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# THE CHARACTERS AND STRUCTURE OF A CLASS OF MODULAR REPRESENTATION ALGEBRAS OF CYCLIC $p$-GROUPS

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**Abstract**

Let $\mathscr{A}^*_{p,m}$ be the modular representation algebra of the cyclic group of order $p^m$ over the prime field $Z_p$. The characters of $\mathscr{A}^*_{p,m}$ are derived. For $p=2$, this provides an alternative proof of a result due to Carlson (1975), that $\mathscr{A}^*_{2,m}$ is a local ring. It is shown that for $p>2$, $\mathscr{A}^*_{p,m}$ is a direct sum of $2^m$ local rings. Their dimensions and primitive idempotents are derived.

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## 1. Introduction

Let $G$ be a cyclic group of order $p^m$, where $p$ is a prime. Let $Z_p$ be the residue field of integers modulo $p$. Let $V_i$ be the isomorphism class of indecomposable $(Z_p, G)$-modules of dimension $i$.

For $0 \leqslant k \leqslant m$, the elements of the set $\{V_i,\ i=1,\ldots,p^k\}$ form a basis for an algebra $\mathscr{A}^*_{p,k}$ over $Z_p$. Products in the algebra are defined by

$$V_i \times V_j = \sum_{l=1}^{p^k} a_{ijl}\, V_l,$$

where $a_{ijl}$ is the number (reduced modulo $p$) of terms of $Z_p$-dimension $l$ in the decomposition of $M_i \otimes_{Z_p} M_j$ into the direct sum of indecomposable modules, $M_i$ and $M_j$ being modules in $V_i$ and $V_j$ respectively.

A character of $\mathscr{A}^*_{p,k}$ is a non-trivial algebra homomorphism from $\mathscr{A}^*_{p,k}$ to $Z_p$. By examining these characters in the case $p=2$ it is shown that $\mathscr{A}^*_{2,m}$ is a local ring, proved by Carlson (1975). For $p>2$ it is shown that $\mathscr{A}^*_{p,m}$ has $2^m$ distinct characters and hence is isomorphic to a direct sum of $2^m$ local rings. Their idempotents and dimensions are derived.



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A major part of this work is based on formulae due to Green (1962), extended by the author (1977). The technique of using Chebyshev polynomials to derive the characters can be used as an alternative method to that of Green in Section 2 of his paper.

NOTATION. When a product decomposition formula is used, the expression $(\bmod p)$ means the coefficients are to be regarded as elements of $Z_p$. Similarly, the expression $\mathrm{res}_p(r)$ means the residue of $r$, modulo $p$.

## 2. The structure of $\mathscr{A}^*_{2,m}$

THEOREM 1. *There exists exactly one character of $\mathscr{A}^*_{2,m}$, and this is defined by*

$$\varphi^m(V_r) = \mathrm{res}_2(r), \quad 1 \leqslant r \leqslant 2^m.$$

PROOF. (All references are to Green, 1962.) $V_1 \times V_1 = V_1$, and by (2.7d), $V_2 \times V_2 = 0$. Hence there exists only one character of $\mathscr{A}^*_{2,1}$, defined by

$$\varphi^1(V_1) = 1, \quad \varphi^1(V_2) = 0.$$

Let $\theta$ be any character of $\mathscr{A}^*_{2,k}$, $1 < k \leqslant m$. Let $q = 2^{k-1}$. By (2.7d),

$$V_q \times V_q = V_{2q} \times V_{2q} = 0.$$

Hence $\theta(V_q) = \theta(V_{2q}) = 0$. By (2.8e), $V_{q+1} \times V_{q+1} = V_1$, and hence $\theta(V_{q+1}) = 1$. By (2.8c), for $1 \leqslant r_1 \leqslant q$,

$$V_{r_1} \times V_{q+1} \equiv V_{q+r_1} + (r_1 - 1)\, V_q \pmod 2$$

and hence $\theta(V_{q+r_1}) = \theta(V_{r_1})$.

Now any character of $\mathscr{A}^*_{2,k}$ is entirely defined by a character of $\mathscr{A}^*_{2,k-1}$: the theorem follows by induction on $k$.

COROLLARY. (Carlson, 1975.) *$\mathscr{A}^*_{2,m}$ is a local ring.*

## 3. Preliminary formulae

The Chebyshev polynomial $S_n$, with integral coefficients, is as defined in Abramovitz and Stegun (1972), Section 22.7: for $x$ indeterminate,

$$S_0(x) = 1, \quad S_1(x) = x, \quad S_n(x) = xS_{n-1}(x) - S_{n-2}(x).$$

The polynomials $A_n$ and $F_n$ are defined by

$$A_0(x) = 1, \quad A_1(x) = x - 1, \quad A_n(x) = (x-1)\, A_{n-1}(x) - A_{n-2}(x),$$
$$F_0(x) = 1, \quad F_1(x) = x, \quad F_n(x) = (x-1) F_{n-1}(x) - F_{n-2}(x).$$

These definitions may be extended to

$$S_{-1}(x) = A_{-1}(x) = 0.$$

Induction on $n$ now gives

(1) $A_n(x) = S_n(x_{-1})$ $(n \geqslant -1)$,

(2) $F_n(x) = A_n(x) + A_{n-1}(x)$ $(n \geqslant 0)$.

Using the results in (2) and (8) of the formulae section of Snyder (1966) it is easy to show that

(3) $F_n(x) = S_{2n}((x+1)^{\frac{1}{2}})$ $(n \geqslant 0)$.

By Section 22.7 in Abramovitz and Stegun and the above sections in Snyder, together with equations (1)–(3) above, the following factorizations are straightforward exercises:

(4) $\quad F_{2n}(x)+1 = F_n(x)\,[F_n(x) - F_{n-1}(x)]$ $(n \geqslant 1)$,

(5) $F_{2n-1}(x)+x = F_n(x)\,[F_{n-1}(x) - F_{n-2}(x)]$ $(n \geqslant 2)$,

(6) $\quad F_{2n}(x)-1 = A_{n-1}(x)\,[F_{n+1}(x) - F_{n-1}(x)]$ $(n \geqslant 1)$,

(7) $F_{2n-1}(x)-x = A_{n-2}(x)\,[F_{n+1}(x) - F_{n-1}(x)]$ $(n \geqslant 1)$.

A further factorization is possible:

(8) $F_{n+1}(x) - F_{n-1}(x) = (-1)^n(x+1)F_n(2-x)$ $(n \geqslant 1)$.

(Direct calculation shows (8) holds at $n = 1, n = 2$ : induction on $n$ yields the general result.)

Let $x \in Z_p$, $p > 2$. *Henceforth all polynomials have their coefficients in* $Z_p$. In Section 4, the solutions (over $Z_p$) of the following pairs of simultaneous equations are required:

(a) $F_{p-1}(x)+1 = 0, \quad F_{p-2}(x)+x = 0$,

(b) $F_{p-1}(x)-1 = 0, \quad F_{p-2}(x)-x = 0$.

For $p = 3$, equations (a) have the solution $x = 0$, while for (b), $x = 2$. For $p > 3$, on setting $p = 2n+1$ and applying equations (4)–(8) above, the equations reduce to

(a′) $F_n(x) = 0$,

(b′) $(x+1)F_n(2-x) = 0$

provided only that:

(i) $[F_n(x) - F_{n-1}(x)]$ and $[F_{n-1}(x) - F_{n-2}(x)]$ cannot be simultaneously zero, and

(ii) $A_{n-1}(x)$ and $A_{n-2}(x)$ cannot be simultaneously zero. These provisos are immediate from the definitions of $F_n$ and $A_n$, using descending induction on the subscripts to reach a contradiction.

It remains to find the solutions to equations (a′) and (b′) above. Two lemmas are required:

Lemma 1

$$S_{p-1}(x) \equiv (x^2-4)^n \quad (mod\, p) \quad (p = 2n+1).$$

Proof. Snyder (1966) gives the result

$$S_{2n}(x) = \sum_{r=0}^{n} (-1)^r \binom{2n-r}{r} x^{2n-2r}$$

Hence on expansion of $(x^2-4)^n$ it is sufficient to show that

$$4^r\binom{n}{r} \equiv \binom{2n-2r}{r} \pmod{p}, \quad 0 \leqslant r \leqslant n, \quad \text{where } p = 2n+1.$$

This is an elementary though tedious exercise and is omitted here.

LEMMA 2

$$F_n(x) \equiv (x-3)^n \quad (\mathit{mod}\, p) \quad (p = 2n+1).$$

PROOF. Apply equations (1) and (3) and Lemma 1.

COROLLARIES.
(i) $S_{p-1}(x) = 0$ *implies* $x = 2$ or $x \equiv -2$ $(\mathit{mod}\, p)$.
(ii) *The solution to* (a′) *is* $x \equiv 3$ $(\mathit{mod}\, p)$.
(iii) *The solution to* (b′) *is* $x \equiv -1$ $(\mathit{mod}\, p)$.
Three further results are required in Section 4:

LEMMA 3. *For* $r \geqslant -1$, $S_r(2) \equiv r+1$ $(\mathit{mod}\, p)$.

PROOF. Apply induction on $r$ using the definition of $S_n$.

LEMMA 4. *For* $a \geqslant 0$, $F_a(-1) \equiv (-1)^a$ $(\mathit{mod}\, p)$.

PROOF. Apply 22.4.5 in Abramovitz and Stegun to equation (3) above.

LEMMA 5. *For* $a \geqslant 0$, $F_a(3) \equiv 2a+1$ $(\mathit{mod}\, p)$.

PROOF. Apply equations (1) and (3) and Lemma 3.

## 4. The characters of $\mathscr{A}^*_{p,m}$

In this section, we assume that $p > 2$. It is shown that $\mathscr{A}^*_{p,m}$ has exactly $2^m$ distinct characters: these are expressed in terms of the $2^{m-1}$ characters of $\mathscr{A}^*_{p,m-1}$, $m \geqslant 2$.

We first derive the characters of $\mathscr{A}^*_{p,1}$:

THEOREM 2. *There exist exactly two characters of* $\mathscr{A}^*_{p,1}$, *and these are defined by*

$$\varphi_0^1(V_r) = \mathrm{res}_p((-1)^{r-1}r),$$
$$\varphi_1^1(V_r) = \mathrm{res}_p(r)$$

*for* $1 \leqslant r \leqslant p$.

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PROOF. By Green (1962) or Renaud (1977), products in $\mathscr{A}^*_{p,1}$ are determined by

$$V_1 \times V_r = V_r \qquad (1 \leqslant r \leqslant p),$$
$$V_2 \times V_r = V_{r-1} + V_{r+1} \qquad (1 \leqslant r < p)$$

and

$$V_r \times V_p \equiv rV_p \pmod p \quad (1 \leqslant r \leqslant p).$$

Let $\theta$ be any character of $\mathscr{A}^*_{p,1}$. Now $\theta(V_1) = 1$, and since $V_p \times V_p = 0$, $\theta(V_p) = 0$. Let $\theta(V_2) = x$. Then for $2 \leqslant r \leqslant p, \theta(V_r) = x\theta(V_{r-1}) - \theta(V_{r-2})$: that is,

$$\theta(V_r) = S_{r-1}(x), S_n$$

as defined in Section 3 above.

The permissible values of $x$ are now the solutions of $S_{p-1}(x) = 0$: by Lemma 1, these are $x = 2$ or $x \equiv -2 \pmod p$. By Lemma 3, the first value gives $\varphi_1^1$, while since $S_n(-x) = (-1)^n S_n(x)$ by 22.4.5 in Abramovitz and Stegun, the second value gives $\varphi_0^1$.

For $k > 1$, we now derive the characters of $\mathscr{A}^*_{p,k}$ in terms of those of $\mathscr{A}^*_{p,k-1}$. Let $r = r_0 q + r_1$, $1 \leqslant r < p^k$, $q = p^{k-1}$, $0 \leqslant r_1 < q$. Let $\varphi_i^{k-1}$, $i = 0, \dots 2^{k-1}-1$, be the characters of $\mathscr{A}^*_{p,k-1}$.

THEOREM 3. *The characters of* $\mathscr{A}^*_{p,k}$ *are defined by*

$$\varphi_i^k(V_r) = (-1)^{r_0} \varphi_i^{k-1}(V_{r_1}) \qquad (0 \leqslant i < 2^{k-2}),$$
$$\varphi_i^k(V_r) = (-1)^{r_0}(2r_0+1)\varphi_i^{k-1}(V_{r_1}) \qquad (2^{k-2} \leqslant i < 2^{k-1}),$$
$$\varphi_i^k(V_r) = (2r_0+1)\varphi_{i-2^{k-1}}^{k-1}(V_{r_1}) \qquad (2^{k-1} \leqslant i < 2^{k-1} + 2^{k-2})$$

and

$$\varphi_i^k(Vr) = \varphi_{i-2^{k-1}}^{k-1}(V_{r_1}) \qquad (2^{k-1} + 2^{k-2} \leqslant i < 2^k).$$

PROOF. (References are to Green, 1962.) Let $\theta$ be any character of $\mathscr{A}^*_{p,k}$. By (2.7d), $\theta(V_q) = \theta(V_{pq}) = 0$. By (2.5b), $V_{q-1} \times V_{q-1} \equiv V_1 + (q-2)V_q \pmod p$ and hence $\theta(V_{q-1}) = 1$ or $-1$. Let $\theta(V_{q+1}) = x$. By (2.9c), for $1 < a \leqslant p$,

$$V_{q-1} \times V_{(a-1)q+1} \equiv V_{aq-1} + (q-2)V_{(a-1)q} \pmod p.$$

By (2.8d), for $2 \leqslant a < p$,

$$V_{q+1} \times V_{(a-1)q+1} \equiv V_{(a-2)q+1} + (q-2)V_{(a-1)q} + V_{aq-1} + V_{aq+1} \pmod p.$$

CASE 1. $\theta(V_{q-1}) = 1$.
Now $\theta(V_{aq-1}) = \theta(V_{(a-1)q+1})$, $1 \leqslant a \leqslant p$. Hence

$$\theta(V_{aq+1}) = (x-1)\theta(V_{(a-1)q+1}) - \theta(V_{(a-2)q+1}), \quad 2 \leqslant a < p,$$

and so $\theta(V_{aq+1}) = F_a(x)$, $0 \leqslant a < p$.

CASE 2. $\theta(V_{q-1}) = -1$.
The same reasoning as in Case 1 gives

$$\theta(V_{aq+1}) = -F_a(-x), \quad 0 \leqslant a < p.$$

By (2.5b), $V_{pq-1} \times V_{pq-1} \equiv V_1 - 2V_{pq} \pmod p$, and hence $\theta(V_{pq-1}) = 1$ or $-1$. Combining all possibilities, four situations arise:

(i) $\theta(V_{q-1}) = 1$, $F_{p-1}(x) = 1$, $F_{p-2}(x) = x$,

(ii) $\theta(V_{q-1}) = 1$, $F_{p-1}(x) = -1$, $F_{p-2}(x) = -x$,

(iii) $\theta(V_{q-1}) = -1$, $F_{p-1}(-x) = -1$, $F_{p-2}(-x) = x$

and

(iv) $\theta(V_{q-1}) = -1, F_{p-1}(-x) = 1, F_{p-2}(-x) = -x$.

By the results obtained in Section 3, the permissible values for $x$ in these cases are:

(i) $x = -1$,

(ii) $x \equiv 3 \pmod p$,

(iii) $x \equiv -3 \pmod p$

and

(iv) $x = 1$.

By Lemma 2.3 in Renaud (1977), an extension of Green's formulae, for $1 \leqslant r < pq$,

$$V_r \equiv V_{r_1} \times V_{r_0 q+1} - (r_1 - 1) V_{r_0 q} \pmod p.$$

Hence $\theta(V_r) = \theta(V_{r_1}) \times \theta(V_{r_0 q+1})$. Induction up to $k-1$ on the characters in the theorem shows that $\varphi_i^{k-1}(V_{q-1}) = 1$ for $0 \leqslant i < 2^{k-2}$, while $\varphi_i^{k-1}(V_{q-1}) = -1$ for $2^{k-2} \leqslant i < 2^{k-1}$.

Now result (i) above gives rise to the first set of characters, (ii) to the third set, (iii) to the second set and (iv) to the fourth set, on applying Lemmas 4 and 5 in Section 3.

COROLLARY. *$\mathscr{A}^*_{p,m}$ has exactly $2^m$ distinct characters. Hence $\mathscr{A}^*_{p,m}$ is isomorphic to a direct sum of $2^m$ local rings.*

## 5. The structure of $\mathscr{A}^*_{p,m}$ $(p > 2)$

For $k = 1, \ldots, m$, let

$$e_{k,0} = \tfrac{1}{2}(V_1 + V_{p^k-1} - V_{p^k})$$

and

$$e_{k,1} = \tfrac{1}{2}(V_1 - V_{p^k-1} + V_{p^k}),$$

where $\frac{1}{2}$ denotes the inverse of 2 in $Z_p$. Application of the appropriate product formulae in Green shows $e_{k,0}$ and $e_{k,1}$ are orthogonal idempotents.

For any integer $j$ such that $0 \leqslant j < 2^m$, express $j$ as its 2-adic expansion

$$j = j_0 + j_1 2 + \ldots + j_{m-1} 2^{m-1}.$$

Define $f_j = e_{1,j_0} e_{2,j_1} \ldots e_{m,j_{m-1}}$. Now clearly $f_j f_{j'} = \delta_{j,j'} f_j$: that is, the set of $2^m f_j$ terms forms a set of primitive orthogonal idempotents in $\mathscr{A}^*_{p,m}$, provided none of these are zero.

Consider the action of the characters $\varphi_j^m$ on $f_{j'}$. Using Theorem 3, it is elementary to show that

$$\varphi_j(V_{p^i-1}) = (-1)^{j_{i-1}}, \quad 1 \leqslant i \leqslant m.$$

Now

$$\begin{aligned}
\varphi_j^m(f_{j'}) &= \varphi_j(e_{1,j_0'}e_{2,j_1'}, \ldots, e_{m,j_{m-1}'}) \\
&= \frac{1}{2^m}\prod_{i=1}^{m}(1+(-1)^{j_{i-1}'}\varphi_j^m(V_{p^i-1})) \\
&= \frac{1}{2^m}\prod_{i=1}^{m}(1+(-1)^{j_{i-1}'}(-1)^{j_{i-1}}) \\
&= \delta_{j,j'}.
\end{aligned}$$

Hence the $f_j$ terms, being distinct and non-zero, form the set of primitive idempotents of $\mathscr{A}^*_{p,m}$.

Let $I_j = f_j\mathscr{A}^*_{p,m}$, $j = 0, 1, \ldots, 2^m-1$. Now $\mathscr{A}^*_{p,m} = I_0 \oplus \ldots \oplus I_{2^m-1}$. We wish to develop a method of calculating the $Z_p$-dimension of these principal ideals, by examining the effect of $f_j$ on the basis elements in $\mathscr{A}^*_{p,m}$.

Consider a heirarchy of blocks of basis elements of $\mathscr{A}^*_{p,m}$, the blocks on the lowest level being of type

$$\{V_{ap+1}, \ldots, V_{(a+1)p}\} \quad \text{for } 0 \leqslant a \leqslant p^{m-1}-1,$$

those on the next level being of type

$$\{V_{bp^2+1}, \ldots, V_{(b+1)p^2}\} \quad \text{for } 0 \leqslant b \leqslant p^{m-2}-1,$$

and so forth. Each block is composed of $p$ blocks from the next lower level, except for the lowest blocks, each of which has one element of type $V_{(a+1)p}$ and $(p-1)$ elements of type $V_{ap+r}$, $r = 1, \ldots, p-1$.

Applying Green's product formulae (1962):

$$\begin{aligned}
e_{1,0}V_{ap+r} &= e_{1,0}V_{(a+1)p-r}, \\
e_{1,0}V_{(a+1)p} &= 0
\end{aligned}$$

and

$$\begin{aligned}
e_{1,1}V_{ap+r} &= V_{ap} + V_{(a+1)p} - e_{1,1}V_{(a+1)p-r}, \\
e_{1,1}V_{(a+1)p} &= V_{(a+1)p}
\end{aligned}$$

for $a$, $r$ as above. Hence if $e_{1,0}$ appears as a factor of $f_j$, each block at the lowest level can contribute only $\frac{1}{2}(p-1)$ basis elements to $I_j$, while if $e_{1,1}$ appears, such blocks can contribute only $\frac{1}{2}(p+1)$ elements. It is shown below that not all blocks contribute to $I_j$.

Consider a higher level, whose blocks are of type

$$\{V_{cp^k+1}, \ldots, V_{(c+1)p^k}\}.$$

Each such block is composed of a central smaller block of type

$$\{V_{cp^k+\frac{1}{2}(p-1)p^{k-1}+r},\ r=1,\dots,p^{k-1}\}$$

and $(p-1)$ other blocks belonging to that level.

Applying Green's formulae:

$$e_{k,0}\,V_{cp^k+s}=e_{k,0}\,V_{(c+1)p^k-s},$$

$$e_{k,0}\,V_{(c+1)p^k}=0$$

and

$$e_{k,1}\,V_{cp^k+s}=V_{cp^k}+V_{(c+1)p^k}-e_{k,1}\,V_{(c+1)p^k-s}$$

$$e_{k,1}\,V_{(c+1)p^k}=V_{(c+1)p^k}$$

for $0\leqslant c\leqslant p^{m-k}-1$, $0<s<p^k$. Hence if $e_{k,0}$ is a factor in $f_j$, the ($k$–1)th level contributes only $\frac{1}{2}(p-1)$ blocks to $I_j$ in each block of the $k$th level, apart from the central block (discussed below). Similarly, if $e_{k,1}$ is a factor in $f_j$, the ($k$–1)th level again contributes $\frac{1}{2}(p-1)$ non-central blocks to $I_j$.

The central blocks may or may not contribute basis elements to $I_j$. Assume $e_{k,0}e_{k+1,0}$ is a factor in $f_j$. The centre block at the $k$th level in each block of the $(k+1)$th level is halved by $e_{k,0}$ and is then unaffected by $e_{k+1,0}$. Hence $I_j$ has $1+\frac{1}{2}(p-1)=\frac{1}{2}(p+1)$ $k$th level blocks in each $(k+1)$th level block.

Assume instead that $e_{k,1}e_{k+1,0}$ is a factor in $f_j$. The central block now vanishes in each $(k+1)$th level block in $I_j$, and there exist only $\frac{1}{2}(p-1)$ $k$th level blocks in each such block. This also holds if $e_{k,0}e_{k+1,1}$ is a factor, while if $e_{k,1}e_{k+1,1}$ is a factor the central block does not vanish.

To summarize: the dimension of $I_j$ is the product of $m$ factors, each being $\frac{1}{2}(p+1)$ or $\frac{1}{2}(p-1)$, where:

(i) the first factor is $\frac{1}{2}(p-1)$ if $e_{1,0}$ is a factor of $f_j$, $\frac{1}{2}(p+1)$ otherwise,

(ii) the $(k+1)$th factor is $\frac{1}{2}(p-1)$ if $e_{k,0}e_{k+1,1}$ or $e_{k,1}e_{k+1,0}$ is a factor, $\frac{1}{2}(p+1)$ otherwise.

EXAMPLE. In $\mathscr{A}^*_{p,4}$, $f_3=e_{1,1}e_{2,1}e_{3,0}e_{4,0}$, and hence $I_3$ has dimension

$$\tfrac{1}{2}(p+1)\tfrac{1}{2}(p+1)\tfrac{1}{2}(p-1)\tfrac{1}{2}(p+1)=\frac{1}{2^4}(p-1)(p+1)^3.$$

The dimension of each summand is calculable by this method.

REMARK. Elementary number theory shows that there exist $\binom{m}{r}$ summands of dimension $(1/2^m)(p-1)^{m-r}(p+1)^r$, for $r=0,1,\dots m$.

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