Seventh Meeting, May 11th, 1894.

Dr C. G. Knott, President, in the Chair.

On the Division of a Parallelepiped into Tetrahedra without making new corners.

By Professor Crum Brown.

The method employed in this paper is first to ascertain in how many ways a cube can be cut into tetrahedra without making new corners, and then, taking each of these divisions of the cube as the type of a genus of divisions of the general parallelepiped, determining the number of species in each genus.

We first fix a system of notation of the corners of the cube. Call any corner A and let the three corners at the other ends of face-diagonals from A, be called B, C, and D, noting them in the positive sense as seen from A. Then let $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$, respectively be the corners at the opposite ends of body-diagonals from A, B, C and D. So that passing along an edge we change both letter and sign, passing along a face-diagonal we change letter but not sign, passing along a body diagonal we change sign but not letter.

The number of distinct forms of tetrahedra which can be cut out of a cube without making new corners is five. We shall give each of them a symbol and indicate one position of each in the cube by noting its four corners. 1st. Ω, ABCD, this is the regular tetrahedron, its volume is $\frac{1}{3}$ of that of the cube. 2nd. Δ, $A\bar{B}\bar{C}\bar{D}$, this tetrahedron has as one of its corners an undivided corner of the cube, its volume is $\frac{1}{6}$ of the volume of the cube. 3rd. I, $A\bar{A}BC$, its volume is $\frac{1}{6}$ of the volume of the cube. 4th and 5th. Two enantiomorph tetrahedra L, $A\bar{A}B\bar{C}$, and Γ, $A\bar{A}\bar{B}C$, each of these has $\frac{1}{8}$ of the volume of the cube. Forms such as $AB\bar{C}\bar{D}$, and $A\bar{A}B\bar{B}$, do not correspond to tetrahedra, because their corners are all in one plane.

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There is only one combination containing Ω, which makes up a cube, in this a Δ is applied to each face of the Ω. This is the only quinquepartite division of the cube into tetrahedra. The other divisions of the cube into tetrahedra without making new corners are necessarily sexpartite, because all our tetrahedra except Ω have $\frac{1}{6}$ of the volume of the cube.

Let i, δ, l, γ represent the number of the tetrahedra, I, Δ, L and Γ respectively in a combination making up a cube. Noting that I has one face which is $\frac{1}{12}$ of the surface of the cube, Δ three such faces, and L and Γ each two such faces, we have, on account of the volume, (1), $i+\delta+l+\gamma=6$, and on account of the surface, (2), $i+3\delta+2l+2\gamma=12$. From these we have at once $i=\delta$, and $l+\gamma$ an even number. Not only is $i=\delta$ but each I has a particular Δ attached to it, the equilateral triangular face of the one fitting to the equilateral triangular face of the other to form a figure which we may call IΔ. This IΔ is $\frac{1}{3}$ of the cube and is an oblique square pyramid with a face of the cube for base and for apex a corner of the cube. It can be cut into an I and a Δ by a plane containing the apex and the two corners of the cube which have the same sign as the apex. It can be cut into two tetrahedra in another way, namely, by a plane through the apex and the two corners of the base which differ in sign from the apex, the two tetrahedra into which it is thus cut are an L and a Γ. So that an IΔ can always be replaced by an LΓ pair.

We shall call those combinations which contain only L's and Γ's *central* combinations because all the tetrahedra in them have an edge bisected by the centre of the cube. Half a cube can be made up either by two L's with a Γ between them or by two Γ's with an L between them, there are therefore three central combinations, namely—3L,3Γ; 4L,2Γ; and 4Γ,2L. The first is uniaxial because all the six tetrahedra meet in one body-diagonal, the other two are biaxial because the three tetrahedra forming one half of the cube meet in one body-diagonal and the three forming the other half of the cube meet in another body-diagonal. From these three central combinations all the others can be derived by replacing LΓ pairs by IΔ's. It is to be noted that when we have two IΔ's we have always two distinct combinations, in one of which the plane between the one I and its Δ is parallel (||) to the plane between the other I and its Δ, while in the other these two planes are inclined ($\nparallel$) to one another.

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