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JOHN ALISON, Esq., M.A. F.R.S.E., Vice-President, in the Chair.

## On the relation between the stereographic projections of points of a plane related to one another by inversion.

By CHARLES TWEEDIE, M.A.

What is meant will best be understood by the following :—

1. Suppose we have an unlimited straight line XX′ (fig. 24) with a finite point O on it. Take two points on the line, C and $C_1$ in the same sense with respect to O such that $OC \cdot OC_1 = a^2$. Then C and $C_1$ are connected by inversion, and O is the centre inversion. Without loss of generality for what follows we may, for convenience, take

$$a^2 = 1^2 = 1,$$

so that
$$OC \cdot OC_1 = 1.$$

Then if we take two points on the line such that OK = OK′ = 1, we obviously divide the straight line into two parts—KK′ and the unlimited part outside KK′—in such a way that to any point C outside KK′ there is one and only one corresponding point $C_1$ within KK′; likewise $C_1$ has only the corresponding point C.

2. Take now a circle with KK′ as diameter and DD′ as diameter perpendicular to KK′. Project on this circle from D the unlimited straight line XX′. To any point on the line there will be one and only one point on the circle. The part of XX′ outside KK′ projects into the semicircle KDK′, while KK′ projects into KD′K′.

Let $\gamma$ be the projection of C, $\gamma_1$ that of $C_1$ the inverse of C. Join $O\gamma$ and $O\gamma_1$. Denote the angle ODC by $\phi$, the angle $ODC_1$ by $\phi_1$, the angle $KO\gamma$ by $\chi$, and the angle $KO\gamma_1$ by $\chi_1$.

Let
$$OC = r,$$
then
$$OC_1 = 1/r.$$
Then
$$\tan\phi = r, \ \tan\phi_1 = 1/r$$

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$\therefore \qquad \phi + \phi_1 = \pi/2,$

$\therefore \qquad D'O\gamma + D'O\gamma_1 = 2\phi + 2\phi_1 = \pi,$

that is, $\qquad \pi/2 + \chi + \pi/2 - \chi_1 = \pi,$

$\therefore \qquad \chi = \chi_1;$

that is, $\gamma_1$ is the reflection in XX′ of $\gamma$; and now from the obvious symmetry we see that, given any point C, to find its inverse $C_1$ all we have to do is to find $\gamma$ by projection and then join D′ to $\gamma$, cutting OX in $C_1$.

Or we may say, if $\gamma$ is characterised by $\chi$ on the circle, then $\gamma_1$ is characterised by $-\chi$.

3. Take now the figure so obtained, and with DD′ as axis give it a complete revolution (fig. 25).

XX′ then generates a plane, the circle KDK′ a sphere; KK′ generates a circular area $K\gamma K'\gamma'$ in the plane, such that to any point of the plane outside it $C(r, \theta)$ there corresponds one inverse point $C_1(1/r, \theta)$ within it, and *vice versa.*

C and $C_1$ project into $\gamma$ and $\gamma_1$ on the sphere which are now characterised by $(\theta, \chi)$ and $(\theta, -\chi)$, so that the one is the reflection, in the plane, of the other. And if C trace out any curve in the plane, while $C_1$ traces out its inverse, then $\gamma$ will trace out a curve $\Gamma$ on the sphere, while $\gamma_1$ will trace out $\Gamma_1$—its reflection in the plane, and therefore an exactly similar though not congruent curve.

From obvious symmetry it is also possible to obtain $\Gamma$ and $\Gamma_1$ from the trace of C by projecting it respectively with respect to D and to D′.

## II.

4. Take now the more important case, and closely connected with the former, of points of an Argand plane (fig. 26) connected by the relation $zz_1 = 1$, $z$ and $z_1$ being the representatives of the two points C and $C_2$ so related.

5. Writing $z$ in the form

$z = r(\cos\theta + i\sin\theta)$, ($r$ and $\theta$ being the polar co-ordinate of C),

we have $\qquad z_1 = 1/r(\cos\theta + i\sin\theta) = (\cos\theta - i\sin\theta)/r;$

and hence the polar co-ordinates of $C_2$ are $(1/r, -\theta)$.

Therefore for every point C outside the circle $K\gamma K'$ we have a

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point $C_2$ within it and only one. In fact, $C_2$ is simply the reflection in XX′ of $C_1$ the inverse of C.

Moreover, since $C_1$ and $C_2$ are obviously situated on the same circle round $O_1$ when we project on the same sphere as before the $\chi_1$ of $C_1$ and the $\chi_2$ of $C_2$ are the same; so that if $(\theta, \chi)$ characterise $\gamma$ the projection of C on the sphere, then $(-\theta, -\chi)$ will characterise $\gamma_2$ the projection of $C_2$.

Therefore $\gamma_2$ is the reflection in the straight line XX′ of $\gamma_1$, and it is only in particular cases of symmetry about XX′ of the traces of C and $C_2$ that the curves $\Gamma$ and $\Gamma_2$ will, taken as a whole, be the reflections of each other in the plane.

But in another sense the symmetry with regard to D and D′ is greater. Looking on $D\gamma$ and $D'\gamma_2$ as indexes tracing out $\Gamma$ and $\Gamma_2$, we see that if $D\gamma$ start to move tracing out any curve $\Gamma$, then the motion of $D'\gamma_2$ looking at it from D′ is exactly the same in every respect in the tracing out of $\Gamma_2$.

6. The case of Inversion may be looked on as a correspondence of points $z'$ and $z$ in an Argand plane through the relation $z'\bar{z} = 1$, where $\bar{z}$ is the conjugate of $z$.

For
$$z' = 1/\bar{z} = 1/r(\cos\theta - i\sin\theta)$$
$$= (\cos\theta + i\sin\theta)/r.$$

so that the polar co-ordinates of $z'$ and $z$ are $(1/r, \theta)$ and $(r, \theta)$ respectively.

7. From the projection on the sphere it is easy to see what curves, taken as a whole, are transformed into themselves. The simplest case is that of the circle, (and it is assumed that circles of the plane project into circles of the sphere, and *vice versa*).

*In the first case, that of Inversion, the only circles which reflect* themselves in the plane are those which cut orthogonally the great circle in which the plane meets the sphere. Their projections in the plane therefore are circles obviously cutting the equatorial circle orthogonally. It is equally obvious that the latter circle goes over point for point into itself.

In the second case, the circles must mirror themselves in XX′.

There are two sets of circles which do this :—

1°. All great circles passing through K and K′; and 2°. all circles of the sphere, whose planes are perpendicular to XX′, and

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therefore cutting the first set, including the equatorial circle orthogonally.

In Projection, the first set goes into circles passing through K and K′; the second set into circles orthogonal to the circle $K\gamma K'$, and whose centres obviously lie on XX′. The two sets being orthogonal on the sphere, their projections on the plane may also be shown to be orthogonal. The only two points which go into themselves are the two K and K′.

These results are thus simply read off from the appearance of the projections on the sphere.

8. We might look on the whole operationally as follows :—

Let V operating on $z$ or $V(z)$ produce $1/z$. Let W denote the corresponding operation on the corresponding point $\gamma$ on the sphere, so that $W(\gamma)$ denotes a mirroring with respect to the straight line XX′ of $\gamma$ or $W(\gamma) = \gamma_2$.

Then $$V^2(z) = V(V)(z) = V(1/z) = z$$

∴ $$V^2 = 1;$$

and obviously $$W^2(\gamma) = W(\gamma_2) = \gamma,$$

since it is simply reflected back,

∴ $$W^2 = 1.$$

Similarly let $\overline{V}$ denote the operation on $z$

$$\overline{V}(z) = (\overline{1/z}) = 1/(x - yi) = 1/r(\cos\theta - i\sin\theta)$$

corresponding to Inversion, and $\overline{W}$ the corresponding operation on $\gamma$ of the sphere, *i.e.*, reflection of $\gamma$ in the equatorial plane into $\gamma_2$.

Then $$\overline{V}^2(z) = \overline{V}(\overline{V}(z)) = \overline{V}(1/z) = z$$

∴ $$\overline{V}^2 = 1,$$

and similarly $$\overline{W}^2 = 1.$$

9. For example; to find the points which go into themselves in Inversion, we have to put $\overline{V}(z) = z$

or $$1/(x - yi) = x + yi$$

∴ $$x^2 + y^2 = 1$$

*i.e.*, every point in the equatorial circle goes over into itself as the operation W obviously allows.

For the second case we have to put

$$V(z) = z \quad i.e., \ (1/z) = z$$

$$\therefore \qquad x^2 - y^2 + 2xyi = 1$$

$$\therefore \qquad xy = 0 \text{ and } x^2 - y^2 = 1$$

with the solutions

$$(1)\ x = 0 \qquad y = \pm i$$
$$(2)\ y = 0 \qquad x = \pm 1.$$

The 1° solution on application of W is easily shown to be false, while the 2° solution, corresponding to K and K′, satisfies the conditions.

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## Experimental Introduction to the Study of Magnetism.

By Professor C. G. Knott, D.Sc., F.R.S.E.

The aim of the scientific teacher is to teach the pupil how to think along scientific lines. By a suitable presentation of the facts of experience he should lead the mind of the learner to form almost intuitively the scientific law or generalisation which embraces them all. We may of course start with the law or formula, and develope it mathematically into all its ramifications. But that reduces itself to mere analytical skill. If carried out faithfully in the elementary teaching of science, it would tend to give the learner an erroneous conception of the whole method of scientific investigation and the meaning of scientific law. On the other hand, if we simply present a series of curious experiments without distinct intellectual linkage, we place the learner in the position somewhat of the old lady who, after reading through the dictionary, remarked that "it was vera interestin' readin', but a wee disconnecked."

In what follows I propose to indicate a course on magnetism which seems to lead naturally to a scientific grasp of the fundamental principles of the science. At the outset I may state briefly what seem to be the faults of the courses usually given in books, and presumably in schools and colleges.

First, from the very outset, a magnet is regarded as made up of two parts. Now there is a great advantage in certain problems of