SEMIRINGS WITH A COMPLETELY 0-SIMPLE ADDITIVE SEMIGROUP

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In a previous paper [2], we gave an explicit description of the structure of all semirings with a completely simple additive semigroup. The next step is then clearly to consider semirings with a completely 0-simple additive semigroup. We are able to classify these semirings according to the multiplicative nature of their additive zero. Let R be a semiring whose additive semigroup is completely 0-simple with zero ∞. First, if $\infty\infty \neq \infty$, then the multiplication of R is trivial. Besides these trivial semirings, another class of semirings with a completely 0-simple additive semigroup can be easily obtained by adjoining an element ∞ which is together an additive zero and a multiplicative zero. We show that every semiring, with a completely 0-simple additive semigroup whose zero ∞ is a multiplicative zero, can be constructed in that manner. If now ∞ is a right (left) zero, but not a left (right) zero of $(R, \cdot)$, then $R - \{\infty\}$ is a subsemiring of $R, xy = \infty y$ for all $x, y \in R$, and the multiplication of R can be completely described in terms of $(R, +) = \mathscr{M}_\infty(G, I, \Lambda, P)$ and idempotent mappings defined on I and Λ. If, finally, ∞ is a multiplicative idempotent, but is neither a left zero, nor a right zero, of $(R, \cdot)$, then the multiplication of R is defined by: $\infty y = x\infty = xy = e$, $\infty\infty = \infty$ for all $x, y \in R - \{\infty\}$, where e is some additive idempotent.

We essentially follow the notation of [1]. However, if R is a semiring with a completely 0-simple additive semigroup, we shall always denote by ∞ the zero of $(R, +)$, and by E the set of additive idempotents. Rees matrix semigroups $S = \mathscr{M}_\infty(G, I, \Lambda, P)$ will be denoted additively, and G^∞ considered as an additive group with zero ∞; elements of S will be written as triples (i, a, λ) with $i \in I$, $\lambda \in \Lambda$, $a \in G^\infty$, and added according to the formula: $(i, a, \lambda) + (j, b, \mu) = (i, a + p_{\lambda, j} + b, \mu)$ for all $i, j \in I$, $\lambda, \mu \in \Lambda$ and $a, b \in G^\infty$. Note that, trivially, if $x, y \in S - \{\infty\}$, then $x + y = y + x$ implies that x and y belong to the same $\mathscr{H}$-class of S.

1. Preliminary results

Let R be a semiring with a completely 0-simple additive semigroup. We easily see that the Green's relations $\mathscr{L}, \mathscr{R}, \mathscr{D}$ and $\mathscr{H}$ of $(R, +)$ are multiplicative congruences. Our first result shows that, except when $R^2 = \{\infty\}$, $R^* = R - \{\infty\}$ is a subsemigroup of $(R, \cdot)$.

THEOREM 1. *Let R be a semiring such that $(R, +)$ is a completely 0-simple semigroup with zero ∞ and that $R^2 \neq \{\infty\}$. Then $xy \neq \infty$ for all $x, y \in R^*$.*

PROOF. Assume that, on the contrary, there exist $x, y \in R^*$ such that $xy = \infty$. Then since $(R, +)$ is 0-bisimple, we have $x \mathscr{D} z$ and $y \mathscr{D} t$ for all $z, t \in R^*$; since $\mathscr{D}$ is a multiplicative congruence, it follows that $xy \mathscr{D} zt$. Therefore $zt = \infty$ for all $z, t \in R^*$. Furthermore, if $z \in R^*$, we have:

$$z\infty = z(\infty + z) = z\infty + zz = z\infty + \infty = \infty,$$

and similarly $\infty z = \infty$. Also

$$\infty\infty = \infty(\infty + x) = \infty\infty + \infty x = \infty\infty + \infty = \infty.$$

We conclude that $R^2 = \{\infty\}$, which contradicts our assumption. Thus we must have $xy \neq \infty$ for all $x, y \in R^*$.

From now on, we shall always assume that $R^2 \neq \{\infty\}$. The next result is fundamental for the study of the structure of semirings with a completely 0-simple additive semigroup, as it provides a first useful classification of these semigroups.

THEOREM 2. *Let R be a semiring such that $(R, +)$ is a completely 0-simple semigroup and that $R^2 \neq \{\infty\}$. Then*

i) *If $\infty\infty \neq \infty$, R is reduced to one element.*

ii) *If $\infty\infty = \infty$, R^* is a subsemiring of R with a completely simple additive semigroup, and R is obtained by adjoining the additive zero ∞ to R^*.*

PROOF. Let us first assume that $\infty\infty \neq \infty$. Then, if $x\infty = \infty$ for some $x \in R^*$, we have:

$$\infty\infty = (\infty + x)\infty = \infty\infty + x\infty = \infty\infty + \infty = \infty,$$

which is a contradiction. Thus we must have $x\infty \neq \infty$ for all $x \in R$. Furthermore

$$x\infty + \infty\infty = (x + \infty)\infty = \infty\infty = (\infty + x)\infty = \infty\infty + x\infty$$

for all $x \in R^*$, and it follows from the fact that $(R, +)$ is completely 0-simple that $x\infty \mathscr{H} \infty\infty$ for all $x \in R^*$. Noting that then $\infty\infty$ and $x\infty$ are two idempotents of $(R, +)$ belonging to the same $\mathscr{H}$-class, we conclude that $x\infty = \infty\infty$. Similarly we have $\infty x = \infty\infty$ for all $x \in R^*$.

Let now $x, y \in R^*$; by the above $x\infty = \infty y = \infty\infty$, so that we have:

$$xy + \infty\infty = xy + \infty y + x\infty + \infty\infty = (x+\infty)(y+\infty) = \infty\infty =$$
$$(\infty + x)(\infty + y) = \infty\infty + x\infty + \infty y + xy = \infty\infty + xy.$$

Since $\infty\infty \neq \infty$ by assumption, and $xy \neq \infty$ by Theorem 1, it follows that $xy\mathscr{H}\infty\infty$. Since $xy + \infty\infty = \infty\infty$ by the above, we must have $xy = \infty\infty$. Therefore we have shown that $R^2 = \{\infty\infty\}$.

To prove ii), let now $\infty\infty = \infty$. By Theorem 1, it is clearly enough to show that R^* is a subsemigroup of $(R, +)$. Suppose that, on the contrary, there exists an element t of R such that $t + t = \infty$. Then we have:

$$t\infty = t(\infty + \infty) = t\infty + t\infty = (t+t)\infty = \infty\infty = \infty.$$

Since $\mathscr{D}$ is a multiplicative congruence, and $x\,\mathscr{D}\,t$ for all $x \in R^*$, it follow that $x\infty = \infty$ for all $x \in R^*$. Let e be an additive idempotent different from ∞; then we have in particular $e\infty = \infty$, whence

$$et = (e+e)t = et + et = e(t+t) = e\infty = \infty,$$

which is a contradiction, since $et \neq \infty$ by Theorem 1. Thus we must have $t + t \neq \infty$ for all $t \in R^*$, which proves that R^* is a subsemiring of R.

2. Structure theorems

From now on we assume that $R^2 \neq \{\infty\}$ and that ∞ is a multiplicative idempotent, thus excluding the trivial case when R^2 is reduced to one element. Clearly, whenever R^* is a semiring with a completely simple additive semigroup, we can construct a semiring with a completely 0-simple additive semigroup by adjoining to R^* an element which is together an additive zero and a multiplicative zero. Conversely, it trivially follows from Theorem 2 ii) that we have the following:

THEOREM 3. *Let R be a semiring such that $(R, +)$ is a completely 0-simple semigroup with zero ∞ such that $R^2 \neq \{\infty\}$ and that ∞ is a multiplicative zero. Then R^* is a subsemiring of R with a completely simple additive semigroup, and R is obtained by trivial adjunction of the additive and multiplicative zero ∞ to R^*.*

Therefore, Theorem 4 of [2], which describes the structure of semirings with a completely simple additive semigroup, trivially extends to give the structure of semirings with a completely 0-simple additive semigroup whose zero is also a multiplicative zero.

We now study the case when ∞ is a multiplicative idempotent, but is not a zero of $(R, \cdot)$. For instance, we assume in the next two lemmas that ∞ is not a left zero of $(R, \cdot)$.

LEMMA 4. *Let R be a semiring such that $(R, +)$ is a completely 0-simple semigroup with zero ∞. and that $\infty\infty = \infty$. If ∞ is not a left zero of $(R.\cdot)$, then $\infty y \neq \infty$ for all $y \in R^*$, and the mapping g defined by: $g(y) = \infty y$ for all $y \in R$, is an idempotent homomorphism of $(R, +)$ into the set E of additive idempotents.*

PROOF. Assume that $\infty x \neq \infty$ for some $x \in R^*$. Then, for all $y \in R^*$, we have: $x \mathscr{D} y$, and, since $\mathscr{D}$ is a multiplicative congruence, it follows that $\infty x \mathscr{D} \infty y$. Thus $\infty y \neq \infty$ for all $y \in R^*$. The last part of the statement is then clear.

LEMMA 5. *Let R be a semiring with a completely 0-simple additive semigroup with zero ∞ such that $\infty\infty = \infty$. If ∞ is not a left zero of $(R, \cdot)$, then we have: $xy = \infty y$ for all $x, y \in R^*$.*

PROOF. Let $x, y \in R^*$. Then $xy \neq \infty$ by Theorem 1, and $\infty y \neq \infty$ by Lemma 4. Therefore we have $xy \mathscr{D} \infty y$. Also

$$xy + \infty y = (x + \infty)y = (\infty + x)y = \infty y + xy,$$

whence $xy \mathscr{H} \infty y$. As an additive idempotent, ∞y is the identity of its $\mathscr{H}$-class, and it follows that $xy = xy + \infty y = \infty y$ for all $x, y \in R^*$.

This last lemma shows that, if $\infty\infty = \infty$, but ∞ is not a left zero of $(R, \cdot)$, then the multiplication of R is completely determined by all products ∞y and $y\infty$ with $y \in R$.

First, if ∞ is neither a left zero, not a right zero, of $(R, \cdot)$, then $x\infty \neq \infty$ and $xy = x\infty$ for all $x, y \in R^*$, by the duals of Lemma 4 and 5. Thus $x\infty = xy = \infty y$ for all $x, y \in R^*$. Thus we easily obtain the following

THEOREM 6. *Let R^* be any completely simple additive semigroup, and R be the semigroup resulting from the adjunction of a zero ∞ to R^*. Then, for every idempotent e of R^*, the multiplication defined by:*

$$x\infty = \infty y = xy = e, \quad \infty\infty = \infty$$

for all $x, y \in R^*$, defines a structure of semiring on R. Conversely, every semiring R, with a completely 0-simple additive semigroup with zero ∞ such that $\infty\infty = \infty$, but ∞ is neither a left, nor a right, zero of $(R, \cdot)$, is obtained in this way.

If now ∞ is a right zero, but not a left zero, of $(R, \cdot)$, then the structure of R can be described as follows:

THEOREM 7. *Let R^* be any completely simple additive semigroup, and R be the semigroup resulting from the adjunction of a zero ∞ to R^*. Given an idempotent homomorphism g of R into the set E of idempotents of R such that $g(\infty) = \infty$, the multiplication defined by: $xy = g(y)$ for all $x, y \in R$,* *defines a structure of semiring on R. Conversely, every semiring R with a completely 0-simple additive semigroup with zero ∞ such that ∞ is a right zero, but not a left zero, of $(R,\cdot)$, is obtained in this way.*

PROOF. Let first R be as in the direct part of the statement, and g be an idempotent homomorphism of R into E. Then we have:

$$(xy)z = g(y)z = g(z) = g^2(z) = xg(z) = x(yx)$$

$$(x+y)z = g(z) = g(z) + g(z) = xz + yz$$

$$z(x+y) = g(x+y) = g(x) + g(y) = zx + zy$$

for all $x, y, z \in R$. Thus R, together with this multiplication is a semiring. The converse follows clearly from Lemmas 4 and 5.

We can use the description of homomorphisms of completely 0-simple semigroups given in [1] to translate this last result in terms of Rees matrix semigroups. We then get:

THEOREM 8. *Let $R = \mathscr{M}_\infty(G, I, \Lambda, P)$ be a completely 0-simple semigroup resulting from the adjunction of the zero to $R^* = \mathscr{M}(G, I, \Lambda, P)$. Let ϕ and ψ be idempotent mappings of I into I and Λ into Λ, respectively, such that $p_{\psi\lambda,\phi i} = v_\lambda + u_i$ for all $i \in I$ and $\lambda \in \Lambda$, where $u_i, v_\lambda \in G$. Then, together with the multiplication defined by:*

$$\infty y = xy = (\phi j, -p_{\psi\mu,\phi j}, \psi\mu), \quad x\infty = \infty\infty = \infty$$

for all $x, y = (j, b, \mu) \in R^*$, R is a semiring. Conversely, every semiring R with a completely 0-simple additive semigroup with zero such that ∞ is a right zero, but not a left zero, of $(R,\cdot)$ is isomorphic to such a semiring.

PROOF. Let R, ϕ and ψ be as in the direct part of the statement, and define a mapping g of R into the set E of idempotents of R by: $g(y) = (\phi j, -p_{\psi\mu,\phi j}, \psi\mu)$ if $y = (j, b, \mu) \in R^*$ and $g(\infty) = \infty$. Then we have

$$\begin{aligned} g(x) + g(y) &= (\phi i, -p_{\psi\lambda,\phi i}, \psi\lambda) + (\phi j, -p_{\psi\mu,\phi j}, \psi\mu) \\ &= (\phi i, -p_{\psi\lambda,\phi i} + p_{\psi\lambda,\phi j} - p_{\psi\mu,\psi j}, \psi\mu) = (\phi i, -u_i - v_\lambda + v_\lambda + u_j \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - u_j - v_\mu, \psi\mu) \\ &= (\phi i, -u_i - v_\mu, \psi\mu) = (\phi i, -p_{\psi\mu,\phi j}, \psi\mu) = g(x+y); \end{aligned}$$

$$g(x) + g(\infty) = g(x) + \infty = \infty = g(\infty) = g(x+\infty),$$

and similarly $g(\infty) + g(x) = g(\infty + x)$; $g(\infty) + g(\infty) = \infty = g(\infty)$; for all $x = (i, a, \lambda), y = (j, b, \mu) \in R^*$. Thus g is an additive homomorphism. That g is idempotent results trivially from the fact that ψ and ϕ are idemponet mappings. Since the multiplication defined on R clearly satisfies $xy = g(y)$ for all $x, y \in R$, *it follows from Theorem 7 that R is a semiring.*

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Conversely, let now R be a semiring, with a completely 0-simple additive semigroup with zero ∞ such that ∞ is a right zero, but not a left zero, of $(R, \cdot)$ Then, by Theorem 7, the mapping g defined by $g(y) = \infty y$ for all $y \in R$, is an idempotent homomorphism of $(R, +)$ into E such that $xy = g(y)$ for all $x, y \in R$. By Theorem 3.11 of [1], we can describe g as follows: Let $i \to u_i'$ and $\lambda \to v_\lambda$ be mappings of I and Λ, respectively, into G^∞. Let ϕ and ψ be mappings of I into I and Λ into Λ, respectively. Let ω be a non-trivial homomorphism of G^∞ into G^∞ such that $\omega(p_{\lambda,i}) = v_\lambda' + p_{\psi\lambda,\phi i} + u_i'$ for all $i \in I$ and $\lambda \in \Lambda$. Then

$$g(j, b, \mu) = (\phi j, u_j' + \omega(b) + v_\mu', \psi\mu)$$

for all $(j, b, \mu) \in R^*$. Since g is idempotent, the mappings ϕ and ψ are clearly idempotent too, Now, for all $y = (j, b, \mu) \in R^*$, $g(y)$ is an additive idempotent contained in $H_{\phi j,\psi\mu}$, so that we must have: $g(y) = (\phi j, -p_{\psi\mu,\phi j}, \psi\mu)$. In particular $u_j' + \omega(b) + v_\mu' = -p_{\psi\mu,\phi j}$ for all $j \in I$, $\mu \in \Lambda$ and $b \in G$. Thus $\omega(b) = \omega(0) = 0$ for all $b \in G$, and $v_\mu' + p_{\psi\mu,\phi j} + u_j' = \omega(p_{\mu,j}) = 0$ for all $j \in I$ and $\mu \in \Lambda$. Therefore, setting $u_i = -u_i'$, $v_\lambda = -v_\lambda'$, we get $p_{\psi\lambda,\phi i} = v_\lambda + u_i$ for all $i \in I$ and $\lambda \in \Lambda$, which completes the proof.

References

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