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PLANAR LATTICES

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A finite partially ordered set (poset) P is customarily represented by drawing a small circle for each point, with a lower than b whenever $a < b$ in P, and drawing a straight line segment from a to b whenever a is covered by b in P (see, for example, G. Birkhoff [**2**, p. 4]). A poset P is *planar* if such a diagram can be drawn for P in which none of the straight line segments intersect.

The main result of this paper is the following characterization of planar lattices (answering Problem 9 of G. Grätzer [**4**, p. 66]). Let

$$\mathscr{L} = \{\mathbf{A}_n | n \geqq 0\} \cup \{\mathbf{B}, \mathbf{B}^d, \mathbf{C}, \mathbf{C}^d, \mathbf{D}, \mathbf{D}^d\} \cup \{\mathbf{E}_n, \mathbf{E}_n{}^d, \mathbf{F}_n, \mathbf{G}_n, \mathbf{H}_n | n \geqq 0\}$$

be the lattices of Figure 1. (The dual of a poset P is denoted by P^d.)

THEOREM 1. *A finite lattice is planar if and only if it does not contain any lattice in* $\mathscr{L}$ *as a subposet. Moreover,* $\mathscr{L}$ *is the minimum such list; that is, if* $\mathscr{F}$ *is a set of lattices such that the first assertion remains true with* $\mathscr{L}$ *replaced by* $\mathscr{F}$, *then* $\mathscr{L} \subseteq \mathscr{F}$.

Theorem 1 is analogous in its statement to K. Kuratowski's characterization of planar graphs [**7**]; however, the corresponding proofs bear little resemblance to each other. C. R. Platt has established a connection between planar lattices and planar graphs [**8**]; this also will have no application in this paper.

The basic concepts for planar lattices are developed in the first section. Section 2 recalls that planar lattices are *dismantlable.* Section 3 describes a procedure for obtaining all planar embeddings of a planar finite lattice from one fixed planar embedding. The purpose of Section 4 is to prove a technical lemma that guarantees the existence of particular subposets at various points in the proof of Theorem 1.

Section 5 consists of the proof of Theorem 1, where essential use is made of our characterization of dismantlable finite lattices [**6**]. In the final section, Theorem 1 is extended to infinite lattices of dimension $\leqq 2$; in addition, we show that in a dismantlable nonplanar finite lattice, there are at least three doubly irreducible elements which are pairwise incomparable.

1. Geometry of planar lattices. Let P be a finite partially ordered set (poset). The relation *a is covered by b* (or *b covers a*) is denoted by $a \prec b$. π_1 and π_2 are the first and second projections of $\mathbf{R}^2$ onto $\mathbf{R}$. A *planar embedding* $e(P)$

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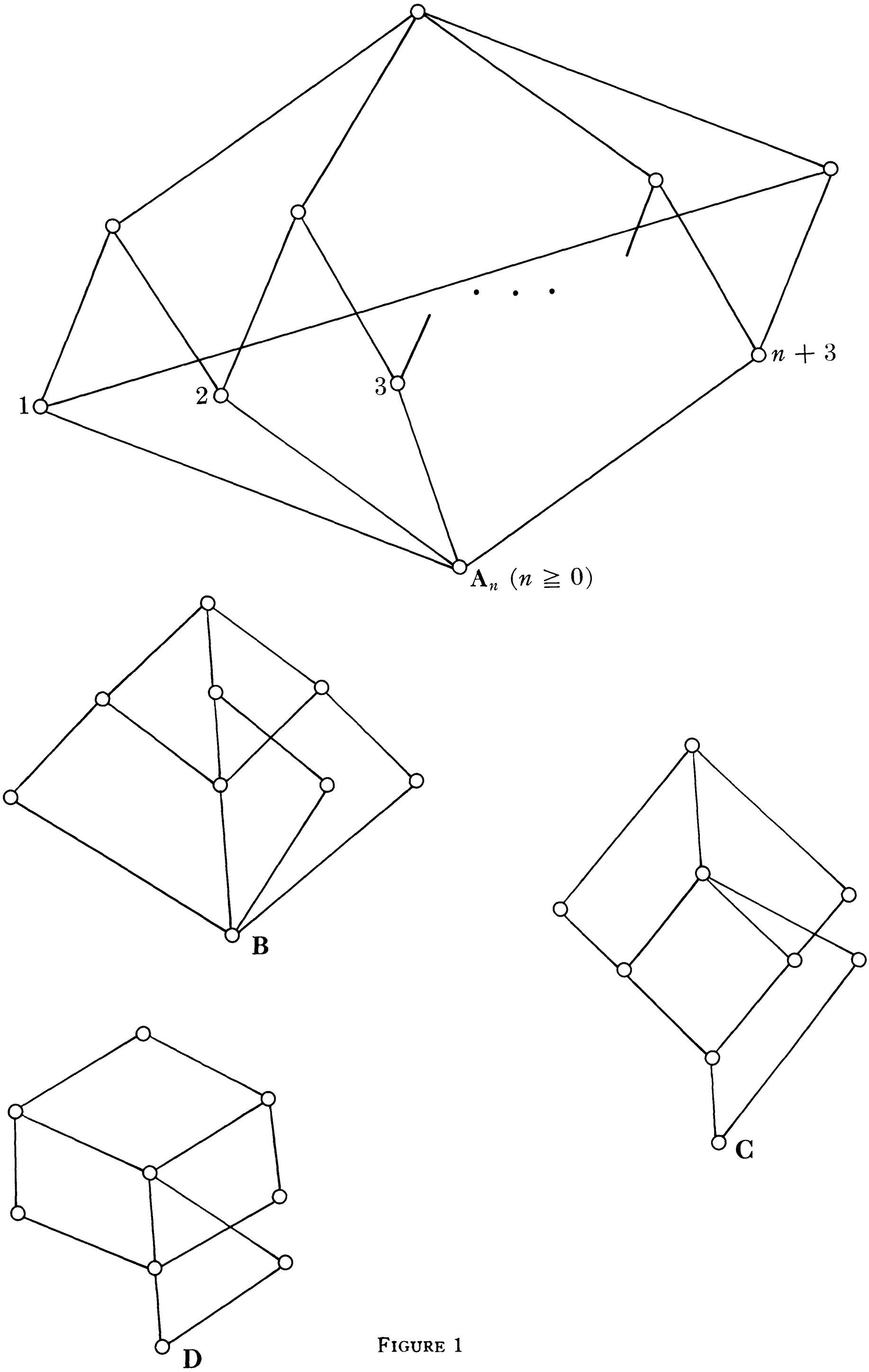
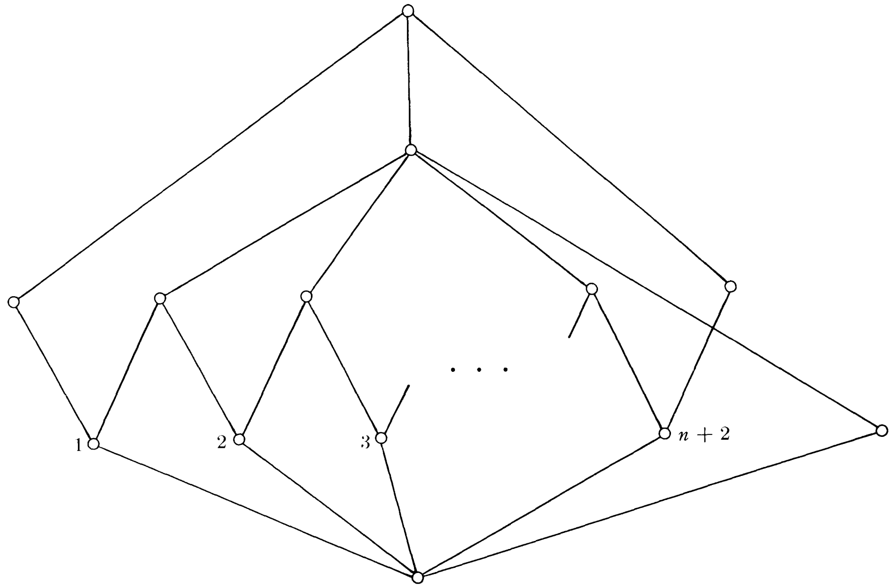
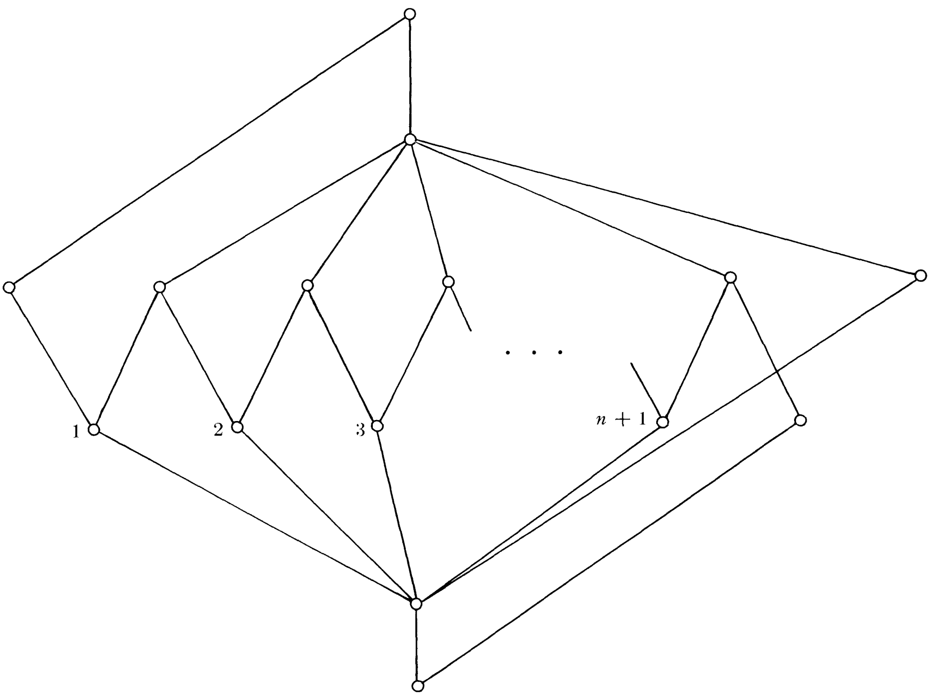


FIGURE 1



$\mathbf{E}_n$ $(n \geqq 0)$



$\mathbf{F}_n$ $(n \geqq 0)$

FIGURE 1.—(Continued)

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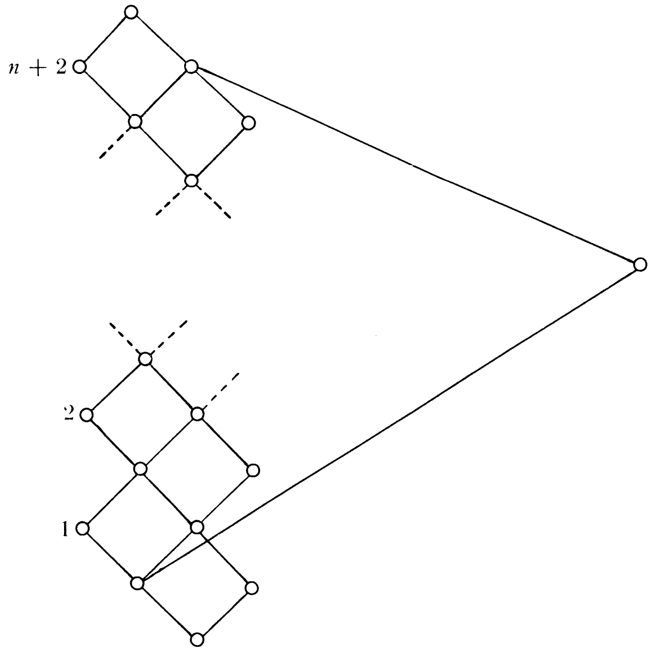
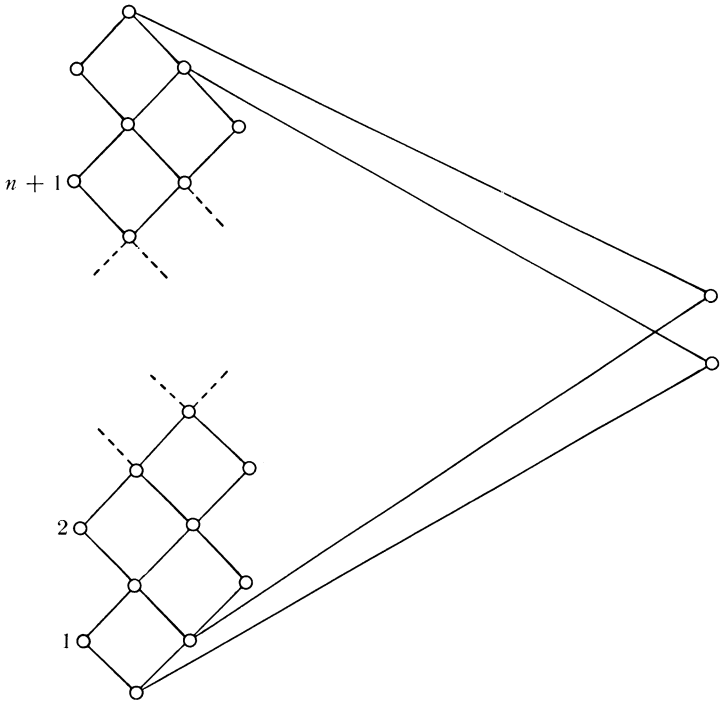


$\mathbf{G}_n$ $(n \geqq 0)$



$\mathbf{H}_n$ $(n \geqq 0)$

FIGURE 1.—(Concluded)

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of P consists of

($P1$) an injection $a \mapsto \bar{a}$ from P to $\mathbf{R}^2$ such that $\pi_2(\bar{a}) < \pi_2(\bar{b})$ whenever $a < b$, and

($P2$) straight line segments $\overline{ab}$, connecting $\bar{a}$ and $\bar{b}$ whenever $a \prec b$ in P; these segments do not intersect, except possibly at their endpoints.

P is *planar* if it has a planar embedding.

We shall soon see the need to consider representations of P which allow more general connecting paths than the straight line segments of ($P2$). A *planar representation* $e(P)$ of P consists of ($P1$) and

($P2'$) paths $y \mapsto \langle f(y), y \rangle$, denoted by $\overline{ab}$, with endpoints $\bar{a}$ and $\bar{b}$ whenever $a \prec b$ in P, where $f:[\pi_2(\bar{a}), \pi_2(\bar{b})] \rightarrow \mathbf{R}$ is a continuous function; these paths do not intersect except possibly at their endpoints.

For $a, b \in P$ such that $a \prec b$, a is a *lower cover* of b (or b is an *upper cover* of a). An element of P is *doubly irreducible in* P if it has at most one lower and at most one upper cover in P. A planar representation $e(P)$ of P induces a planar representation of $P - \{c\}$, where c is a doubly irreducible element in P, and $a \prec c \prec b$, by defining $\overline{ab}$ to be $\overline{ac} \cup \overline{cb}$ if $a \prec b$ in $P - \{c\}$, and deleting $\overline{ac} \cup \overline{cb}$ otherwise; this induced planar representation of $P - \{c\}$ is denoted by $e(P - \{c\})$. The fact that this induced planar representation $e(P - \{c\})$ need not be a planar embedding of $P - \{c\}$, even if $e(P)$ were a planar embedding, shows the need to consider planar representations.

Actually, it is shown in D. Kelly [**5**] that the existence of a planar *representation* of P is equivalent to the planarity of P; Theorem 2.5 will prove this equivalence in case P is a lattice.

For each $a \in P$, a planar representation $e(P)$ of P induces a strict linear ordering on the set $U(a)$ of upper covers of a defined by: for $x, y \in U(a)$, x is *to the left of* y if and only if $\overline{ax}(m) < \overline{ay}(m)$ where $m = \min \{\pi_2(\bar{x}) | x \in U(a)\}$; the ordering of the lower covers of a point is defined dually. Two planar representations of P are *similar* if, for each $a \in P$, the upper (lower) covers of a have the same ordering with respect to the two representations.

In the remainder of this section, L is assumed to be a finite lattice with a planar representation $e(L)$. All the geometric concepts to be introduced will depend on the choice of the planar representation $e(L)$, although the dependence will not always be stated explicitly. Most such concepts will, however, be invariant with respect to similar planar representations.

A *maximal chain* from a to b (with $a \leqq b$ in L) is a sequence $a = x_0, x_1, \ldots, x_n = b$ of points of L with $x_i \prec x_{i+1}$ $(0 \leqq i \leqq n - 1)$; if a and b are not mentioned, $a = 0$ and $b = 1$ are understood. The *function* φ: $[\pi_2(\bar{a}), \pi_2(\bar{b})] \rightarrow \mathbf{R}$ *corresponding to* such a maximal chain is defined by: $\varphi(y) = f_i(y)$ where f_i is the function on $[\pi_2(\bar{x}_{i-1}), \pi_2(\bar{x}_i)]$ representing $\overline{x_{i-1}x_i}$ and y is in this interval. (If $a = b$, φ is $\{ \langle \pi_2(\bar{a}), \pi_1(\bar{a}) \rangle \}$.)

The only result that we need from analysis is the following immediate consequence of the intermediate value theorem.

LEMMA 1.1. *Let $p, q \in \mathbf{R}$, $p < q$, and let φ_1, φ_2 be continuous functions from $[p, q]$ into $\mathbf{R}$. If $\varphi_1(p) \leqq \varphi_2(p)$ and $\varphi_1(q) \geqq \varphi_2(q)$, then there is $r \in [p, q]$ such that $\varphi_1(r) = \varphi_2(r)$.*

If C and D are two maximal chains between a and b, $(a < b)$, in L such that $C \cap D = \{a, b\}$, and φ, ψ are their corresponding functions then, by Lemma 1.1, the infimum of $\{\varphi, \psi\}$ is φ or ψ since any crossing of φ and ψ would correspond to a common element of C and D; in case inf $\{\varphi, \psi\} = \varphi$, we call C the *infimum*, and D the *supremum* of C and D (of course, not to be confused with the join and meet in L).

In general, for maximal chains C and D between a and b $(a < b)$ in L, there are elements $a = x_0 < x_1 < \ldots < x_n = b$ of L and maximal chains C_i and D_i from x_{i-1} to x_i, $(1 \leqq i \leqq n)$, such that $C = \bigcup_{i=1}^n C_i$, $D = \bigcup_{i=1}^n D_i$ and $C_i \cap D_i = \{x_{i-1}, x_i\}$. Let φ, ψ and φ_i, ψ_i be the functions corresponding to C, D and C_i, D_i, respectively, $1 \leqq i \leqq n$. Clearly, $\varphi = \bigcup_{i=1}^n \varphi_i$, $\psi = \bigcup_{i=1}^n \psi_i$ and inf $\{\varphi, \psi\} = \bigcup_{i=1}^n$ inf $\{\varphi_i, \psi_i\}$; that is, inf $\{\varphi, \psi\}$ is the function corresponding to the maximal chain $\bigcup_{i=1}^n$ inf $\{C_i, D_i\}$. The *region* R defined by C and D is the subposet of L consisting of all elements of L in the area of the plane bounded by φ and ψ; that is, $x \in R$ if and only if $\pi_2(\bar{a}) \leqq \pi_2(\bar{x}) \leqq \pi_2(\bar{b})$ and $\langle \pi_2(\bar{x}), \pi_1(\bar{x}) \rangle$ lies between inf $\{\varphi, \psi\}$ and sup $\{\varphi, \psi\}$. The *left* (*right*) *boundary* of R is inf $\{C, D\}$ (sup $\{C, D\}$); the *boundary* of R is $C \cup D$, and the *interior* of R is $R - (C \cup D)$.

Correspondingly, the *left* (*right*) *boundary* of L is the infimum (supremum) of all maximal chains in L. The *left* (*right*) *side* of a maximal chain C in L is the region defined by the left (right) boundary of L and C. An element x on the left (right) side of C is also said to be *on the left* (*right*) *of* C. Equivalently, x is on the left of C whenever $\pi_1(\bar{x}) \leqq \varphi(\pi_2(\bar{x}))$, where φ is the function corresponding to the maximal chain C of L. Obviously, every element of L is either on the left of or right of C, and it is on both sides precisely when it is an element of C.

LEMMA 1.2. *Let $x \leqq y$ in L. If x and y are on different sides of a maximal chain C in L, then there is $z \in C$ such that $x \leqq z \leqq y$.*

Proof. Suppose x and y are on the left and right, respectively, of C. Let φ_1 be the function corresponding to a maximal chain between x and y, and let φ_2 be the restriction to $[\pi_2(\bar{x}), \pi_2(\bar{y})]$ of the function corresponding to C, and apply Lemma 1.1.

LEMMA 1.3. *If, in a region R, a and b are the least and greatest elements of the boundary of R, then a and b are the least and greatest elements of R; that is, $R \subseteq [a, b]$.*

Proof. Let φ_1 (φ_2) be the function corresponding to the left (right) boundary of R. For an element $x \in R$, let ψ be the function corresponding to a maximal chain between x and 1. Without loss of generality, we may assume that $\varphi_2(\pi_2(\bar{b})) \leqq \psi(\pi_2(\bar{b}))$. Since $\psi(\pi_2(\bar{x})) \leqq \varphi_2(\pi_2(\bar{x}))$, it follows from Lemma **1.1** that $x \leqq b$.

For a region R and elements $a, b \in R$ as in Lemma 1.3 we call a and b the *bounds* of R. A sublattice S of L is *cover-preserving* if $a \prec b$ in S implies $a \prec b$ in L.

PROPOSITION 1.4. *A region of L is a cover-preserving sublattice of L.*

Proof. Let R be a region of L and suppose that $x, y \in R$ but $x \vee y \notin R$. We may assume that $x \vee y$ is on the right of a maximal chain in L containing the right boundary of R. By Lemma 1.2, there are $x', y' \in C$ such that $x \leqq x' < x \vee y$ and $y \leqq y' < x \vee y$. Without loss of generality, $x' \leqq y'$, and therefore $x, y \leqq y' < x \vee y$, a contradiction. Thus, R is a sublattice of L. Furthermore, if $x, y \in R$, $x < y$, $x < z < y$ for some $z \in L$, but $(x, y) \cap R = \emptyset$ then again by Lemma 1.2 there are z_1, z_2 on the right boundary of R, say, such that $x \leqq z_1 < z < z_2 \leqq y$. Then $x = z_1$ and $y = z_2$ which is impossible since the right boundary of a region is a maximal chain in L.

It is evident from Proposition 1.4 that, for a region R, the association of each a in R to $\bar{a}$ in $e(L)$, and of each cover $a \prec b$ in R to $\overline{ab}$ in $e(L)$ determines a planar representation of R; this induced representation of R is denoted by $e(R)$.

LEMMA 1.5. *A closed interval of L is a region of L.*

Proof. Let $a < b$ in L, let $C(D)$ be the infimum (supremum) of all maximal chains from a to b, and let R be the region defined by C and D. Clearly, $[a, b] \subseteq R$, and by Lemma 1.3, $R \subseteq [a, b]$.

For $x, y \in L$, x is *incomparable* with y in L $(x \| y)$ whenever $x \nleqq y$ and $x \ngeqq y$. We define the relation λ on L (with respect to $e(L)$) by: $x\lambda y$ if and only if $x \| y$ and there are lower covers x' and y' of $x \vee y$ such that $x \leqq x'$, $y \leqq y'$, and x' is to the left of y' (with respect to $e(L)$).

PROPOSITION 1.6. *If $x\lambda y$, then x is on the left of any maximal chain through y. If $x \| y$ and x is on the left of some maximal chain through y, then $x\lambda y$.*

Proof. First, let us observe that if $x \| y$ and x is on the left of a maximal chain C through y, then x is on the left of every maximal chain through y. Indeed, if x were on the right of some maximal chain D through y then x would be in the region defined by $C \cap [y, 1]$ and $D \cap [y, 1]$ or in the region defined by $C \cap [0, y]$ and $D \cap [0, y]$. But then, in view of Lemma 1.3, x and y are comparable.

If $x\lambda y$ then there are lower covers x', y' of $x \vee y$ such that $x \leqq x'$, $y \leqq y'$ and x' is to the left of y'. Let φ_1 and φ_2 be the functions corresponding to $x' \prec x \vee y$ and C, where C is a maximal chain through $y \leqq y' < x \vee y$. Since $\varphi_1(\pi_2(\overline{x \vee y})) = \varphi_2(\pi_2(\overline{x \vee y}))$ and $\varphi_1(m) < \varphi_2(m)$, where $m = \max$

$\{\pi_2(\bar{w})|w \prec z\}$, it follows from Lemma 1.1 that $\pi_1(\bar{x}) = \varphi_1(\pi_2(\bar{x})) < \varphi_2(\pi_2(\bar{x}))$ so that x' is on the left of C. By Lemma 1.3, x is also on the left of C.

Finally, let $x \| y$, x', y' be lower covers of $x \vee y$ with $x \leqq x'$, $y \leqq y'$, and let C be a maximal chain through $y \leqq y' < x \vee y$. If x is on the left of some maximal chain through y, then x is on the left of C. Therefore, x' is on the left of C and x' is to the left of y'.

Clearly, two planar representations of L are similar if and only if they induce the same λ.

It follows from Proposition 1.6 that we get the same relation λ if we define $x\lambda y$ dually in terms of upper covers of $x \wedge y$. In particular, two planar representations of a *lattice* are similar if and only if the induced orderings on the sets of upper covers are identical.

The next result is due to J. Zilber [**2**, p. 32, ex. 7(c)].

PROPOSITION 1.7. *λ is a strict partial order on L. Moreover, if $x \| y$, then $x\lambda y$ or $y\lambda x$.*

Proof. The preceding proposition shows that $x \| y$ implies that exactly one of $x\lambda y$ or $y\lambda x$ holds. It only remains to establish the transitivity of λ. Let $x\lambda y$ and $y\lambda z$, and let D be a maximal chain through z; then y is on the left of D. In view of Proposition 1.4, there is a maximal chain C through y on the left side of D. Therefore, x is on the left of C, and in particular, on the left of D.

Thus, if x and y are comparable, $x\lambda z$, and $y \| z$, then $y\lambda z$.

Since λ is an order relation extending the ordering of the set of upper (or lower) covers of any element of L, it is reasonable to read $x\lambda y$ as x is "to the left of" y.

A *connection* between elements c and d in a partially ordered set P is a sequence $c = x_0, x_1, \ldots, x_n = d$ of elements of P such that $x_{i-1} \prec x_i$ or $x_i \prec x_{i-1}$ for every $i = 1, 2, \ldots, n$. A *fence* is a partially ordered set $\{x_1, x_2, \ldots, x_k\}$ in which the comparabilities that hold are precisely $x_1 < x_2$, $x_2 > x_3, \ldots, x_{2i-1} < x_{2i}, x_{2i} > x_{2i+1}, \ldots$ or $x_1 > x_2, x_2 < x_3, \ldots, x_{2i-1} > x_{2i}$, $x_{2i} < x_{2i+1}, \ldots$. We usually denote a fence F by $(x_1, x_2, \ldots, x_k)$ and use the terms *down* and *up* to indicate which comparability holds between x_1 and $x_2(x_{k-1}$ and $x_k)$; for example, F is *down-up* if $x_1 < x_2 > x_3 \ldots > x_{k-1} < x_k$. It is easy to verify that any connection between c and d contains (as a subposet of P) a fence $(x_1, x_2, \ldots, x_k)$ with $x_1 = c$ and $x_k = d$.

Let $a < b$ in L. An *$\langle a, b\rangle$-component* of L is a connected component of the undirected graph corresponding to the covering relation in (a, b); that is, an $\langle a, b\rangle$-component is a maximal subset of (a, b) in which, between every pair of elements, there is a connection in (a, b). In particular, if x and y are in different $\langle a, b\rangle$-components, then $x \| y$. It is also obvious that an $\langle a, b\rangle$-component is a convex subset of L. An $\langle a, b\rangle$-component is *proper* if whenever $y \leqq x (y \geqq x)$ for some $x \in C$ and $y \in L - C$, then $y \leqq a (y \geqq b)$.

Let C be an $\langle a, b\rangle$-component of L. Although $C \cup \{a, b\}$ is obviously not an $\langle a, b\rangle$-component, we will, for brevity, call it a *bounded $\langle a, b\rangle$-component.*

LEMMA 1.8. *A bounded component of L is a region of L.*

Proof. Let $a < b$ in L and let C be a bounded $\langle a, b\rangle$-component of L. Furthermore, let $D(E)$ be the infimum (supremum) of all maximal chains from a to b contained in C, and let R be the region defined by D and E. Clearly, $C \subseteq R$. If $R \not\subseteq C$, let x be a minimal element of $R - C$ and let y be a lower cover of x in R. If $y > a$, then $y \in C$ so that $x \in C$. Otherwise, $a \prec x$. Let d and e be upper covers of a in D and E respectively, let F be a maximal chain in R from a to b through x, and let $d = z_0, z_1, \ldots, z_n = e$ be a connection between d and e in (a, b). Let k be the least index i such that z_i is not in the region defined by D and F. Since $z_k \in C$ and $C \subseteq R$, z_k is not on the left side of a maximal chain extending F; thus, by Lemma 1.2, $z_{k-1} \in F$. Then $d = z_0, z_1, \ldots, z_{k-1}, y_1, y_2, \ldots, y_n = x$ is a connection between d and x, where $z_{k-1} \succ y_1 \succ y_2 \succ \ldots \succ y_n = x$ $(n \geqq 0)$; therefore $x \in C$, a contradiction.

Since the intersection of two distinct bounded $\langle a, b\rangle$-components is $\{a, b\}$, the ordering of the functions corresponding to the left boundaries of the bounded $\langle a, b\rangle$-components induces a strict linear ordering $C_1 \lambda\, C_2 \lambda \ldots \lambda C_n$ on the $\langle a, b\rangle$-components. This ordering can be defined by: $C_i \lambda\, C_j$ if and only if, for any $x \in C_i$ and $y \in C_j$, $x \lambda y$. The left boundary of $[a, b]$ is clearly the left boundary of $C_1 \cup \{a, b\}$.

Let R be a region with bounds $a < b$. A *left up-dangle* (*down-dangle*) *on* R is an element z such that $z \lambda b (z \lambda a)$ and $z > x$ $(z < x)$ for some $x \in R - \{a, b\}$. *Right dangles* are defined analogously. The *attachment point* of an up-dangle (*down-dangle*) z is the greatest (least) element of R less (greater) than z; clearly, the attachment point is distinct from a and b.

PROPOSITION 1.9. *The attachment point of a left dangle on a region R always exists and is on the left boundary of R.*

Proof. Let z be a left up-dangle on R, let w be the greatest element on the left boundary C of R that is less than z, and let C_1 be a maximal chain extending C. Let $x \in R$ be such that $z > x$. By Proposition 1.6, z is on the left of C_1; since x in on the right of C_1, there is $y \in C$ such that $x \leqq y < z$ (by Lemma 1.2). Then $x \leqq w$, and therefore, w is the attachment point of z.

COROLLARY 1.10 *Let the $\langle a, b\rangle$-components of L be $C_1 \lambda\, C_2 \lambda \ldots \lambda C_n$, where $a < b$ but a is not covered by b in L, and let C_i' $(1 \leqq i \leqq n)$ be the corresponding bounded components. The only bounded component that can have a left* (*right*) *dangle z is C_1' (C_n'); the corresponding attachment point is $z \wedge b$ $(z \vee a)$ if z is an up-dangle* (*down-dangle*). *In particular, all components C_i for $i \neq 1$ or n are proper.*

Proof. A left up-dangle z on any bounded $\langle a, b\rangle$-component is a left up-dangle on $[a, b]$, and therefore, has attachment point w on the left boundary of

$[a, b]$, which is the left boundary of C_1'; clearly, $w = z \wedge b$. For $i \neq 1$ and any $x \in C_i$, $x \| w$ so that z cannot be a dangle on C_i.

A *face* is a region of L whose interior is empty and contains no paths of $e(L)$, and whose bounds are the only elements common to both its left and right boundary. It is easy to verify that any two incomparable elements x and y in a face uniquely determine the face. In fact, if $x\lambda y$, the left (right) boundary of any face containing x and y must be the supremum (infimum) of all maximal chains between $x \wedge y$ and $x \vee y$ that pass through $x(y)$. If the $\langle a, b \rangle$-components of L are $C_1\lambda\, C_2\lambda \ldots \lambda C_n$, then the region defined by the right boundary of $C_i \cup \{a, b\}$ and the left boundary of $C_{i+1} \cup \{a, b\}$ is a face (for $1 \leqq i < n$). Indeed, if there were an element in the interior of this region it would be in (a, b) and therefore, in some $\langle a, b \rangle$-component; if there were only a path in the interior of this region, then elements from different $\langle a, b \rangle$-components would be comparable.

LEMMA 1.11. *If $x, z \in [a, b] \subseteq L$ with $x\lambda z$ then there is $y \in [a, b]$ such that x and y are in a common face with $x\lambda y$.*

Proof. Since x is not on the right boundary of $[a, b]$, there is a first maximal chain which a horizontal ray from $\bar{x}$ to the right intersects, and an element y on this maximal chain such that $x\lambda y$. Let $C(D)$ be the supremum (infimum) of all maximal chains from $x \wedge y$ to $x \vee y$ that pass through $x(y)$. The region defined by C and D is a face containing both x and y.

For $a < b$ in L, b is *visible* from a (with respect to $e(L)$) if and only if there is a continuous function φ: $[\pi_2(\bar{a}), \pi_2(\bar{b})] \to \mathbf{R}$ such that the path $y \mapsto \langle \varphi(y), y \rangle$, $y \in [\pi_2(\bar{a}), \pi_2(\bar{b})]$), intersects $e(L)$ only at $\bar{a}$ and $\bar{b}$; φ is called a *visibility function* (for a and b). The next result will play an important role in the proof of Theorem 1.

THEOREM 1.12. *For $a < b$ in L, b is not visible from a if and only if a is not covered by b, there is exactly one $\langle a, b \rangle$-component, and $[a, b]$ has both a left and a right dangle.*

Proof. Let $p = \pi_2(\bar{a})$ and $q = \pi_2(\bar{b})$, and let us suppose that $[a, b]$ has no right dangle. Let φ_1 be the function corresponding to the right boundary C of $[a, b]$. Define ψ: $[p, q] \to \mathbf{R}$ by $\psi(x) = \varphi_1(x) + (x - p)(q - x)$, and let φ_2: $[p, q] \to \mathbf{R}$ be defined by $\varphi_2(x) = \min\{\psi(x), f(x)\}$, where f is the restriction to $[p, q]$ of the function corresponding to the first maximal chain to the right of C ($f(x) = \infty$ if both a and b are the right boundary of L). A visibility function is $\frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$.

If a is not covered by b and there are $\langle a, b \rangle$-components $C_1\lambda C_2\lambda \ldots \lambda C_n$ with $n \geqq 2$, let φ_1 be the function corresponding to the right boundary of $C_1 \cup \{a, b\}$ and let φ_2 be the function corresponding to the left boundary of $C_2 \cup \{a, b\}$. A visibility function is $\frac{1}{2}\varphi_1 + \frac{1}{2}\varphi_2$.

Let us now suppose that a is not covered by b and there is exactly one $\langle a, b\rangle$-component such that $[a, b]$ has both a left and a right dangle. Let $p = \pi_2(\bar{a})$, $q = \pi_2(\bar{b})$, and let φ_1 and φ_2 be the functions corresponding to the left and right boundaries C, D, respectively, of $[a, b]$. Let us, furthermore, assume that there is a visibility function ψ for a and b.

Let z be a left up-dangle on $[a, b]$ with attachment point w, and let δ be the function corresponding to a maximal chain between w and 1 through z. If $\psi(x) < \varphi_1(x)$ for all $x \in (p, q)$, then $\psi(r) < \delta(r)$, where $r = \pi_2(\bar{w})$, and $\delta(q) < \psi(q)$ imply that ψ and δ cross between r and q, which is impossible.

Thus, we may assume that $\varphi_1(x) < \psi(x) < \varphi_2(x)$ for all $x \in (p, q)$. We now proceed as in the proof of Lemma 1.8. Let c and d be upper covers of a in C and D, respectively, and let $c = z_0, z_1, \ldots, z_n = d$ be a connection between c and d in (a, b). Let k be the least index i such that $\langle \pi_2(\overline{z_i}), \pi_1(\overline{z_i})\rangle$ is not between the functions φ_1 and ψ; then, $\langle \pi_2(\overline{z_{k-1}}), \pi_1(\overline{z_{k-1}})\rangle$ is between φ_1 and ψ. Let $r = \pi_2(\overline{z_{k-1}})$ and $s = \pi_2(\overline{z_k})$. We will consider only the case that $r < s$; let δ be the function corresponding to $z_{k-1} \prec z_k$. Since $\delta(r) \leqq \psi(r)$ and $\delta(s) > \psi(s)$, δ crosses ψ, a contradiction.

2. Dismantlability of planar lattices. Let P be a finite partially ordered set. P is *dismantlable* if $P = \{x_1, x_2, \ldots, x_n\}$ where x_i is doubly irreducible in $\{x_1, x_2, \ldots, x_i\}$ for $1 \leqq i \leqq n$. The notion of dismantlability was first introduced for finite lattices in I. Rival [**9**].

PROPOSITION. 2.1. *A dismantlable finite bounded poset is a lattice.*

Proof. Let c be a doubly irreducible element of a dismantlable finite bounded poset P; then $P - \{c\}$ is a dismantlable bounded poset, and therefore a lattice by induction on $|P|$. It is then immediate that P is a lattice.

We note that the concepts of "left (right) side of a maximal chain" and "left (right) boundary of a region" extend in the natural way to planar finite bounded posets, and that Lemmas 1.2 and 1.3 remain valid in this context.

The first statement of the next result is due to K. A. Baker, P. C. Fishburn, and F. S. Roberts [**1**].

PROPOSITION 2.2. *A planar finite bounded poset P with $|P| \geqq 3$ contains a doubly irreducible element $c \neq 0$, 1 on the left boundary. Moreover, $P - \{c\}$ is planar.*

COROLLARY 2.3. *A planar finite bounded poset is dismantlable.*

COROLLARY 2.4. *A planar finite bounded poset is a lattice.*

Corollary 2.4 appears as an exercise in G. Birkhoff [**2**, p. 32, ex. 7(a)]. Proposition 2.2 is an immediate consequence of the following theorem which establishes the equivalence (for lattices) between planarity and the existence of a planar representation. This equivalence was already established by C. R. Platt [**8**].

THEOREM 2.5. *Every finite bounded poset P with a planar representation $e_1(P)$ has a planar embedding $e_2(P)$ which is similar to $e_1(P)$. Furthermore, if $|P| \geqq 3$, then P contains a doubly irreducible element distinct from* 0 *and* 1 *on its left boundary.*

Proof. We may obviously assume that $|P| \geqq 3$. Let B be the left boundary of P, and let c be the maximum element of $B - \{1\}$ which has a unique lower cover a in P. Let $c \prec b$ in B and suppose that c also has an upper cover b_1 distinct from b. In view of the choice of c, b has a lower cover c_1 distinct from c. Now, let $B_0 = B \cap [0, c]$, $B_1 = B \cap [c, 1]$, and C be a maximal chain from c to 1 through b_1. Since $c_1 \| c$ we have, by Lemma 1.3, that c_1 is not in the region defined by B_1 and C; thus, c_1 is on the right of $B_0 \cup C$. But b is on the left of $B_0 \cup C$; therefore, by Lemma 1.2, there exists $x \in B_0 \cup C$ such that $c_1 < x < b$ which contradicts $c_1 \prec b$. Thus, c is a doubly irreducible element in P. By induction on $|P|$, there is a planar embedding $e_3(P - \{c\})$ similar to $e_1(P - \{c\})$. If $a \prec b$ in $P - \{c\}$, choose $\bar{c}$ as the midpoint of the line segment $\overline{ab}$ to obtain $e_2(P)$. Otherwise, since a and b are on the left boundary of $e_3(P - \{c\})$, we can adjoin $\bar{c}$ to $e_3(P - \{c\})$ to form a planar embedding $e_2(P)$ by taking $\pi_2(\bar{c}) = \frac{1}{2}\pi_2(\bar{a}) + \frac{1}{2}\pi_2(\bar{b})$, and $\pi_1(\bar{c})$ a sufficiently small real number.

PROPOSITION 2.6. *If d is not on the left boundary B of a planar finite lattice L, there is a doubly irreducible element $c \in B$ which is incomparable with d.*

Proof. Let $u(v)$ be the greatest (least) element of B that is $< d$ $(> d)$, let $C = B \cap [u, v]$, and let D be a maximal chain from u to v through d. By Proposition 2.2, there is $c \in C - \{u, v\}$ which is doubly irreducible in the region R defined by C and D; clearly $c \| d$. An application of Lemma 1.2 shows that, if $x \in L - R$ and $x > c$, then there is $y \in R$ such that $x > y > c$; therefore, c is doubly irreducible in L.

3. Transformations of planar lattice embeddings. The points of a planar embedding of a finite poset can be moved "slightly" without destroying the planarity of the embedding.

LEMMA 3.1. *If $e(P)$ is a planar embedding of a finite poset P, there is $\epsilon > 0$ such that if each point $\bar{a}$ of $e(P)$ is replaced by a point $\hat{a}$ with $|\hat{a} - \bar{a}| \leqq \epsilon$, then joining each $\hat{a}$ and $\hat{b}$ by a straight line segment whenever $a \prec b$ in P also defines a planar embedding.*

Proof. It suffices to take an $\epsilon > 0$ less than the ϵ given by the next lemma and less than $\frac{1}{2}(\pi_2(\bar{b}) - \pi_2(\bar{a}))$ whenever $a \prec b$ in P.

LEMMA 3.2. *If G is a finite planar graph in $\mathbf{R}^2$ with straight line edges, there is $\epsilon > 0$ such that if, for every vertex x of G, a point x' is chosen such that $|x' - x| \leqq \epsilon$, and the graph G' consists of the vertices x' and straight line edges connecting x' and y' whenever x and y are connected by an edge in G, then G' is a planar graph.*

Proof. For each edge $\overline{xy}$ in G, choose $0 < \epsilon_{xy} < \frac{1}{2}|\overline{xy}|$ such that $\{p \in \mathbf{R}^2 |$ there is q on $\overline{xy}$ with $|p - q| \leqq \epsilon_{xy}\}$ contains no vertices of G except x and y, and no part of an edge unless it is incident with x or y. It is enough to take $\epsilon = \frac{1}{2} \min \{\epsilon_{xy} | \overline{xy}$ in $G\}$.

Let p and q be points in $\mathbf{R}^2$ with $\pi_2(p) < \pi_2(q)$. A *diamond* with bottom and top points p and q is the area of $\mathbf{R}^2$ bounded by two paths $\overline{pr} \cup \overline{rq}$ and $\overline{ps} \cup \overline{sq}$ intersecting only at p and q such that $\pi_2(r), \pi_2(s) \in (\pi_2(p), \pi_2(q))$, and each of $\overline{pr}$, $\overline{rq}$, $\overline{ps}$ and $\overline{sq}$ are straight line segments; p and q are also called the extreme points of the diamond. In other words, a diamond corresponds to a planar embedding of $\mathbf{2}^2$.

LEMMA 3.3. *Let L be a finite lattice with a planar embedding $e(L)$, and let D be a diamond in $\mathbf{R}^2$ with bottom and top points p and q, respectively. There is a planar embedding $e'(L)$ of L similar to $e(L)$ and contained in D such that $\bar{0} = p$ and $\bar{1} = q$.*

Proof. Actually, we shall show slightly more, namely, that all elements on the left boundary of L can be taken on the left boundary D_1 of D. Without loss of generality, $|L| \geqq 3$. Let $c \neq 0, 1$ be a doubly irreducible element on the left boundary of L and let $a \prec c \prec b$. By induction on $|L|$, there is a planar embedding $e'(L - \{c\})$ with the desired properties which is similar to $e(L - \{c\})$. If $a \prec b$ in $L - \{c\}$, we can adjoin c to $e'(L - \{c\})$ by choosing $\bar{c}$ on $\overline{ab}$; if a is not covered by b in $L - \{c\}$, we can use Lemma 3.1 to shift any elements on D_1 strictly between a and b slightly to the right, and then suitably place c on D_1 so it can be joined to a and b with straight line segments in D_1.

PROPOSITION 3.4. *Let $a < b$ but a is not covered by b in a finite lattice L with a planar embedding $e(L)$, and let $C_1 \lambda C_2 \lambda \ldots \lambda C_n$ $(n \geqq 1)$ be the proper $\langle a, b\rangle$-components with respect to $e(L)$. If σ is a permutation of $\{1, 2, \ldots, n\}$ and $\tau: \{1, 2, \ldots, n\} \to \{0, 1\}$, then there is a planar embedding $e'(L)$ of L with the "to the left of" relation λ' such that for $x \| y$ in L:*

$x\lambda' y$ if and only if $x \lambda y$, whenever
(i) *x or $y \notin \bigcup_{i=1}^n C_i$, or*
(ii) *for some $1 \leqq i, j \leqq n$, $x \in C_i$, $y \in C_j$, and $(i - j)(\sigma(i) - \sigma(j)) > 0$, or*
(iii) *$x, y \in C_i$ and $\tau(i) = 0$ for some $1 \leqq i \leqq n$;*
$x\lambda' y$ if and only if $y \lambda x$, whenever
(iv) *for some $1 \leqq i, j \leqq n$, $x \in C_i$, $y \in C_j$, and $(i - j)(\sigma(i) - \sigma(j)) < 0$, or*
(v) *$x, y \in C_i$ and $\tau(i) = 1$ for some $1 \leqq i \leqq n$.*

Remark. Under the conditions of the proposition, $C_{\sigma(1)} \lambda' C_{\sigma(2)} \lambda' \ldots \lambda' C_{\sigma(n)}$, and $e'(C_i)$ is similar to $e(C_i)$ if $\tau(i) = 0$, and similar to the reflection of $e(C_i)$ if $\tau(i) = 1$. We say that $e'(L)$ is obtained (up to similarity) by *permuting* the proper $\langle a, b\rangle$-components of $e(L)$ according to σ, and *reflecting* them according to τ. We also write $e'(L) \equiv T_{\sigma,\tau}^{a,b}\, e(L)$ (using $\equiv$ to indicate similarity), and call $T_{\sigma,\tau}^{a,b}$ an *elementary transformation* (with respect to L).

Proof. As in the argument of Theorem 1.12, there are visibility functions for a and b, one on the left of C_1 and one on the right of C_n. Adding one point to each of these paths gives a planar representation of the lattice $L^* = L \cup \{\alpha, \beta\}$ where $a \prec \alpha$, $\beta \prec b$; let $e^*(L^*)$ be a similar planar embedding. Then $e^*(L)$ is similar to $e(L)$, and $\bar{\alpha}$, $\bar{\beta}$, $\bar{a}$ and $\bar{b}$ form a diamond D in $e^*(L)$ which includes exactly the elements $\{a, b\} \cup \bigcup_{i=1}^{n} C_i$ of L. We now delete all the lines and points in the interior of D, and draw n smaller diamonds inside D with extreme points $\bar{a}$ and $\bar{b}$ which intersect only at $\bar{a}$ and $\bar{b}$. Applying Lemma 3.3 to the ith inner diamond, we obtain a planar embedding of $C_{\sigma(i)} \cup \{a, b\}$ inside this diamond similar to $e(C_{\sigma(i)} \cup \{a, b\})$ if $\tau(\sigma(i)) = 0$, and to its reflection if $\tau(\sigma(i)) = 1$. Finally, deleting $\bar{\alpha}$, $\bar{\beta}$ and the edges of D yields a planar embedding $e'(L)$ with the desired properties.

It is clear that any region R with bounds $a < b$ and without dangles is the union of "consecutive" proper $\langle a, b\rangle$-components $C_k, C_{k+1}, \ldots, C_m$ and the bounds a, b. A new planar embedding is obtained by reversing the order of $C_k, C_{k+1}, \ldots, C_m$ and reflecting each C_i ($k \leqq i \leqq l$). We say that this new planar embedding is obtained by *reflecting* R.

THEOREM 3.5. *Let L be a finite planar lattice. If $e(L)$ and $e'(L)$ are two planar embeddings of L, there are elementary transformations $T_1, T_2, \ldots, T_n$ such that*

$$e'(L) \equiv T_n \ldots T_2 T_1\, e(L).$$

Proof. Let $|L| \geqq 3$ and let c be a doubly irreducible element in L such that $a \prec c \prec b$. By induction on $|L|$, there is a sequence $S_1, S_2, \ldots, S_m$ of elementary transformations (with respect to $L - \{c\}$) such that $e'(L - \{c\}) \equiv S_m \ldots S_2 S_1\, e(L - \{c\})$. We inductively define a sequence $T_1, T_2, \ldots, T_n$ of elementary transformations with respect to L, with a corresponding sequence of planar embeddings $e_0(L), e_1(L), \ldots e_m(L)$, where $e_0(L) = e(L)$ and $e_i(L) \equiv T_i\, e_{i-1}(L)$ for $1 \leqq i \leqq m$. If S_i is $T_{\sigma,\tau}^{x,y}$ and $\{x, y\} \neq \{a, b\}$, then T_i is $T_{\sigma,\tau}^{x,y}$. On the other hand, if S_i is $T_{\sigma,\tau}^{a,b}$, if $\{c\}$ is the jth proper $\langle a, b\rangle$-component of L with respect to $e_{i-1}(L)$, and there are k proper $\langle a, b\rangle$-components in $L - \{c\}$ then T_i is $T_{\sigma',\tau'}^{a,b}$, where $\sigma' = \alpha\sigma\alpha^{-1} \cup \{\langle j, j\rangle\}$, $\tau' = \alpha\tau\alpha^{-1} \cup \{\langle j, 0\rangle\}$, and α: $\{1, 2, \ldots, k\} \rightarrow \{1, 2, \ldots, k+1\}$ is defined by $\alpha(t) = t$ for $t < j$, and $\alpha(t) = t + 1$ for $t > j$. By induction, it is easy to show that $e_i(L - \{c\}) \equiv S_i \ldots S_2 S_1\, e(L - \{c\})$ for $0 \leqq i \leqq m$. Therefore, $e_m(L - \{c\}) \equiv e'(L - \{c\})$; hence, $e'(L) \equiv T_{\sigma,\tau}^{a,b} e_m(L)$ for a suitable σ and a zero function τ. Finally, if $T_{m+1} = T_{\sigma,\tau}^{a,b}$, then $e'(L) \equiv T_{m+1} \ldots T_2 T_1\, e(L)$, which completes the proof.

4. Dangles on indecomposable intervals. A *splitting element* of a poset P is an element comparable with every element of P. P is (*linearly*) *decomposable* if it contains a splitting element which is not a universal bound of P; otherwise,

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P is (*linearly*) *indecomposable*. If $0 = d_0 < d_1 < \ldots < d_n = 1$ are all the splitting elements of a nontrivial planar finite lattice L, then $L = \bigcup_{i=1}^{n}[d_{i-1}, d_i]$ and each $[d_{i-1}, d_i]$ is indecomposable. If $x \in L$ is not on the left boundary of L, x is not a splitting element since there exists $y \in L$ such that $y\lambda x$. Thus, the splitting elements of L are precisely those elements common to both boundaries of L.

LEMMA 4.1. *If $[u, v]$ is an indecomposable interval in a join-semilattice S and $w \geqq u$, $w \in S$, is incomparable with v, then $[u, v \vee w]$ is indecomposable.*

Let S be a join-semilattice. We call a sequence

$$x_0, x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1} \quad (n \geqq 0)$$

of elements of S a *join-extension sequence* if the following three conditions are satisfied:

(i) $x_0 < x_1$;

(ii) $x_i \| y_i \quad (1 \leqq i \leqq n)$;

(iii) $x_{i+1} = x_i \vee y_i \quad (1 \leqq i \leqq n)$.

We note that $y_i \| y_{i+1}$ and $x_{i+2} = y_i \vee y_{i+1}$ $(1 \leqq i \leqq n - 1)$. A *meet-extension sequence* is defined dually.

LEMMA 4.2. *Let L be an indecomposable planar finite lattice and let $a \neq 0$ be an element on the left boundary of L. Then there is a join-extension sequence*

$$x_0 = 0, x_1 = a, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1} = 1$$

such that x_i is on the left (right) boundary of L for odd (even) i, and y_i is on the left (right) boundary of L for even (odd) i.

Proof. If $x_i < 1$ has been chosen and x_i is, say, on the left boundary of L, then, since L is indecomposable, x_i is not on the right boundary of L. We can now take x_{i+1} to be the minimum element on the right boundary $> x_i$ and y_i a lower cover of x_{i+1} on the right boundary.

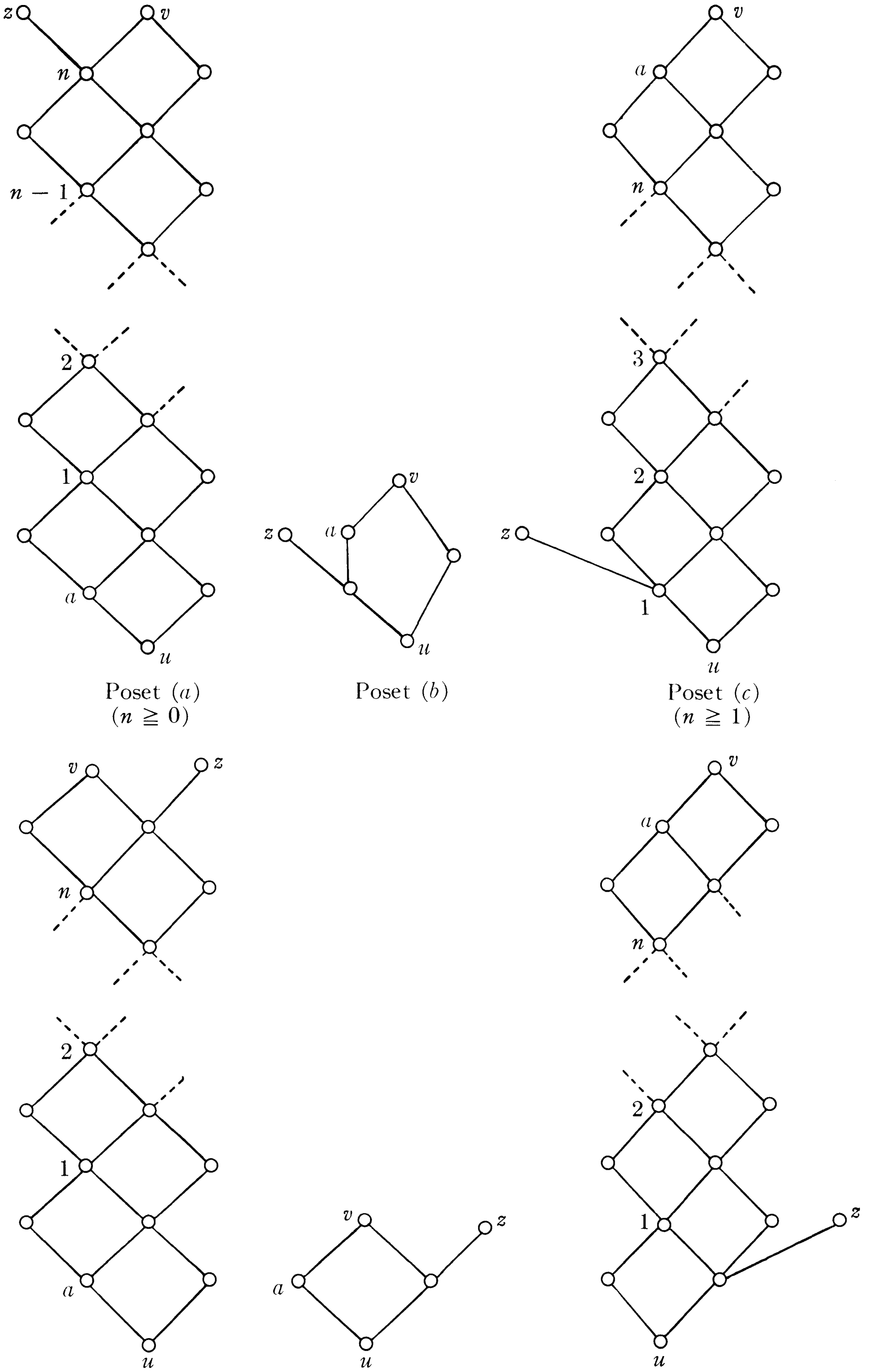
In a planar finite lattice, whenever there is a dangle on an indecomposable interval with a distinguished element on one boundary of this interval, then, as we show in the next lemma, one of a certain class of posets must occur. This result is applied repeatedly in the proof of Theorem 1.

LEMMA 4.3. *Let L be a planar finite lattice, let $[u, v] \subseteq L$ be an indecomposable interval with $a \neq u, v$ on the left boundary of $[u, v]$, and let z be an up-dangle on $[u, v]$ with attachment point $w = z \wedge v$. Then one of the following six cases will occur; in each case, one of the posets of Figure 2 listed for that case will be isomorphic to a subposet of L containing z, a, u, and v.*

(i) *z is left dangle and $w \geqq a$*: Poset (a).

(ii) *z is left dangle and $w = a$*: Poset (a) for $n = 0$.

(iii) *z is left dangle and $w < a$*: Poset (b), Poset (c).



Poset (a)
($n \geqq 0$)

Poset (b)

Poset (c)
($n \geqq 1$)

Poset (d)
($n \geqq 0$)

Poset (e)

Poset (f)
($n \geqq 0$)

FIGURE 2

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(iv) *z is right dangle and $w > a$*: Poset (d).
(v) *z is right dangle and $w \| a$*: Poset (e).
(vi) *z is right dangle and $w < a$*: Poset (f).

Proof. One of these cases must occur for an up-dangle z because, if z is a left (right) dangle, then w is on the left (right) boundary by Proposition 1.9, and $w \neq a$ for a right dangle because $[u, v]$ is indecomposable. We now analyse each case separately to determine the posets which appear.

(i) *z is left dangle and $w \geqq a$*: In view of Lemma 4.2, there is a join-extension sequence $x_0 = u, x_1 = a, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1} = v$ such that x_i is on the left (right) boundary of $[u, v]$ for odd (even) i, and y_i is on the left (right) boundary of $[u, v]$ for even (odd) i. Since w is on the left boundary of $[u, v]$, there is odd k such that $w \in [x_k, x_{k+2})$ (or $w \in [x_k, x_{k+1})$ in case $x_{k+1} = v$). Then $\{z, u, x_1, y_1, x_2, y_2, \ldots, x_k, y_k, v\}$ is isomorphic to Poset (a). For example, if $w \in [x_k, x_{k+2})$, then $z \geqq y_k$ is impossible because it would imply either $y_k \leqq y_{k+1}$ if $z \leqq y_{k+1}$, or $z \geqq y_k \vee y_{k+1} = x_{k+2}$ if $z \geqq y_{k+1}$.

(ii) *z is left dangle and $w = a$*: This is trivial.

(iii) *z is left dangle and $w < a$*: Let $x_0 = v, x_1 = a, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1} = u$ be a meet-extension sequence with x_i on the left (right) boundary of $[u, v]$ for odd (even) i, and y_i on the left (right) boundary of $[u, v]$ for even (odd) i. If $w \in (x_2, a)$ with $x_2 = u$ or $w \in (x_3, a)$, then $\{z, u, w, a, y_1, v\}$ is isomorphic to Poset (b). Otherwise, there is odd $k \geqq 3$ such that $w \in (x_{k+2}, x_k]$ (or $w \in (x_{k+1}, x_k]$ when $x_{k+1} = u$); then $\{v, x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}, y_k, w, u, z\}$ is isomorphic to Poset (c), since, for example, $w \leqq y_k$ and $w \leqq y_{k+1}$ would imply $w \leqq x_{k+2}$.

(iv) *z is right dangle and $w > a$*: Let $x_0, x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}$ be the join-extension sequence of (i). Since w is on the right boundary of $[u, v]$, $w \geqq x_2$. If, for even $k \geqq 2$, $w \in [x_k, x_{k+2})$, or $w \in [x_k, x_{k+1})$ with $x_{k+1} = v$, then $\{u, x_1, y_1, x_2, y_2, \ldots, x_k, y_k, v, z\}$ is isomorphic to Poset (d).

(v) *z is right dangle and $w \| a$*: This is trivial.

(vi) *z is right dangle and $w < a$*: Let $x_0, x_1, y_1, x_2, y_2, \ldots, x_n, y_n, x_{n+1}$ be the meet-extension sequence of (iii). There is even $k \geqq 2$ such that either $w \in (x_{k+2}, x_k]$, or $w \in (x_{k+1}, x_k]$ when $x_{k+1} = u$. Then

$$\{v, x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}, y_k, w, u\}$$

is isomorphic to Poset (f).

5. Proof of Theorem 1. It is easy to verify that $\mathbf{A}_n$ ($n \geqq 0$) is a lattice. Since $\mathbf{A}_n$ is not dismantlable, it follows from Corollary 2.3 that it cannot be planar. On the other hand, if P is one of the other posets of Figure 1, then there is a doubly irreducible element c in P such that $L = P - \{c\}$ is planar; hence, by Corollary 2.3, P is dismantlable and, consequently, a lattice by Corollary 2.4. In each such poset P, if c is chosen so that the diagram in Figure 1 given for P induces a planar embedding $e(L)$ of the lattice L, then for all $u < v$ in

L with u not covered by v, there is at most one proper $\langle u, v \rangle$-component C; either C consists of a doubly irreducible element of L, or $L = C \cup \{u, v, 0, 1\}$. Therefore, by Theorem 3.5, any planar embedding of L is similar to $e(L)$ or its reflection. If $a \prec c \prec b$ in P, then b is not visible from a by Theorem 1.12; by the next lemma, this means that P is nonplanar. Therefore, every poset in Figure 1 is a nonplanar lattice.

LEMMA 5.1. *Let L be a planar finite lattice, $a < b$ in L, and $M = L \cup \{c\}$ be defined by setting $a \prec c \prec b$ in M. M is planar if and only if there is a planar embedding of L in which b is visible from a.*

Proof. If $e(M)$ is a planar embedding of M, then b is visible from a in $e(M - \{c\})$ with visibility path $\overline{ac} \cup \overline{cb}$. Now, let $e(L)$ be a planar embedding of L in which b is visible from a. If $a \prec b$, add $\bar{c}$ on the midpoint of $\overline{ab}$ to give a planar embedding of L. Otherwise, add $\bar{c}$ to the visibility path for a and b, forming a planar representation of M, and take a similar planar embedding.

Let M be a finite lattice which contains a poset P in Figure 1. In [**5**], it is shown that a finite poset which contains a nonplanar *lattice* is nonplanar; therefore, M is nonplanar, and the proof of one direction of Theorem 1 is complete. The nonplanarity of M also follows from a consideration of *dimension* 2 lattices. The *dimension* of an arbitrary poset is the least number of linear orders whose intersection is the partial ordering of the poset (see B. Dushnik and E. W. Miller [**3**]). As observed in [**1**], the following characterization of planar lattices is a combination of results of J. Zilber [**2**, p. 32, ex. 7(c)] and B. Dushnik and E. W. Miller [**3**, Theorem 3.61].

PROPOSITION 5.2. *Let L be a finite lattice. L is planar if and only if* dimension $(L) \leqq 2$.

Therefore, dimension $(M) \geqq$ dimension $(P) > 2$ so that M is nonplanar.

Now, let M be a nonplanar finite lattice. We will show that M must contain one of the posets (or their duals) of Figure 1. To this end, we suppose that M does not contain any of these posets; it will eventually be shown that this is impossible. At various stages of the proof, certain hypotheses will be shown to be untenable by exhibiting an occurrence in M of one of these posets.

By the characterization of dismantlable lattices established in D. Kelly and I. Rival [**6**], M is dismantlable since it does not contain $\mathbf{A}_n$ $(n \geqq 0)$. If M is dismantled (by removing doubly irreducible elements one at a time), a planar lattice will eventually be obtained; therefore, we can, without loss of generality, assume that $M = L \cup \{c\}$, where L is planar and $a \prec c \prec b$ in M. By Lemma 5.1, b is not visible from a with respect to *any* planar embedding of L.

Let $e_1(L)$ be a planar embedding of L. In the course of the proof, various planar embeddings of L will be introduced; at any point of the proof, it is to be understood that all statements are with respect to the most recently introduced planar embedding.

By Theorem 1.12, a is not covered by b, there is exactly one $\langle a, b\rangle$-component, and there are both left and right dangles on $[a, b]$. We first suppose that there are left and right dangles z_1 and z_2 on $[a, b]$ with attachment points w_1 and w_2, respectively, such that $w_1 \| w_2$. Since w_1 and w_2 are in the same $\langle a, b\rangle$-component, there is a fence $F = (w_1 = x_1, x_2, \ldots, x_k = w_2)$ in (a, b) with $k \geqq 3$.

Let both z_1 and z_2 be up-dangles. $P = F \cup \{a, c, b, z_1, z_2, 1\}$ is a subposet of M. We show that P contains $\mathbf{C}$ or $\mathbf{E}_n$ for some n as a subposet. If $w_1 \wedge w_2 > a$, then $\mathbf{C} \cong \{a, w_1 \wedge w_2, w_1, w_2, c, b, z_1, z_2, 1\}$. For example, $w_1 < z_2$ would imply that $w_1 \leqq z_2 \wedge b = w_2$. We can now assume that $w_1 \wedge w_2 = a$. If F is down-down, then, using the definition of an attachment point and taking into account the incomparabilities that hold in a fence, we have that

$$\mathbf{E}_n \cong \{a, x_1, x_2, \ldots, x_k, c, b, z_1, z_2, 1\}$$

with $n = \frac{1}{2}(k - 3)$. If F is down-up, then

$$\mathbf{E}_n \cong \{a, x_1, x_2, \ldots, x_{k-1}, c, b, z_1, z_2, 1\}$$

with $n = \frac{1}{2}(k - 4)$. If F is up-up, then $k \geqq 5$ since $w_1 \wedge w_2 = a$, and therefore,

$$\mathbf{E}_n \cong \{a, x_2, x_3, \ldots, x_{k-1}, c, b, z_1, z_2, 1\}$$

with $n = \frac{1}{2}(k - 5)$.

By symmetry, we can now assume that z_1 is an up-dangle and z_2 is a down-dangle. We show that $Q = F \cup \{0, z_2, a, c, b, z_1, 1\}$ contains a poset in $\mathscr{L}$. If F is down-up, then

$$\mathbf{F}_n \cong \{0, z_2, a, x_1, x_2, \ldots, x_k, c, b, z_1, 1\}$$

with $n = \frac{1}{2}(k - 2)$. If F is up-up, then

$$\mathbf{F}_n \cong \{0, z_2, a, x_2, x_3, \ldots, x_k, c, b, z_1, 1\}$$

with $n = \frac{1}{2}(k - 3)$. The remaining case is that F is up-down; then

$$\mathbf{F}_n \cong \{0, z_2, a, x_2, x_3, \ldots, x_{k-1}, c, b, z_1, 1\}$$

with $n = \frac{1}{2}(k - 4)$. Thus, we have shown that w_1 and w_2 must be comparable.

If $z_1(z_2)$ were an up-dangle (down-dangle) on $[a, b]$ with attachment point $w_1(w_2)$ such that $w_1 < w_2$, then $\mathbf{F}_0 \cong \{0, z_2, a, w_1, w_2, c, b, z_1, 1\}$; for example, $z_2 < z_1$ would imply that $w_2 = z_2 \vee a \leqq z_1 \wedge b = w_1$. If z_1 and z_2 are up-dangles on $[a, b]$ with attachment points $w_1 \leqq w_2$ and $z_1 \vee b \| z_2 \vee b$, then

$$\mathbf{D} \cong \{a, w_1, c, b, z_1, z_1 \vee b, z_2, z_2 \vee b, 1\};$$

if $w_1 < w_2$ and $z_1 \vee b < z_2 \vee b$, then

$$\mathbf{F}_0 \cong \{a, c, w_1, w_2, b, z_1, z_1 \vee b, z_2, z_2 \vee b\}.$$

Therefore, there is an element $d \in (a, b)$ such that if z is an up-dangle and z' is a down-dangle on $[a, b]$ then $z > d > z'$. Moreover, the set

$$S = \{[z \wedge b, z \vee b] | z \text{ up-dangle on } [a, b]\}$$

is a chain (with respect to $\subseteq$) of closed intervals. Let $[r, s]$ be the maximum element of S.

We now show that there is a planar embedding $e_2(L)$, obtained from $e_1(L)$ by elementary transformations $T_{\sigma,\tau}^{x,y}$, $x, y \geqq d$, in which b is on the boundary of $[r, s]$. Let $[u, v]$ be the minimum element of S such that b is not on the boundary of $[u, v]$ with respect to $e_1(L)$. We show that there is a planar embedding $e_1'(L)$ obtained from $e_1(L)$ by elementary transformations $T_{\sigma,\tau}^{x,y}$, $x, y \geqq d$ in which b is on the boundary of $[u, v]$. Iteration of this procedure will provide the desired planar embedding $e_2(L)$. We can assume that b is on the left boundary of $[u', v']$ with respect to $e_1(L)$ for every $[u', v'] \in S$ such that $[u', v'] \subset [u, v]$. Let z be an up-dangle on $[a, b]$ such that $u = z \wedge b$ and $v = z \vee b$. If $b\lambda z$ for all such z, then b would be on the left boundary of $[u, v]$. Choose z so that $z\lambda b$. By Lemma 1.11, there is $z_1 \in [u, v]$ such that $z_1\lambda b$ and z_1 and b are in a common face. Clearly, z_1 is an up-dangle on $[a, b]$, $z_1 \wedge b = u$ and $z_1 \vee b = v$. Let $C_1\lambda C_2\lambda \ldots \lambda C_n$ be the $\langle u, v\rangle$-components. If z_1 and b were in the same $\langle u, v\rangle$-component, then, in any connection between z_1 and b in (u, v), there would be two consecutive elements that are on different boundaries of the common face containing z_1 and b. Since this is impossible, $b \notin C_1$.

We first consider the case that $b \notin C_n$. Suppose that both C_1 and C_n are not proper. Let $x_1(x_2)$ be a left (right) dangle on $[u, v]$ with attachment point $y_1(y_2)$. If both x_1 and x_2 are up-dangles, then $\mathbf{C} \cong \{a, u, y_1, y_2, c, v, x_1, x_2, 1\}$; if both are down-dangles, then $\mathbf{B} \cong \{0, u, c, b, x_1, y_1, x_2, y_2, v\}$; if x_1 is an up-dangle and x_2 is a down-dangle, then $\mathbf{C}^d \cong \{0, u, c, b, x_2, y_2, v, x_1, 1\}$. Therefore, C_1 (or C_n) is proper. Since b is on the left boundary of the $\langle u, v\rangle$-component B containing b, C_1 (or C_n) can be permuted with (the reflection of) B, giving a planar embedding in which b is on the left (right) boundary of $[u, v]$.

We can now assume that $b \in C_n$. Let D be the maximum indecomposable subinterval of $C_n' = C_n \cup \{u, v\}$ which contains b. Let $p < q$ be the universal bounds of D; obviously, p and q are consecutive splitting elements of C_n'. If there are no dangles on D, then, since b is on the left boundary of D, reflecting D will give a planar embedding of L in which b is on the right boundary of $[u, v]$. We complete the proof of the existence of the planar embedding $e_1'(L)$ by showing that there can be no dangles on D. Let us suppose that there is an up-dangle z on D. If $z < v$, then $z \in C_n$ and q would not be a splitting element of C_n'. Thus, $z \parallel v$, and z is a right up-dangle on $[u, v]$. Let $w = z \wedge v$; since b is not on the right boundary of $[u, v]$, $w \neq b$. If $w > b$, let P be Poset (d) of Figure 2 with a replaced by b. By Lemma 4.3, P is a subposet of L for some $n \geqq 0$. Then, $\mathbf{G}_n \cong (P - \{p, q\}) \cup \{a, c, u, z_1, v, 1\}$. If $w < b$, then $\mathbf{F}_0 \cong \{0, c, p, w, b, z_1, q, z, 1\}$. Therefore, $w \parallel b$; let F be a fence in C_n connecting w and b. In this case, we obtain the same subposets obtained previously for the bounded component $[a, b]$ with left and right dangles having incomparable attachment points. (The poset Q introduced there is isomorphic to $F \cup \{0, c, p, z_1, q, z, 1\}$.)

Now, suppose there is a down-dangle z on D. As above, z must be a right down-dangle on $[u, v]$. Let $w = z \vee u$. If $w < b$, let P be the subposet of L that is isomorphic to the dual of Poset (d) for some $n \geqq 0$ with a, u, and v replaced by b, q, and p, respectively. Then, $\mathbf{H}_n \cong (P - \{p, q\}) \cup \{0, u, c, z_1, v\}$. If $w \| b$, we obtain the duals of the subposets that we had for the case of a bounded component $[a, b]$ with two up-dangles and incomparable attachment points. (If P is the poset introduced there, $P^d \cong F \cup \{0, c, z, p, z_1, q\}$, where F is a fence in (p, q) that connects b and w). Finally, if $w > b$, let P be the subposet of L that is isomorphic to the dual of Poset (f) for some $n \geqq 0$ with a, u, and v replaced by b, q, and p, respectively. In this case, $\mathbf{H}_n \cong (P - \{p, q\}) \cup \{0, c, u, z_1, v\}$. Thus, we conclude that D can have no dangles.

We have now shown how to obtain a planar embedding $e_2(L)$ of L for which b is on the boundary of $[r_2, s_2]$, the maximum element of S (which was previously denoted by $[r, s]$). Applying the dual of this procedure to $e_2(L)$ for the down-dangles on $[a, b]$, we arrive at a planar embedding $e_3(L)$ of L for which the following conditions are satisfied:

(a) $r_1 \leqq s_1 \leqq r_2 \leqq s_2$, where $r_1 = z_1 \wedge a$, $s_1 = z_1 \vee a$, $r_2 = z_2 \wedge b$, and $s_2 = z_2 \vee b$ for some down-dangle z_1 (up-dangle z_2) on $[a, b]$;

(b) all down-dangles on $[a, b]$ are in $[r_1, s_1]$ and all up-dangles on $[a, b]$ are in $[r_2, s_2]$;

(c) a is on the boundary of $[r_1, s_1]$ and b is on the boundary of $[r_2, s_2]$.

Clearly, both $[r_1, s_1]$ and $[r_2, s_2]$ are indecomposable intervals. We now choose intervals $[u_1, v_1]$, $[u_2, v_2]$ containing $[r_1, s_1]$, $[r_2, s_2]$ subject to the following conditions:

(1) $u_1 \leqq r_1 \leqq s_1 \leqq v_1 \leqq u_2 \leqq r_2 \leqq s_2 \leqq v_2$;

(2) $[u_1, v_1]$ and $[u_2, v_2]$ are indecomposable;

(3) u_1 and u_2(v_1 and v_2) are minimal (maximal) with respect to (1) and (2).

In fact, the following four properties are also satisfied:

(4) all down-dangles (up-dangles) on $[a, b]$ are in $[u_1, v_1]$ ($[u_2, v_2]$);

(5) a is on the boundary of $[u_1, v_1]$ and b is on the boundary of $[u_2, v_2]$;

(6) there are no down-dangles on $[u_1, v_1]$ and no up-dangles on $[u_2, v_2]$;

(7) if x is a down-dangle on $[u_2, v_2]$ (up-dangle on $[u_1, v_1]$), then $x \| v_1$ ($x \| u_2$).

Indeed, (4) follows trivially from (b). If a is not on the left boundary of $[u_1, v_1]$, then there is $z \in [u_1, v_1]$ with $z \lambda a$. Since z is a left dangle on $[a, b]$, $z \in [r_1, s_1]$ which, in turn, implies that a could not be on the left boundary of $[r_1, s_1]$; thus, (5) now follows by (c). If there were a down-dangle y on $[u_1, v_1]$, then $[y \wedge u_1, v_1]$ would also be an indecomposable interval, contradicting (3). Also, if x were a down-dangle on $[u_2, v_2]$ with x comparable with v_1, then $x \geqq v_1$ which, since $[u_2 \wedge x, v_2]$ is indecomposable, contradicts (3).

If a and b were both on the left (right) boundary of $[u_1, v_1]$ and $[u_2, v_2]$, respectively, then $[a, b]$ would have no left (right) dangles; hence, b would be visible from a. Therefore, without loss of generality, a is on the left boundary of $[u_1, v_1]$ and b is on the right boundary of $[u_2, v_2]$. Furthermore, neither $[u_1, v_1]$ nor $[u_2, v_2]$ can be reflected, since a and b would then be on the same boundary of $[u_1, v_1]$ and $[u_2, v_2]$, respectively. Thus, there must be a dangle on both $[u_1, v_1]$ and $[u_2, v_2]$. In view of (6), there must be an up-dangle on $[u_1, v_1]$ and a down-dangle on $[u_2, v_2]$. In order to complete the proof of the first assertion of Theorem 1, we show that, whichever way these dangles occur, M contains a lattice in $\mathscr{L}$ as a subposet.

We first suppose that there is a left down-dangle z on $[u_2, v_2]$ with $w_2 = z \vee u_2$ and $w_1 = z \wedge v_1$.

(i) $w_2 \leqq b$: In this case $w_1 \geqq a$, since otherwise z would be a down-dangle on $[a, b]$. Let P be a subposet of L, guaranteed by Lemma 4.3, which is isomorphic to Poset (a) for $n = k$ with u and v replaced by u_1 and v_1, respectively; and let Q be a subposet of L which is isomorphic to Poset (d) for $n = m$ with v deleted, and a and u replaced by b and v_2, respectively. Note that $P \cap Q = \{z\}$. $P \cup Q \cup \{c\}$ is a subposet of M which is isomorphic to $\mathbf{G}_{k+m}$.

(ii) $w_2 \| b$ *and* $z > a$: If $w_1 > a$, then z would be an up-dangle on $[a, b]$; hence, $w_1 = a$. Let P and Q be the subposets of L provided by Lemma 4.3 which correspond to Poset (a) for $n = 0$ and the dual of Poset (e), respectively, with the replacements and deletion as in (i). Then, $\mathbf{D} \cong P \cup Q \cup \{c\}$.

(iii) $w_2 \| b$ *and* $u_1 < w_1 < a$: Let $P_1(P_2)$ and Q be the subposets of L that correspond to Poset (b) (Poset (c) for $n = k$) and the dual of Poset (e), respectively, with the replacements and deletion as in (i). Then,

$$\mathbf{D} \cong (P - \{a\}) \cup Q \cup \{c\} \ (\mathbf{G}_{k-1} \cong (P - \{v_1\}) \cup (Q - \{b\}) \cup \{c\}).$$

(iv) $w_2 > b$ *and* $z > a$: As in (ii), $w_1 = a$. If P and Q are the subposets of L that correspond to Poset (a) for $n = 0$ and the dual of Poset (f) for $n = m$, respectively, with the replacements and deletion as in (i), then $\mathbf{G}_m \cong P \cup Q \cup \{c\}$.

(v) $w_2 > b$ *and* $u_1 < w_1 < a$: Let $P_1(P_2)$ and Q be the subposets of L that correspond to Poset (b) (Poset (c) for $n = k$) and the dual of Poset (f) for $n = m$, respectively, with the replacements and deletion as in (i). In this case,

$$\mathbf{G}_m \cong (P_1 - \{a\}) \cup Q \cup \{c\} (\mathbf{G}_{k+m} \cong P \cup Q \cup \{c\}).$$

We have shown that no left down-dangle on $[u_2, v_2]$ satisfies any of the conditions (i) to (v), and, by duality, that no right up-dangle on $[u_1, v_1]$ satisfies any of the corresponding dual conditions. Let z_2 be a left down-dangle on $[u_2, v_2]$ with attachment point $w_2 = z_2 \vee u_2$; by (i), $w_2 \nleqq b$. There is also an up-dangle z_1 on $[u_1, v_1]$ with attachment point $w_1 = z_1 \wedge v_1$.

We now show that z_1 must be a right up-dangle on $[u_1, v_1]$. It suffices to show that, if z_1 were a left up-dangle on $[u_1, v_1]$, then $z_1 \leqq w_2$ because z_1 would then be a left down-dangle on $[u_2, v_2]$ satisfying one of (i) to (v), contrary to assumption. Let C be a maximal chain from 0 to w_2 through z_2, let D_1 be a maximal chain passing through the left boundaries of $[u_1, v_1]$ and $[u_2, v_2]$, and let $D = D_1 \cap [0, w_2]$. Suppose $z_1 \nleqq w_2$; then, by Lemma 1.3, z is not in the region defined by C and D. Since z_1 is on the left of D_1, z_1 is on the left of $C_1 = C \cup (D_1 \cap [w_2, 1])$. Let $x \in C_1$ be such that $z_1 \geqq x \geqq w_1$. Since $z_1 \ngeqq u_2$ and

$z_2 \| u_2$, $x \| u_2$; x is thus a left down-dangle on $[u_2, v_2]$ that satisfies one of (i) to (v).

Since, by duality, we can assume that $w_2 > b$ implies $w_1 < a$, there are the following three cases to consider.

(vi) $w_2 \| b$ *and* $w_1 \| a$: Let P be Poset (e) with u, v, and z replaced by 0, v_1, and z_1, respectively, and let Q be the dual of Poset (e) with v deleted, and a, u and z replaced by b, 1 and z_2, respectively, which, by Lemma 4.3, occur as subposets of L. $P \cup (Q - \{b\}) \cup \{c\}$ is a subposet of M which is isomorphic to $\mathbf{E}_0$.

(vii) $w_2 \| b$ *and* $w_1 < a$: Let P and Q be the subposets of L that correspond to Poset (f) for $n = k$ and the dual of Poset (e), respectively, with the replacements and deletion as in (vi). Then, $\mathbf{H}_k \cong P - \{v_1\} \cup (Q - \{b\}) \cup \{c\}$.

(viii) $w_2 > b$ *and* $w_1 < a$: Let P and Q be the subposets of L that correspond to Poset (f) for $n = k$ and the dual of Poset (f) for $n = m$, respectively, with the replacements and deletion as in (vi). Then, $\mathbf{H}_{k+m-1} \cong P \cup Q \cup \{c\}$.

We can now assume there are no right dangles on $[u_1, v_1]$ and no left dangles on $[u_2, v_2]$. Let z_1 be a left up-dangle on $[u_1, v_1]$ with attachment point $w_1 = z_1 \wedge v_1$ and let z_2 be a right down-dangle on $[u_2, v_2]$ with attachment point $w_2 = z_2 \vee u_2$. For all $x \in [u_2, v_2)$, $z_1 \nleq x$, since we have already observed that $z_1 \| u_2$, and otherwise z_1 would be a left dangle on $[u_2, v_2]$; similarly, $z_2 \ngeq x$ for all $x \in (u_1, v_1]$. It follows that $w_1 \leqq a$ and $w_2 \geqq b$ since otherwise z_1 or z_2 would be a dangle on $[a, b]$. By duality, there are only three cases left to consider.

(ix) $w_2 = b$ *and* $w_1 = a$: Let P and Q be the subposets of L that correspond to Poset (a) for $n = 0$ and the dual of Poset (a) for $n = 0$, respectively, with the replacements and deletion as in (vi). Then, $\mathbf{H}_0 \cong P \cup Q \cup \{c\}$.

(x) $w_2 = b$ *and* $w_1 < a$: Let $P_1(P_2)$ and Q be the subposets of L that correspond to Poset (b) (Poset (c) for $n = k$) and the dual of Poset (a) for $n = 0$, respectively, with the replacements and deletion as in (vi). Then, $\mathbf{H}_0 \cong (P_1 - \{a\}) \cup Q \cup \{c\}$ ($\mathbf{H}_k \cong P_2 \cup Q \cup \{c\}$).

(xi) $w_2 > b$ *and* $w_1 < a$: Let $P_1(P_2)$ and $Q_1(Q_2)$ be the subposets of L that correspond to Poset (b) (Poset (c) for $n = k$) and the dual of Poset (b) (the dual of Poset (c) for $n = m$), respectively, with the replacements and deletion as in (vi). By duality, M has one of the following subposets: $\mathbf{H}_0 \cong (P_1 - \{a\}) \cup (Q_1 - \{b\}) \cup \{c\}$; $\mathbf{H}_k \cong (P_1 - \{a\}) \cup Q_2 \cup \{c\}$; or $\mathbf{H}_{k+m} \cong P_2 \cup Q_2 \cup \{c\}$.

This completes the proof of the fact that a finite lattice is planar if and only if it does not have a subposet isomorphic to a lattice in $\mathscr{L}$. Furthermore, it is easy to verify that no lattice is repeated in the list given for $\mathscr{L}$.

Let $\mathscr{F}$ be a set of finite lattices such that a lattice is planar if and only if it does not have a subposet isomorphic to a lattice in $\mathscr{F}$. We will show that $\mathscr{L} \subseteq \mathscr{F}$. Obviously, every lattice in $\mathscr{F}$ is nonplanar. If $L \in \mathscr{L}$, there is a subposet K of L that is isomorphic to a lattice in $\mathscr{F}$. If we show that no proper subposet of L is a nonplanar lattice, it will follow that $K = L$, and therefore, $\mathscr{L} \subseteq \mathscr{F}$. An element of a finite lattice will be called *irreducible* if it is a join-irreducible element of the lattice distinct from 0, or a meet-irreducible element distinct from 1. We will apply the following general observation.

PROPOSITION 5.3. *Let L be a finite lattice and K be a subposet of L which is a lattice. If $K \subset L$, then there is $x \in L - K$ that is irreducible in L. Moreover, $L - \{x\}$ is a lattice.*

Proof.† Let $y \in L - K$ and suppose that K contains every irreducible element of L. Then, y can be expressed as

$$y = \bigvee_L (a_i | 1 \leqq i \leqq m) = \bigwedge_L (b_j | 1 \leqq j \leqq n),$$

where $a_i (1 \leqq i \leqq m)$ are join-irreducibles of L distinct from 0, and $b_j (1 \leqq j \leqq n)$ are meet-irreducibles of L distinct from 1. By assumption, all the a_i's and b_j's are in K. Hence, $\bigvee_K (a_i | 1 \leqq i \leqq m) \geqq y \geqq \bigwedge_K (b_j | 1 \leqq j \leqq n)$. If both m and n are nonzero, then $a_i \leqq b_j$ for all i and j which implies that $\bigvee_K (a_i | 1 \leqq i \leqq m) \leqq \bigwedge_K (b_j | 1 \leqq j \leqq n)$. Therefore, $y = \bigvee_K (a_i | 1 \leqq i \leqq m) \in K$, contrary to assumption. If $m = 0$, then $y = 0 \in L$ and $n \geqq 1$. In this case, $\bigwedge_K (b_j | 1 \leqq j \leqq n) \leqq 0$; since equality must hold, we again obtain a contradiction. The second statement is trivial.

For every lattice L in Figure 1, it is easy to check that $L - \{x\}$ is planar for any irreducible element x of L. If the lattice K is a proper subposet of $L \in \mathscr{L}$, then, by the above proposition, K is a subposet of $L - \{x\}$ for an irreducible x. Since $L - \{x\}$ is planar, so is K. This completes the proof of Theorem 1.

6. Some results related to Theorem 1. For finite lattices, Proposition 5.2 showed that planarity and dimension $\leqq 2$ are equivalent properties. Using the compactness property of finite dimension, Theorem 1 can be extended to all lattices of dimension $\leqq 2$.

THEOREM 6.1. *A lattice has dimension $\leqq 2$ if and only if it does not contain any lattice in $\mathscr{L}$ as a subposet. Moreover, $\mathscr{L}$ is the minimum such list of lattices.*

Proof. Since the dimension of each lattice in $\mathscr{L}$ exceeds 2 (in fact, equals 3), one direction is immediate. If K is a lattice whose dimension exceeds 2, then there is a finite subset S of K whose dimension exceeds 2. The join-semilattice L of K generated by $S \cup \{\bigwedge S\}$ is a finite lattice of dimension > 2; hence, by Proposition 5.2, L is nonplanar. Since, by Theorem 1, L contains a lattice in $\mathscr{L}$ as a subposet, so does K. The second statement of the theorem follows immediately from the corresponding statement of Theorem 1.

K. A. Baker has shown that the dimension of the lattice obtained by completion by cuts of a poset of dimension n also has dimension n (cf. [**1**, Theorem 4.1]).

†Simplified by B. Wolk.

COROLLARY 6.2. *A poset has dimension* $\leqq 2$ *if and only if its completion by cuts does not contain any lattice in* $\mathscr{L}$ *as a subposet.*

The following result is proved in R. Wille [**10**].

THEOREM 6.3. *A modular lattice has dimension* $\leqq 2$ *if and only if it does not contain* $\mathbf{A}_0$, $\mathbf{B}$, $\mathbf{B}^d$, $\mathbf{C}$ *or* $\mathbf{C}^d$ *as a subposet.*

We will prove a more general theorem in which only four *modular* lattices are mentioned. In particular, this will show that the list in Theorem 6.3 is redundant since one of $\mathbf{B}$ or $\mathbf{B}^d$ can be omitted. To this end, let $\mathbf{I}$ and $\mathbf{J}$ be the modular lattices illustrated in Figure 3. Since $\mathbf{I}$ contains $\mathbf{B}$ (and $\mathbf{B}^d$) and $\mathbf{J}$ contains $\mathbf{C}$, both $\mathbf{I}$ and $\mathbf{J}$ are nonplanar.

A sublattice S of a finite lattice L *can be obtained by dismantling* L if there is a sequence

$$L = L_0 \supset L_1 \supset \ldots \supset L_n = S$$

of sublattices of L satisfying $|L_i| = |L_{i+1}| + 1$ for $0 \leqq i \leqq n - 1$.

LEMMA 6.4. *Let* M *be a finite modular lattice and let* M_1 *be a sublattice of* M *that can be obtained by dismantling* M. *If* P *is a cover-preserving sublattice of* M_1 *such that* x *and* y *are not both splitting elements of* P *whenever* $x \prec y$ *in* P, *then* P *is a cover-preserving sublattice of* M.

Proof. By induction, we can assume that $M = M_1 \cup \{c\}$, where c is doubly irreducible in M and $a \prec c \prec b$ in M. Let P be a cover-preserving sublattice of M_1 that satisfies the condition of the lemma. The only cover of P that c could destroy is $a \prec b$. Suppose that $a \prec b$ in P. Without loss of generality, there is a cover d of a in P with $d \neq b$; hence, $\{a, c, b, d, b \vee d\}$ would be a nonmodular sublattice of M, a contradiction.

THEOREM 6.5. *A finite modular lattice is planar if and only if it does not contain* $\mathbf{A}_0$, $\mathbf{I}$, $\mathbf{J}$ *or* $\mathbf{J}^d$ *as a cover-preserving sublattice. Moreover, if* $\mathscr{F}$ *is a set of finite modular lattices such that a modular lattice is planar if and only if it does not contain any lattice in* $\mathscr{F}$ *as a subposet, then* $\mathscr{F}$ *contains* $\mathbf{A}_0$, $\mathbf{I}$, $\mathbf{J}$ *and* $\mathbf{J}^d$.

Proof. As in the proof of Theorem 1, one direction is immediate since $\mathbf{A}_0$, $\mathbf{I}$ and $\mathbf{J}$ are nonplanar.

Let M be a nonplanar finite modular lattice. If M does not contain $\mathbf{A}_0$ as a cover-preserving sublattice, then it follows from the proof of Theorem 3.5 of [**6**] that M is dismantlable. M can be dismantled down to a nonplanar sublattice M_1 so that $M_1 = L \cup \{c\}$ for a planar lattice L. By Lemma 6.4, we can assume that $M = M_1$. If $a \prec c \prec b$ in M, then, by virtue of Lemma 5.1, b is not visible from a in any planar embedding of L. Since M is modular, there is d in L such that $a \prec d \prec b$. Let $r_1 \lambda r_2 \lambda \ldots \lambda r_m$ ($s_1 \lambda s_2 \lambda \ldots \lambda s_n$) be all the lower (upper) covers of d in L with respect to a planar embedding $e(L)$ of L. The join of any two distinct s_j's is the same since otherwise $\mathbf{A}_0$ would be a

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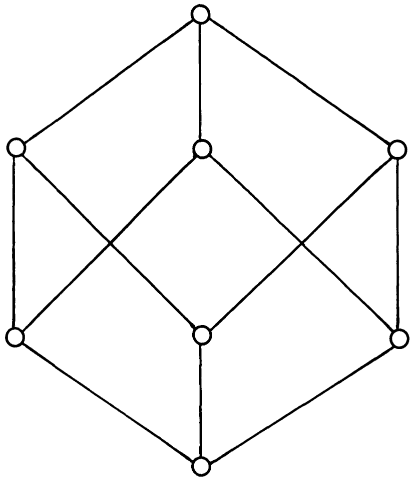
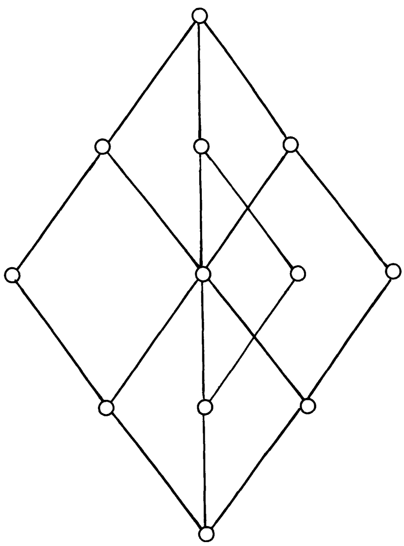
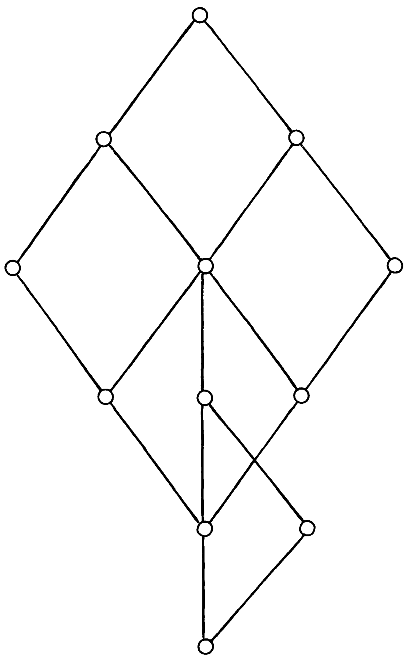


$\mathbf{A}_0$



I



J

FIGURE 3

cover-preserving sublattice of M; the dual statement holds for the r_i's. If b is not s_1 or s_n, and there is both a left dangle z_1 and a right dangle z_2 on $[d, s_1 \vee s_n]$, then a cover-preserving sublattice of M is isomorphic to **I**, **J** or $\mathbf{J}^d$. For example, if z_1 and z_2 are both down-dangles, they can be chosen so that $z_1 \prec s_1$ and $z_2 \prec s_n$. If $a = r_1$, then $\{a, z_1, c, b, s_1 \vee s_n\}$ would be a nonmodular sublattice of M which is impossible; therefore,

$$\mathbf{I} \cong \{r_1 \wedge r_m, r_1, a, r_m, d, z_1, c, z_2, s_1, b, s_n, s_1 \vee s_n\}.$$

We now show that the above situation or its dual actually must occur. Otherwise, $a = r_1 \neq r_m$ and $b = s_n \neq s_1$ in some planar embedding of L. For example, if there is no right dangle on $[d, s_1 \vee s_n]$ then b and s_n can be interchanged, giving a planar embedding of L in which b is visible from a. If d were a splitting element of L, then b would be visible from a in some planar embedding of L, a contradiction. We can assume that there is y in L with $y\lambda d$. By Lemma 1.11, there is a face containing d and some x in L with $x\lambda d$. In order that this face be modular, $x \wedge d \prec x$, $d \prec x \vee d$; then, since $a = x \wedge d$, $\{a, x, c, b, s_1 \vee b\}$ is a nonmodular sublattice of M, a contradiction.

None of the lattices $\mathbf{A}_0$, **I**, **J** or $\mathbf{J}^d$ contain one of the others as a subposet since each of the latter three consist of 12 elements and are dismantlable. The second statement now follows immediately.

In order to extend Theorem 6.5 to infinite lattices, we need the following lemma.

LEMMA 6.6. *If M is a modular lattice that does not contain $\mathbf{A}_0$, **I**, **J** or $\mathbf{J}^d$ as a subposet, then any finitely generated sublattice of M is finite.*

This lemma and its proof are based on an idea of R. Wille [**10**, Theorem 5]. The only real novelty here is Lemma 6.8.

LEMMA 6.7 (cf. [**10**, Lemma 4]). *Let M be a modular lattice with no subposet isomorphic to **I**. If $\{a, c_1, c_2, c_3, e\}$ is a nondistributive sublattice of M with $a < c_i < e$, then $a \prec c_i \prec e$ $(i = 1, 2, 3)$.*

Proof. If the conclusion were false, then, by modularity, there is b_1 in M with $a < b_1 < c_1$. Let $b_2 = (b_1 \vee c_3) \wedge c_2$, $b_3 = (b_1 \vee c_2) \wedge c_3$, $c_4 = (b_1 \vee c_2) \wedge (b_1 \vee c_3)$, and $d_i = c_i \vee c_4$ for $i = 1, 2, 3$. As noted in Lemma 4 of [**10**], $a < b_i < c_i$ and $c_i \wedge c_4 = b_i$ for $i = 1, 2, 3$. Since for $i = 2$ or 3, $d_i = b_1 \vee c_i$, it follows similarly that $c_i < d_i < e$. Also, $c_1 < d_1 < e$ since otherwise $c_1 \vee c_4 = e$, implying that both c_4 and d_2 are comparable relative complements of c_1 in $[b_1, e]$. Therefore, $\{a, b_1, b_2, b_3, c_1, c_2, c_3, c_4, d_1, d_2, d_3, e\}$ is isomorphic to **I**.

LEMMA 6.8. *Let M be a modular lattice with no subposet isomorphic to $\mathbf{A}_0$. If a, b_1, b_2, c, z_1 and z_2 are elements of M such that $b_1 \| b_2$, $b_1 \wedge b_2 = a$, $b_1 \vee b_2 = c$, $z_1 \wedge c = b_1$ and $z_2 \wedge c = b_2$, then $z_1 \vee c \| z_2 \vee c$.*

Proof. Note that $z_1 \| b_2$ and $z_2 \| b_1$. If $z_1 \wedge z_2 \| c$, then

$$\{a, z_1, b_1, c, b_2, z_2, z_1 \wedge z_2, z_1 \vee z_2 \vee c\}$$

would be isomorphic to $\mathbf{A}_0$. Therefore, $z_1 \wedge z_2 \leqq c$; hence, $z_1 \wedge z_2 = (z_1 \wedge c) \wedge (z_2 \wedge c) = b_1 \wedge b_2 = a$. Suppose $z_1 \vee c \leqq z_2 \vee c$. Then, $z_2 \vee z_1 = z_2 \vee b_2 \vee z_1 \vee b_1 = z_2 \vee c$, and $z_2 \vee b_1 = z_2 \vee b_2 \vee b_1 = z_2 \vee c$. Hence, both b_1 and z_1 are comparable relative complements of z_2 in $[a, z_2 \vee c]$, a contradiction.

Proof of Lemma 6.6. If $M_5 = \{a, b_1, b_2, b_3, c\}$ is a sublattice of M with $a \prec b_i \prec c$ $(i = 1, 2, 3)$, then some b_i is doubly irreducible in M. For example, if all the b_i's are join-reducible, then there are z_1, z_2, z_3 in M such that $z_i \wedge c = b_i$ $(i = 1, 2, 3)$. By Lemma 6.7, $z_1 \vee c$, $z_2 \vee c$, and $z_3 \vee c$ are distinct pairwise incomparable elements; therefore,

$$\mathbf{I} \cong \{a, b_1, b_2, b_3, c, z_1, z_2, z_3, z_1 \vee c, z_2 \vee c, z_3 \vee c, z_1 \vee z_2 \vee z_3 \vee c\}.$$

The other cases are similar.

Let S be a finite subset of M. Let T be the set of all elements x of S that are doubly irreducible in M and appear as some b_i in some sublattice of M of the form M_5 as above; set $x_0 = a$ and $x_1 = c$. Let $T_k = \{x_k | x \in T\}$ for $k = 0, 1$. By Lemmas 6.7 and 6.8, $(S - T) \cup T_0 \cup T_1$ generates a distributive sublattice N of M. Then, the finite sublattice $N \cup T$ of M includes S.

THEOREM 6.9. *A modular lattice has dimension $\leqq 2$ if and only if it does not contain one of the modular lattices $\mathbf{A}_0$, $\mathbf{I}$, $\mathbf{J}$ or $\mathbf{J}^d$ as a subposet. Moreover, this is the minimum such list of modular lattices.*

Proof. The proof of Theorem 6.1 is used with Theorem 6.5 replacing Theorem 1. The only other difference is that L is the *sublattice* generated by S; L is finite by Lemma 6.6.

In [**6**], we proved that every finite dismantlable lattice which is not a chain contains two incomparable doubly irreducible elements. We now show that this can be sharpened for nonplanar finite dismantlable lattices. To this end, we need the following lemma.

LEMMA 6.10. *Let S be a sublattice of a finite lattice L that can be obtained by dismantling L. If S contains n pairwise incomparable doubly irreducible elements, then so does L.*

Proof. Let $a_1, a_2, \ldots, a_n$ be n pairwise incomparable doubly irreducible elements in S. By induction, we can assume that $L = S \cup \{c\}$, where c is doubly irreducible in L. All the elements $a_1, a_2, \ldots, a_n$ would be doubly irreducible in L unless c covers or is covered by one of them. Therefore, we can assume that $a_1 \prec c$; a_1 is thus the unique lower cover of c in L. It follows from $a_1 \| a_i$ that $c \| a_i$ for every $i = 2, 3, \ldots, n$. Thus $\{a_2, a_3, \ldots, a_n, c\}$ is an n-element set of pairwise incomparable doubly irreducible elements in $S \cup \{c\}$.

THEOREM 6.11. *Any nonplanar finite dismantlable lattice contains at least three pairwise incomparable doubly irreducible elements.*

Proof. Let M be a nonplanar finite dismantlable lattice. By virtue of Lemma

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6.10, we can assume that $M = L \cup \{c\}$, where c is doubly irreducible in M, $a \prec c \prec b$ in M, b is not visible from a in L, and L is planar with a planar embedding $e(L)$. We denote the left (right) boundary of a region R in L by $l(R)$ $(r(R))$.

Let

$$C = l([0, a] \cup [a, b] \cup [b, 1]),\ D = r([0, a] \cup [a, b] \cup [b, 1]),$$

S be the left side of C, and T be the right side of D. It suffices to show that $S \cup T - ([0, a] \cup [b, 1])$ contains two incomparable doubly irreducible elements x and y in L, since $\{x, y, c\}$ would then consist of three pairwise incomparable doubly irreducible elements in M. We now carry out the proof in a sequence of simple steps, each of which we elaborate upon at most briefly.

(i) $l(S) = l(L)$.

(ii) *There is* $d \in \{a, b\}$ *such that* $d \notin l(L)$. Since b is not visible from a, there is a left dangle on $[a, b]$.

(iii) *There is* $z \in l(S) - C$ *which is doubly irreducible in* L. By Proposition 2.6, there is $z \in l(L)$ such that z is doubly irreducible in L and $z \parallel d$.

Let u be a minimal such z and choose v analogously in $r(T) - D$.

(iv) *Without loss of generality*, $u > v$. If $u \parallel v$, we are done.

Let $E = r([v, u])$, $h \in C \cap E$, and choose k maximal in $D \cap E$. Let

$$H = (C \cap [0, h]) \cup (E \cap [h, u]) \cup (l(L) \cap [u, 1])$$

and

$$K = (r(L) \cap [0, v]) \cup (E \cap [v, k]) \cup (D \cap [k, 1]).$$

(v) $b > h \geqq k > a$.

(vi) $l(L) = H$. If there were $x \in H$, $x < u$, and $x \notin l(L)$, then there would be a $y \in l(S) - C$, doubly irreducible in L, $y \parallel x$, and $y < u$, contradicting the minimality of u.

(vii) *Without loss of generality*, $r(L) = K$. If $r(L) \neq K$, there is $x \in K$ such that $x \notin r(L)$. Then $x > v$ and there is $y \in r(T) - D$, doubly irreducible in L, $y \parallel x$, and $y > v$. If $y \parallel u$, we are done. If $y < u$, then $y \in [v, u]$ which is impossible since $y \notin K$; if $y > u$, then a maximal chain from u to y crosses C at $z \leqq b$ so that $u \in [a, b]$, which is impossible.

(viii) *Without loss of generality*, $[k, b]$ *is a chain*. Otherwise, there is $y \in (k, b)$ such that $y \notin r([k, b])$. By the proof of Proposition 2.6, there is a doubly irreducible element x of L such that $x \parallel y$ and $x \in r(L)$; thus, $x \in r([k, d]) \subseteq D$. Since $x > k$, $x \parallel u$ by the choice of k and we would be done.

(ix) *h is a splitting element of L; that is*, $L = [0, h] \cup [h, 1]$.

Finally, by reflecting $[h, 1]$ in $e(L)$ we obtain a planar embedding of L in which b is visible from a, a contradiction which completes the proof.

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