(A, t) \in O_m'$. Since $\dot{T}_{n,k}{}^m = (X/O_m', Y/O_m')$ it then follows that the composition $(O_m, O_{m-k}') \times I \to (X, Y) \to \dot{T}_{n,k}{}^m$ factors through $O_{m,k}' \times I$. The adjoint of this factorization is defined to be λ.

Next let Λ^* be the adjoint of the composition

$$O_{m,k}' \wedge O_{n,k} \xrightarrow{\lambda \wedge 1} \Omega \dot{T}_{n,k}{}^m \wedge O_{n,k} \xrightarrow{J} \Omega O_{m+n,k}' \xrightarrow{\varphi} \Omega O_{m+n,k}'$$

where $\varphi(w)(t) = w(1 - t)$ and J is the induced map of Proposition 1.2. We can now state a theorem conjectured by Bott [**2**] and proved by Husseini [**3**].

THEOREM 4.3. *The composition*

$$O_{m,k}' * O_{n,k} \xrightarrow{\pi} O_{m,k} \wedge S^1 \wedge O_{n,k} \xrightarrow{\Lambda^*} O_{m+n,k}$$

where π is the natural projection is homotopic to the intrinsic map, h, except for the orthogonal groups when n, m, and k are all odd in which case the obstruction to their being homotopic is a column operation (4), *of odd order.*

If σ_n is the composition

$$\pi_r(O_{m,k}') \xrightarrow{\lambda} \pi_r(\Omega \dot{T}_{n,k}{}^m) \xrightarrow{\partial^{-1}} \pi_{r+1}(\dot{T}_{n,k}{}^m)$$

we then obtain the needed relationship between the transfer product and the intrinsic join in the following.

THEOREM 4.4. *If $a \in \pi_r(O_{m,k}')$, $b \in \pi_s(O_{n,k})$ then $\langle \sigma_n(a), b\rangle = -V(a * b)$ where $V : O_{m+n,k} \to O_{m+n,k}$ is the identity except for the orthogonal groups when m, n and k are all odd in which case V is a column operation of odd order.*

Proof. This is an immediate consequence of Proposition 1.2 (ii) and Theorem 4.3. The (-1) can be traced to φ in the definition of Λ^*.

With V as in Theorem 4.4 we have:

COROLLARY 4.5. *With $a \in \pi_p(O_m)$, $b \in \pi_q(O_m)$ and $c \in \pi_r(O_m)$*

$$V(a^* \langle b, c\rangle) = \langle a^*b, c\rangle + (-1)^{q(p+1)} \langle b, a^*c\rangle$$

where $\langle\ ,\ \rangle$ denotes the Samelson product on the left side of the identity and the relative Samelson product on the right.

Proof. Consider (see Theorem 3.2) $\dot{T}_{m,m}{}^m$ as included in $\dot{T}_{2m,m}{}^m$ and let $\bar{a} = \sigma_m(a) \in \pi_{p+1}(\dot{T}_{m,m}{}^m)$. Then from Proposition 2.1 (iii) with $L = O_m$ and $(R, S) = \dot{T}_{m,m}{}^m$ we have

$$(-1)^{(p+1)r} \langle \bar{a}, \langle b, c\rangle\rangle + (-1)^{rq} \langle c, \langle \bar{a}, b\rangle\rangle + (-1)^{q(p+1)} \langle b, \langle c, \bar{a}\rangle\rangle = 0.$$

From Proposition 1.1, we have $\langle c, \bar{a}\rangle = (-1)^{r(p+1)-1} \langle \bar{a}, c\rangle$ and consequently Theorem 4.4 gives the identity

$$(-1)^{\alpha} V(a^* \langle b, c\rangle) + (-1)^{\beta} \langle c, U(a^*b)\rangle + (-1)^{\gamma} \langle b, W(a^*c)\rangle = 0$$

where U and W correspond to the V of Theorem 4.4 and where $\alpha = (p + 1)r$, $\beta = rq$ and $\gamma = (q + r)(p + 1) - 1$. Since a^*b and a^*c both lie in $\pi_*(O_{2m,m})$ we see that $U = W = 1$. The stated identity then follows from the commutativity relation $\langle c, a^*b\rangle = (-1)^{\delta} \langle a^*b, c\rangle$ with $\delta = r(p + q + 1) - 1$.

5. The unitary groups. With the notation of Section 3 we have:

LEMMA 5.1. *For the unitary groups with $F = F_{n,1}{}^1$ we have $\pi_1(F) = 0$ and $\pi_2(F) = Z$.*

Proof. For the unitary groups it is readily seen that

$$C_{n+1}(n-1, n) = (U_n \times U_2')/U_1''; \quad C_{n+1}(n) = S^1 I_n \times S^1.$$

Consequently $C_{n+1}(n)$ acts freely on $C_{n+1}(n - 1, n)$ from both the left and

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the right. The left coset space is by definition, F, and we will denote the right coset space by $\hat{F}$. Then

$$\pi_*(F) = \pi_*(C_{n+1}(n-1, n), C_{n+1}(n)) = \pi_*(\hat{F}).$$

To compute $\pi_*(\hat{F})$ define a projection

$$p : \hat{F} \to PU_n \times G_{2,1}$$

by $p(i(A)i'(B)) = [Ai'(\det(A^{-1}))] \times [i(\det(A))B]$ where the brackets denote classes in the projective unitary group, PU_n, and $G_{2,1} = S^2$. To evaluate the fiber of p we see that $Ai'(\det(A^{-1})) \in S^1 I_n$ if and only if $\det(A) = 1$. Hence $A \in Z_n I_n$ and there is a fibration

$$Z_n \to \hat{F} \to PU_n \times G_{2,1}$$

from which the lemma follows.

THEOREM 5.2. *If $\beta_n \in \pi_2(F)$ is an appropriate generator and $\hat{B}_n : \pi_r(S^{2n-1}) \to \pi_{r+2}(S^{2n+1})$ is the map $\hat{B}_n(a) = \langle \beta_n, a \rangle$, then $\hat{B}_n = n\Sigma^2$ (Σ = suspension).*

Proof. With the notation of Lemma 5.1 there is a fibration of right coset spaces

$$\hat{P} \to \hat{F} \to \hat{T}$$

where $\hat{P} = C_{n+1}(n, n)/C_{n+1}(n)$ and $\hat{T} = C_{n+1}(n-1, n)/C_{n+1}(n, n)$. Now $C_{n+1}(n, n) = U_n \times S^1$ acts freely from the left on $C_{n+1}(n-1, n)$ while $C_{n+1}(n)$ acts freely from both the right and the left on $C_{n+1}(n, n)$. Consequently from Theorem 3.2 we see that $\pi_*(\hat{T}) = \pi_*(T) = \pi_*(G_{2,1}) = \pi_*(S^2)$ where $T = T_{n,1}{}^1 = (C_{n+1}(n-1, n)/C_{n+1}(n), C_{n+1}(n, n)/C_{n+1}(n))$ is the T-pair of left cosets. Since it is also readily seen that $\hat{P} = PU_n$, we have the equivalences $\pi_*(\hat{P}) = \pi_*(PU_n)$, $\pi_*(\hat{F}) = \pi_*(F)$, and $\pi_*(T) = \pi_*(\hat{T}) = \pi_*(S^2)$ which together with the above fibration and the above lemma yields the short exact sequence

$$O \longrightarrow \pi_2(F) \xrightarrow{\pi} \pi_2(T) \longrightarrow \pi_1(PU_n) \longrightarrow 0$$

where π is induced from the inclusion $F \subset T$. Finally $\pi_1(PU_n) = Z_n$ and so π is multiplication by n. The stated result then follows from Theorem 4.4 with $m = k = 1$.

THEOREM 5.3. *If $B_n : \pi_r(U_n) \to \pi_{r+2}(U_{n+1})$ is the map $B_n(a) = \langle \beta_n, a \rangle$ then*

$$B_n \langle a, b \rangle = \langle B_n a, b \rangle + \langle a, B_n b \rangle$$

so B_n maps as a derivation on the Samelson ring.

Proof. This is an easy consequence of Proposition 2.1 (iii).

We now define a generator of $\pi_2(C_{n+1}(n-1, n), C_{n+1}(n)) = \pi_2(F)$. Let $r(t)$ be the 2×2 rotation matrix used in defining λ in Section 4 when $m = k = 1$.

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Set $u(s, t) = [i(e^{i2\pi s}), r(t)] = [i'(e^{-i2\pi s}), r(t)] \in U_2$ and inductively define $\gamma_{n+1}(s, t) = i(\gamma_n(s, t)i'(u(ns, t))$ in U_{n+1} where $\gamma_2(s, t) = u(s, t)$. Then $\gamma_{n+1}(0, t) = \gamma_{n+1}(1, t) = \gamma_{n+1}(s, I) = I_{n+1}$ and $\gamma_{n+1}(s, 0) = i(e^{i2\pi s}I_n)i'(e^{-i2\pi n s}) \in C_{n+1}(n)$. We leave it as an exercise to the reader to trace γ_{n+1} through Theorem 4.4 to verify that it is a representative of β_n with the proper orientation.

For the transfer pairing, γ_{n+1}, has the important property that for $A \in U_{n-1}$, $[\gamma_{n+1}, i(A)] = i[\gamma_n, iA]$ and so induces a map of n-tuples

$$(S^2 \wedge U_n, S^2 \wedge U_{n-1}, \ldots, S^2 \wedge U_1) \to (U_{n+1}, U_n, \ldots, U_2)$$

of which an immediate consequence is the following.

THEOREM 5.4. *There is a homotopy ladder of long exact sequences with commutative squares* ($k \leqq l \leqq n$)

$$\begin{array}{ccccccc}
\ldots \to & \pi_r(U_l, U_k) & \to & \pi_r(U_n, U_k) & \to & \pi_r(U_n, U_l) & \to \ldots \\
& \downarrow B_l & & \downarrow B_n & & \downarrow B_n & \\
\ldots \to & \pi_{r+2}(U_{l+1}, U_{k+1}) & \to & \pi_{r+2}(U_{n+1}, U_{k+1}) & \to & \pi_{r+2}(U_{n+1}, U_{l+1}) & \to \ldots
\end{array}$$

and when $l = n - 1$, $B_n = n\Sigma^2$.

THEOREM 5.5. *The map* $B_n : \pi_r(U_n) \to \pi_{r+2}(U_{n+2})$ *is an isomorphism for* $r \leqq 2n - 1$ *and when* $r = 2n$, B_n *maps a generator of* $\pi_{2n}(U_n) = Z_{n!}$ *to* $n + 1$ *times a generator of* $\pi_{2(n+1)}(U_{n+1}) = Z_{(n+1)!}$.

Proof. This is a straightforward induction using Theorem 5.4 with $k = 0$ and $l = n - 1$.

Consequently B_n has all the properties of Lundell's deformation [**6**, **7**] of Bott's suspension.

THEOREM 5.6. *For* $a \in \pi_r(U_n, U_k)$ *and* $b \in \pi_s(U_k)$

$$B_n\langle a, b\rangle = \langle B_n(a), i_1(b)\rangle + \langle i_2(a), B_n(b)\rangle$$

where all products are relative Samelson products and i_1, i_2 *are the inclusions*

$$i_1 : \pi_s(U_k) \to \pi_s(U_{k+1}),\ i_2 : \pi_r(U_n, U_k) \to \pi_r(U_{n+1}, U_{k+1}).$$

Thus, when $k = n - 1$,

$$B_n\langle a, b\rangle = n\langle \Sigma^2(a), i_1(b)\rangle.$$

Proof. This is a straightforward application of Proposition 2.1 (iv).

Remarks. We remark that there are transfer pairings for the orthogonal and sympletic groups which factor the usual Bott suspensions, however, even these factorizations do not seem to go far enough for non-stable homotopy theory. In any event it does seem significant that for all the classical groups, the James join, the Bott suspension, the Samelson product, and Jacobi identities fit together in such an intricate manner.

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