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CARATHÉODORY'S THEOREM

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Despite the abundance of generalizations of Carathéodory's theorem occurring in the literature (see [1]), the following simple generalization involving infinite convex combinations seems to have passed unnoticed. Boldface letters denote points of R^n and Greek letters denote scalars.

THEOREM. *If a point of R^n is an infinite convex combination of a sequence of points of R^n, then it can be represented as a convex combination of at most $n+1$ points of the sequence.*

Proof. Let $\mathbf{z} \in R^n$ be such that

$$\mathbf{z} = \sum_{i=1}^{\infty} \lambda_i \mathbf{x}_i$$

where $\lambda_i \geq 0$ and $\sum_{i=1}^{\infty} \lambda_i = 1$, and let X denote the convex hull of the set $\{\mathbf{x}_1, \ldots, \mathbf{x}_m, \ldots\}$. We may, without loss of generality, assume $\lambda_i > 0$ for all i and that the sequence $\mathbf{x}_1, \ldots, \mathbf{x}_m, \ldots$ does not lie in any hyperplane of R^n. We prove $\mathbf{z} \in X$. The proof is then completed by a standard application of Carathéodory's theorem. For each m write

$$\mathbf{y}_m = \left(\sum_{i=1}^{m} \lambda_i \mathbf{x}_i\right) \Big/ \left(\sum_{i=1}^{m} \lambda_i\right).$$

Then $\mathbf{y}_m \in X$ for all m and $\mathbf{y}_m \to \mathbf{z} (m \to \infty)$. This shows that $\mathbf{z} \in \bar{X}$. If $\mathbf{z} \notin X$, then $\mathbf{z}$ is a boundary point of the convex set X and so there exists a nonzero $\mathbf{a}$ and a scalar α such that

$$\mathbf{a}.\mathbf{z} = \alpha \quad \text{and } \mathbf{a}.\mathbf{x} \geq \alpha(\mathbf{x} \in X).$$

However,

$$\mathbf{a}.\mathbf{z} = \sum_{i=1}^{\infty} \lambda_i(\mathbf{a}.\mathbf{x}_i) > \alpha,$$

since there is some m for which $\mathbf{a}.\mathbf{x}_m > \alpha$ and $\lambda_m > 0$. This contradiction proves that $\mathbf{z} \in X$ and so completes the proof.

REFERENCE

1. J. R. Reay, *Generalizations of a theorem of Carathéodory*, Memoirs Amer. Math. Soc. **54**, 1965.

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