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RELATIONS BETWEEN THE AMALGAMATION PROPERTY AND ALGEBRAIC EQUATIONS

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Abstract

A variety $\mathfrak{V}$ is called solutionally complete if any system of algebraic equations over an algebra A in $\mathfrak{V}$ has a solution in A provided it is solvable in $\mathfrak{V}$ and has at most one solution in any extension of A in $\mathfrak{V}$. $\mathfrak{V}$ is called solutionally compatible if every solvable system of equations over an algebra in $\mathfrak{V}$ is also solvable over any extension of that algebra. It is shown that solutional compatibility is equivalent with the amalgamation property and that a weaker form of the strong amalgamation property is sufficient but not necessary for equational completeness.

1. Introduction

Let A be a universal algebra in the variety $\mathfrak{V}$, $X = \{x_1, \cdots, x_n\}$ a finite set of indeterminates, and $W(A, X)$ the word algebra over $A \cup X$ as defined by Lausch and Nöbauer (1973). A system of algebraic equations in X over A is a family $\langle(p_i, q_i)\rangle_{i \in I}$ with $p_i, q_i \in W(A, X)$. The equations are often written in the form $p_i = q_i$ which of course does not mean that the two words are identical. Every $p \in W(A, X)$ induces in a natural way an n-ary polynomial function on any extension B of A and we write $p(b_1, \cdots, b_n)$ for the image of $(b_1, \cdots, b_n)$ under the function induced by p.

Let $S = \langle(p_i, q_i)\rangle_{i \in I}$. Then an n-tuple $(b_1, \cdots, b_n) \in B^n$, where B is an extension of A, is called a "solution of S in B" if $p_i(b_1, \cdots, b_n) = q_i(b_1, \cdots, b_n)$ holds for every $i \in I$. The systems S is "solvable over $(A, \mathfrak{V})$" if there exists a solution of S in some $\mathfrak{V}$-extension B of A (i.e. an algebra $B \in \mathfrak{V}$ which has A as a subalgebra). The following questions have been proposed by Lausch and Nöbauer (1973):

I. If S is a system of equations solvable over $(A, \mathfrak{V})$ which has at most one solution in each $\mathfrak{V}$-extension of A, is then the (unique) solution of S in A?

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II. If B is a $\mathfrak{V}$-extension of A and S a system of algebraic equations solvable over $(A, \mathfrak{V})$, is S then solvable over $(B, \mathfrak{V})$?

Question I has been investigated by Hule (1976) and Question II by Hule and Müller (1976). It has been shown in these papers that the answer to both questions is negative in general but affirmative in certain varieties. Such varieties are called "solutionally complete" ("lösungsvollständig") in the case of Question I and "solutionally compatible" in the case of Question II. The following sufficient conditions for solutional completeness and solutional compatibility have been established:

(i) If $\mathfrak{V}$ has the amalgamation property in the sense of Jónsson (1961), then $\mathfrak{V}$ is solutionally compatible.

(ii) If $\mathfrak{V}$ has the strong amalgamation property in the sense, for instance, of Grätzer (1971), then $\mathfrak{V}$ is solutionally complete.

The *amalgamation property* means: If B and C are $\mathfrak{V}$-extensions of A, then there exists a $\mathfrak{V}$-extension D of A with subalgebras B' and C' such that $A \subseteq B' \cap C'$, B' is isomorphic to B over A (i.e. under an isomorphism which fixes A), and C' is isomorphic to C over A. The *strong amalgamation property* means the following: If B and C are $\mathfrak{V}$-extensions of A, then there exists an algebra $D \in \mathfrak{V}$ which has B and C as subalgebras.

One main result of the present article is that the amalgamation property is also a necessary condition for solutional compatibility. The author hoped to find a similar result for solutional completeness. It turns out, however, that even a very weak form of the strong amalgamation property (which, for instance, does not imply the common amalgamation property) is still sufficient but not necessary for solutional completeness.

2. Solutional compatibility

THEOREM 1. *A variety is solutionally compatible if and only if it has the amalgamation property.*

PROOF. The "if" part has been shown in Hule and Müller (1976). Now suppose that the amalgamation property does not hold in the variety $\mathfrak{V}$. Then there exist an algebra $A \in \mathfrak{V}$ and $\mathfrak{V}$-extensions B and C of A such that no $\mathfrak{V}$-extension D of A has subalgebras B' and C' isomorphic with B and C respectively over A. According to Lausch and Nöbauer (1973) there exist polynomial algebras $A(Y, \mathfrak{V})$, $A(Z, \mathfrak{V})$ (where we assume $Y \cap Z = \varnothing$, without loss of generality) and onto homomorphisms $\varphi: A(Y, \mathfrak{V}) \to B$, $\psi: A(Z, \mathfrak{V}) \to C$ which fix A. Let β be the kernel of φ and γ the kernel of ψ. Now define $D = A(Y \cup Z, \mathfrak{V})/\delta$ where δ is the congruence generated by β and γ. First suppose that for all $p, q \in A(Y, \mathfrak{V})$, $p \equiv q\ (\delta)$ implies $p \equiv q\ (\beta)$, and for all $p, q \in A(Z, \mathfrak{B})$, $p \equiv q\ (\delta)$ implies $p \equiv q\ (\gamma)$. Then, in particular, δ separates A, and D can thus be considered as a $\mathfrak{B}$-extension of A. Clearly the congruence classes of the elements of $A(Y, \mathfrak{B})$ form a subalgebra of D isomorphic to B over A and in the same way we get a subalgebra of D isomorphic to C over A, contradicting the assumption. Therefore there exist $p, q \in A(Y, \mathfrak{B})$ such that $p \equiv q\ (\delta)$, $p \not\equiv q\ (\beta)$, or there exist $p, q \in A(Z, \mathfrak{B})$ such that $p \equiv q\ (\delta)$, $p \not\equiv q\ (\gamma)$. Let us assume the first case. Since δ is the least congruence of $A(Y \cup Z, \mathfrak{B})$ including β and γ, there exists a sequence $w_0, w_1, \cdots, w_k$ of elements of $W(A, Y \cup Z)$ such that w_0 represents the polynomial p, w_k represents q, and for $i = 1, \cdots, k$ either w_{i-1} and w_i represent the same polynomial or w_i is obtained from w_{i-1} replacing a subword u_i of w_{i-1} by v_i where u_i and v_i represent polynomials which are congruent under β or under γ. (Obviously the relation on $A(Y \cup Z, \mathfrak{B})$ defined by such sequences of words is a congruence and the least congruence which includes β and γ.) Now let $\{z_1, \cdots, z_n\}$ be the (finite) set of elements of Z which occur in $w_0, w_1, \cdots, w_k$, and consider the family $S = \langle (u_i, v_i) \rangle_{i \in I}$ where I is the set of indices $i \in \{1, \cdots, k\}$ such that an occurrence of γ was used in the formation of w_i from w_{i-1}, i.e. u_i, v_i represent polynomials congruent under γ. Since γ separates A, S considered as a system of equations in $\{z_1, \cdots, z_n\}$ over A is solvable over $(A, \mathfrak{B})$. Suppose that S is also solvable over $(B, \mathfrak{B})$ and let $(e_1, \cdots, e_n)$ be a solution of S in some $\mathfrak{B}$-extension E of B. Now for $i = 1, \cdots, k$ let $\bar{w}_i$ be the element of E obtained from w_i by substituting each $y \in Y$ by $\varphi y \in B$ and each $z_j \in Z$ by e_j and performing the operations indicated in w_i. Then, $\varphi p = \bar{w}_0 = \bar{w}_1 = \cdots = \bar{w}_k = \varphi q$ holds and on the other hand $\varphi p \neq \varphi q$ because of $p \not\equiv q\ (\beta)$. This contradiction shows that $\mathfrak{B}$ is not solutionally compatible.

The restriction to systems of equations in finitely many indeterminates might perhaps appear somewhat artificial. Actually we could admit arbitrary sets of indeterminates. The preceding theorem shows that this would not change the definition of solutionally compatible varieties. On the other hand, the proof of the theorem shows that it is even sufficient to consider finite systems of equations.

We want to consider another consequence of Theorem 1. Suppose that $\mathfrak{B}$ satisfies the following "weak" amalgamation property: If B and C are $\mathfrak{B}$-extensions of A with β the inclusion of A into B and γ the inclusion of A into C, then there exists an algebra $D \in \mathfrak{B}$, a monomorphism $\varphi: B \to D$ and a homomorphism $\psi: C \to D$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "\gamma"'] & B \arrow[d, "\varphi"] \\ C \arrow[r, "\psi"'] & D \end{tikzcd}$$

Note that the difference to the common amalgamation property lies in admitting an arbitrary homomorphism $\psi : C \to D$ instead of a monomorphism. Without loss of generality we can assume that D is an extension of B (then $\varphi\beta a = \psi\gamma a = a$ for any $a \in A$). Now suppose that a system of equations S over A has a solution $(c_1, \cdots, c_n)$ in C. Then $(\psi c_1, \cdots, \psi c_n)$ is obviously a solution of S in D. This means that the "weak" amalgamation property implies solutional compatibility and hence, by Theorem 1, is equivalent with the usual amalgamation property. This interesting fact could also be obtained as a consequence of Theorem 13.18 in Grätzer (1971) (the proof of which is quite complicated) or proved directly in the following simple way: Consider the diagram above. By symmetry there exist also an algebra $D' \in \mathfrak{V}$, a homomorphism $\varphi' : B \to D'$ and a monomorphism $\psi' : C \to D'$ such that the diagram with φ, ψ, D replaced by φ', ψ', D' also commutes. Then define $E = D \times D'$ and note that $B' = \{(\varphi b, \varphi' b) \mid b \in B\}$ and $C' = \{(\psi c, \psi' c) \mid c \in C\}$ are subalgebras of E isomorphic to B resp. C which satisfy the condition required for the amalgamation property.

3. Solutional completeness

We already know that the strong amalgamation property implies solutional completeness. Actually, in the proof of this fact, the strong amalgamation property is only used in the following "symmetric" form: If B and C are $\mathfrak{V}$-extensions of A which are isomorphic over A, then there exists an algebra $D \in \mathfrak{V}$ which has B and C as subalgebras. We shall show that even a weaker condition is sufficient but still not necessary for solutional completeness. In analogy to classical field theory, an extension B of A is called a simple extension if there exists an element $b \in B$ such that B is generated by $A \cup \{b\}$. We then write $B = A(b)$. Now let us define the following condition on $\mathfrak{V}$:

(*) If B and C are simple $\mathfrak{V}$-extensions of A which are isomorphic over A, then there exists an algebra $D \in \mathfrak{V}$ which is a common extension of B and C.

THEOREM 2. *If a variety $\mathfrak{V}$ satisfies condition (*) then $\mathfrak{V}$ is solutionally complete. The converse is not true in general.*

PROOF. Suppose $\mathfrak{V}$ satisfies (*). In Hule (1976) it was shown that a variety is solutionally complete if the definition is satisfied for systems of equations in one indeterminate. So let S be a system over A in one indeterminate x and suppose S has a solution b in the $\mathfrak{V}$-extension C of A such that $b \notin A$. Let $B = A(b)$ be the subalgebra of C generated by $A \cup \{b\}$. Then we can construct an isomorphic copy $B' = A(b')$ of B such that b' is also a solution of S and $b' \neq b$. By (*), there exists a common $\mathfrak{V}$-extension of B and B' where S has two different solutions b and b'. This shows that $\mathfrak{V}$ is solutionally complete. To show that solutional completeness does not imply condition (*), consider the following example. Let $\mathfrak{V}$ be the class of algebras with one multiplicatively written binary operation, one unary operation g and one nullary operation 0, satisfying the following laws:

$$xy = yx, xg(y) = yg(x),$$

$$x(yz) = (xy)z = g(xy) = g(g(x)) = g(x)g(y) = 0.$$

Let $B = \{0, a, b, c\}$ with $ab = ba = c$, $uv = 0$ for $(u, v) \in B^2 - \{(a, b), (b, a)\}$, $g(b) = a$, $g(u) = 0$ for $u \in B - \{b\}$. We verify easily that B is in $\mathfrak{V}$ and that $A = \{0, a\}$ is a subalgebra of B with $B = A(b)$. Let $B' = \{0, a, b', c'\}$ be an isomorphic copy of B such that the isomorphism fixes A and takes b to b', c to c', and $B \cap B' = A$. Suppose that there exists an algebra $D \in \mathfrak{V}$ which is a common extension of B and B'. Then in D,

$$c = ba = bg(b') = b'g(b) = b'a = c'$$

holds, a contradiction. Hence $\mathfrak{V}$ does not satisfy condition (*). Now we show that $\mathfrak{V}$ is solutionally complete. Assume the contrary. Then there exists an algebra $A \in \mathfrak{V}$ and a system of equations S in one indeterminate x such that S has a solution b in some $\mathfrak{V}$-extension B of A, $b \notin A$, and no $\mathfrak{V}$-extension of A contains more than one solution of S. Without loss of generality we assume $B = A(b)$ (otherwise take the subalgebra $A(b)$ of B instead of B). Then there exists a polynomial algebra $A(\{y\}, \mathfrak{V})$ and a homomorphism φ from $A(\{y\}, \mathfrak{V})$ onto B which fixes A and such that $\varphi y = b$. Let z be another indeterminate different from y and $\chi : A(\{z\}, \mathfrak{V}) \to A(\{y\}, \mathfrak{V})$ the bijection defined by $\chi w(z) = w(y)$. Let β be the kernel of φ and γ the congruence of $A(\{z\}, \mathfrak{V})$ defined by $p \equiv q(\gamma) \Leftrightarrow \chi p \equiv \chi q(\beta)$. Let further δ be the congruence of $A(\{y, z\}, \mathfrak{V})$ generated by β and γ. If δ does not separate A, then we have two different elements a_1 and a_2 of A and a sequence of words w_0, $w_1, \cdots, w_k$ (like in the proof of Theorem 1) such that $w_0 = a_1$, $w_k = a_2$ and for $i \in \{1, \cdots, k\}$ w_{i-1} and w_i either represent the same polynomial or differ only by subwords u_i, v_i which represent polynomials congruent under β or under γ. If we substitute z by y, then every occurrence of γ is converted into an occurrence of β, hence the contradiction $a_1 \equiv a_2(\beta)$ follows. Consequently, δ separates A and A can thus be identified with a subalgebra of $A(\{y, z\}, \mathfrak{V})/\delta$. Now the congruence classes of y and of z are solutions of S in $A(\{y, z\}, \mathfrak{V})/\delta$ considered as an extension of A. By assumption they cannot be different. Hence there is a sequence of words as described above, leading from y to z.

Since no single indeterminate occurs in any law of $\mathfrak{V}$, the step from $w_0 = y$ to w_1 can only be performed by using an occurrence of β. It is easy to see that every element of $A(\{y\}, \mathfrak{V})$ is of one of the following forms with $a \in A$:

$$a, y, g(y), ay, ag(y), yy, yg(y).$$

(Note that by the laws of $\mathfrak{V}$ any product of three or more factors and any expression where g appears twice or applied to a product equal 0.) Thus y is congruent under β to one of the polynomials a, $g(y)$, ay, $ag(y)$, yy, $yg(y)$. We observe

$$y \equiv g(y) \Rightarrow y \equiv g(g(y)) = 0,$$

$$y \equiv ay \Rightarrow y \equiv a(ay) = 0,$$

$$y \equiv ag(y) \Rightarrow y \equiv ag(ag(y)) = a0 = a(g(a)g(a)) = 0,$$

$$y \equiv yy \Rightarrow y \equiv y(yy) = 0,$$

$$y \equiv yg(y) \Rightarrow y \equiv (yg(y))g(y) = 0.$$

Thus in any case $y \equiv a\,(\beta)$ for some $a \in A$. But this implies

$$b = \varphi y = \varphi a = a \in A,$$

contrary to the assumption. Hence $\mathfrak{V}$ is solutionally complete.

We finally observe that the properties solutionally complete and solutionally compatible are independent of one another. For instance, the variety of distributive lattices was shown in Hule (1976) not to be solutionally complete. However it is solutionally compatible since it satisfies the amalgamation property. On the other hand, consider the variety of groupoids with the laws

$$((xx)y)z = z((xx)y) = (xx)y.$$

Ježek (1976) showed that this variety satisfies the "symmetric" strong amalgamation property (hence condition (*)) but not the amalgamation property. Consequently it is solutionally complete but not solutionally compatible. Of course, there are also varieties which are both solutionally complete and solutionally compatible (for instance, the varieties of groups and of lattices and others with the strong amalgamation property) and varieties which have none of the two properties (for instance, the varieties of semigroups and of rings).

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