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Shub's example revisited

CHAO LIANG†, RADU SAGHIN‡, FAN YANG§ and JIAGANG YANG¶

† *School of Statistics and Mathematics, Central University of Finance and Economics, Beijing 100081, China*
(e-mail: chaol@cufe.edu.cn)
‡ *Instituto de Matemática, Pontificia Universidad Católica de Valparaíso, Blanco Viel 596, Cerro Barón, Valparaíso, Chile*
(e-mail: radu.saghin@pucv.cl)
§ *Department of Mathematics, Wake Forest University, Winston-Salem, NC, USA*
(e-mail: yangf@wfu.edu)
¶ *Departamento de Geometria, Instituto de Matemática e Estatística, Universidade Federal Fluminense, Niterói, Brazil*
(e-mail: yangjg@impa.br)

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Abstract. For a class of robustly transitive diffeomorphisms on $\mathbb{T}^4$ introduced by Shub [Topologically transitive diffeomorphisms of T^4. *Proceedings of the Symposium on Differential Equations and Dynamical Systems (Lecture notes in Mathematics, 206)*. Ed. D. Chillingworth. Springer, Berlin, 1971, pp. 39–40], satisfying an additional bunching condition, we show that there exists a C^2 open and C^r dense subset $\mathcal{U}^r$, $2 \le r \le \infty$, such that any two hyperbolic points of $g \in \mathcal{U}^r$ with stable index 2 are homoclinically related. As a consequence, every $g \in \mathcal{U}^r$ admits a unique homoclinic class associated to the hyperbolic periodic points with index 2, and this homoclinic class coincides with the whole ambient manifold. Moreover, every $g \in \mathcal{U}^r$ admits at most one measure of maximal entropy, and every $g \in \mathcal{U}^\infty$ admits a unique measure of maximal entropy.

Key words: partially hyperbolic diffeomorphisms, homoclinic classes, equilibrium states
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1. *Introduction and results*

Shub introduced in [26] an example of a diffeomorphism on $\mathbb{T}^4$ which is very important in smooth dynamics: it is the first example of a diffeomorphism which is robustly transitive and it is not uniformly hyperbolic. Later, Mañé [19] also built an example of a robustly transitive but non-hyperbolic diffeomorphism, this time on $\mathbb{T}^3$. Both examples belong to the class of partially hyperbolic diffeomorphisms, Shub's example has center dimension 2, while Mañe's example has center dimension 1 (for the definition of partial hyperbolicity, see §1.1).

There are many works addressing further properties of Mañe's examples, and there is a fairly good understanding of their dynamics. The Shub's example was also studied, but mainly under the restrictive condition that the center bundle has a dominated splitting into two one-dimensional sub-bundles. In this paper, we are interested in the general Shub's examples, in particular, we do not assume that the maps admit a further domination of the center bundle. This lack of further domination makes it an interesting class of maps, because we cannot use one-dimensional techniques; however, we will see that we may have enough hyperbolicity within these systems to obtain a good understanding of their ergodic properties.

In this paper, we will consider a slightly more general class than the original setting of the Shub's example, a precise definition is the following.

1.1. *Shub class*

Definition 1.1. A diffeomorphism $f : M \to M$ is called *partially hyperbolic* if the tangent bundle admits a continuous Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$ such that there exist continuous functions $0 < \lambda_s(x) < \lambda_c^-(x) \leq \lambda_c^+(x) < \lambda_u(x)$, with $\lambda_s(x) < 1 < \lambda_u(x)$, satisfying the following conditions:

(1) $\|Df(x)v^s\| \leq \lambda_s(x)$;
(2) $\lambda_c^-(x) \leq \|Df(x)v^c\| \leq \lambda_c^+(x)$;
(3) $\|Df(x)v^u\| \geq \lambda_u(x)$,

for every $x \in M$ and unit vectors $v^i \in E^i(x)(i = s, c, u)$.

A partially hyperbolic diffeomorphism is called *dynamically coherent* if there exist invariant foliations $\mathcal{F}^{cs}$ and $\mathcal{F}^{cu}$ tangent to $E^{cs} = E^c \oplus E^s$ and $E^{cu} = E^c \oplus E^u$. In this case, $\mathcal{F}^{cs}$ is subfoliated by the stable and central foliations $\mathcal{F}^s$ and $\mathcal{F}^c$, while $\mathcal{F}^{cu}$ is subfoliated by the unstable and center foliations $\mathcal{F}^u$ and $\mathcal{F}^c$.

Let A, B be two linear Anosov automorphisms on $\mathbb{T}^2$ such that $1 < |\lambda_B| < |\lambda_A|$, where λ_A and λ_B are the unstable eigenvalues of A and B. Then $f_{A,B} : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2 \times \mathbb{T}^2$

$$f_{A,B}(x, y) = (A(x), B(y))$$

is an Anosov automorphism, which can also be seen as a partially hyperbolic diffeomorphism with two-dimensional center bundle, and one-dimensional stable and unstable bundles.

Definition 1.2. Let $\mathrm{PH}_{A,B}$ be the set of partially hyperbolic diffeomorphisms isotopic to $f_{A,B}$, all of them having the same dimension (that is, one dimension) of the stable and unstable bundle, and let $\mathrm{PH}^0_{A,B}$ be the connected component of $\mathrm{PH}_{A,B}$ containing $f_{A,B}$.

It is easy to see that $\mathrm{PH}^0_{A,B}$ is an open set of diffeomorphisms of $\mathbb{T}^4$. The following proposition is known.

PROPOSITION 1.3. (Fisher, Potrie, and Sambarino [11]) *If $f \in \mathrm{PH}^0_{A,B}$, then f is dynamically coherent and admits a center foliation where all central leaves are C^1 two-dimensional tori, and f is center leaf conjugate to $f_{A,B}$.*

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Definition 1.4. The *Shub class* $\mathcal{SH} \subset \bigcup_{1<\lambda_B<\lambda_A} \mathrm{PH}^0_{A,B}$ is the set of partially hyperbolic diffeomorphisms f of $\mathbb{T}^4$ such that f belongs to some $\mathrm{PH}^0_{A,B}$ and there exists a fixed point $p_f = f(p_f) \in \mathbb{T}^4$, such that $f\mid_{\mathcal{F}^c_f(p_f)}$ is an Anosov diffeomorphism, where $\mathcal{F}^c_f(p_f)$ is the (fixed) center leaf passing through p_f. Also let

$$\mathcal{SH}^r := \{f \in \mathcal{SH} : \ f \text{ is } C^r\}, \quad r \geq 1.$$

Although this part will not be used in the proof, through analyzing the induced map on the fundamental group, it is easy to show that $f\mid_{\mathcal{F}^c_f(p_f)}$ is topological conjugate to B. Shub proved the following.

THEOREM 1.5. (Shub [26]) *$\mathcal{SH}$ is C^1 open and every $f \in \mathcal{SH}$ is transitive.*

Shub proved this result for some specific examples, but the proof can be adapted for the Shub class of diffeomorphisms with minor modifications. In this article, we consider the class of Shub diffeomorphisms which also satisfy some bunching conditions.

Definition 1.6. The *bunched Shub class* $\mathcal{SH}^r_b$ is the set of partially hyperbolic diffeomorphisms $f \in \mathcal{SH}^r$ which also satisfy the following bunching conditions:

(a) global bunching,

$$\lambda_s(x) < \frac{\lambda_c^-(x)}{\lambda_c^+(x)} \leq \frac{\lambda_c^+(x)}{\lambda_c^-(x)} < \lambda_u(x) \quad \text{for all } x \in \mathbb{T}^4; \tag{1}$$

(b) stronger local bunching at the fixed center leaf $\mathcal{F}^c_f(p_f)$,

$$\lambda_s(x) < (\lambda_c^-(x))^2 \leq (\lambda_c^+(x))^2 < \lambda_u(x) \quad \text{for all } x \in \mathcal{F}^c_f(p_f) \tag{2}$$

and

$$\lambda_s(x) < \frac{\lambda_c^-(x)}{(\lambda_c^+(x))^2} \leq \frac{\lambda_c^+(x)}{(\lambda_c^-(x))^2} < \lambda_u(x) \quad \text{for all } x \in \mathcal{F}^c_f(p_f). \tag{3}$$

Clearly, $\mathcal{SH}^r_b$ is a C^1 open set.

Remark 1.7. The condition in equation (1) implies (see [24]) that if f is C^2, then the stable and unstable bundles are C^1 when restricted to the center-stable and center-unstable leaves, and, as a consequence, the strong stable and strong unstable holonomies between the center leaves are of class C^1 (when restricted to the center-stable respectively center-unstable leaves). We will see later that, in fact, these holonomies depend continuously in the C^1 topology with respect to the points (or the center leaves) and with respect to the map f (in the C^2 topology).

Remark 1.8. The condition in equation (2) is the standard 2-bunching condition, and [13] implies that if f is C^2, then $\mathcal{F}^c_f(p_f)$, $\mathcal{F}^{cs}_f(p_f)$, and $\mathcal{F}^{cu}_f(p_f)$ are of class C^2. If the central bounds are symmetric, or $\lambda_c^-\lambda_c^+ = 1$, then it is equivalent to the global bunching condition in equation (1).

The condition in equation (3) gives us better regularity of the strong foliations corresponding to the fixed Anosov leaf $\mathcal{F}^c_f(p_f)$. In particular, if f is C^3, then the strong stable foliation $\mathcal{F}^s_f$ restricted to the center-stable manifold $\mathcal{F}^{cs}_f(p_f)$ is of class C^2, and the strong unstable foliation $\mathcal{F}^u_f$ restricted to the center-unstable manifold $\mathcal{F}^{cu}_f(p_f)$ is also of class C^2 (see [24]).

1.2. *Results.* The homoclinic intersections between hyperbolic periodic points were first observed by Poincaré, and, since then, they play an important role in the theory of dynamical systems. Smale [27] used them to define homoclinic classes.

Definition 1.9. Given two hyperbolic periodic points p, q of the diffeomorphism f, with the same stable index, we say that they are *homoclinically related* if their stable and unstable manifolds intersect transversally:

$$W^s(p) \pitchfork W^u(q) \neq \emptyset \quad \text{and} \quad W^s(q) \pitchfork W^u(p) \neq \emptyset. \tag{4}$$

We say that $\mathrm{Orb}(p)$ and $\mathrm{Orb}(q)$ are *homoclinically related* if

$$W^s(\mathrm{Orb}(p)) \pitchfork W^u(\mathrm{Orb}(q)) \neq \emptyset \quad \text{and} \quad W^s(\mathrm{Orb}(q)) \pitchfork W^u(\mathrm{Orb}(p)) \neq \emptyset. \tag{5}$$

This is an equivalence relation between hyperbolic periodic orbits. The *homoclinic class of* $\mathrm{Orb}(p)$, $HC(\mathrm{Orb}(p))$, is the closure of the equivalence class of $\mathrm{Orb}(p)$.

For diffeomorphisms in the Shub class, the center bundle may not admit a dominated splitting, which means that the diffeomorphisms may not have a dominated splitting of index 2. If a diffeomorphism has no dominated splitting of index 2, it seems unexpected that any two hyperbolic points of stable index 2 are homoclinically related to each other. Indeed, the sizes of stable and unstable manifolds of the hyperbolic periodic points are non-uniform, and the intersection in equation (5) can be empty. However, even if the intersection is non-empty, the intersection may not be transverse, because of the lack of domination (see [25]).

The main result of this paper is the following.

THEOREM A. *For any $2 \leq r \leq \infty$, there exists a C^2 open and C^r dense subset $\mathcal{U}^r \subset \mathcal{SH}^r_b$, such that for any $f \in \mathcal{U}^r$, holds the following: every pair of hyperbolic periodic points of f with stable index 2 are homoclinically related.*

As a consequence, any diffeomorphism $f \in \mathcal{U}^r$ admits a unique homoclinic class associated to the hyperbolic periodic points of index 2. Denote by p_f a hyperbolic fixed point of $f \in \mathcal{U}^r$.

COROLLARY B. *For any $f \in \mathcal{U}^r$, f admits a unique homoclinic class $H(p_f, f)$ associated to the hyperbolic periodic points of index 2, and the homoclinic class coincides with the ambient manifold.*

For a continuous potential ϕ and a continuous map f, an f-invariant probability measure μ is called an *equilibrium measure* for the potential ϕ, if

$$h_\mu(f) + \int \phi \, d\mu = P_{\mathrm{top}}(\phi),$$

where $P_{\mathrm{top}}(\phi) := \sup_{\nu \in \mathcal{M}_e(f)} \{h_\nu(f) + \int \phi \, d\nu\}$.

The equilibrium states do not necessarily exist. Assuming entropy expansiveness, Bowen [4] proved the equilibrium states do exist. It was shown by Liao, Viana, and Yang [18] that any diffeomorphism away from homoclinic tangencies is entropy expansive. Yomdin [31] (see also Buzzi [6]) proved also that for any C^∞ diffeomorphism, equilibrium states always exist.

The uniqueness of equilibrium states is a more subtle problem. Recently, Climenhaga and Thompson [9] (see also Pacifico, Yang, and Yang [23]) gave a criterion based on Bowen property and specification. Another method used by Buzzi, Crovisier, and Sarig [7] (see also Ben Ovadia [2, 3]) is based on the use of the homoclinic class of measures.

Definition 1.10. Suppose f is a C^r diffeomorphism for some $r > 1$. For two ergodic hyperbolic measures μ_1 and μ_2 of f, we write $\mu_1 \preceq \mu_2$ if and only if there exist measurable sets $A_1, A_2 \subset M$ with $\mu_i(A_i) > 0$ such that for any $x_1 \in A_1$ and $x_2 \in A_2$, the manifolds $W^u(x_1)$ and $W^s(x_2)$ have a point of transverse intersection.

Here, μ_1, μ_2 are *homoclinically related* if $\mu_1 \preceq \mu_2$ and $\mu_2 \preceq \mu_1$. We write $\mu_1 \overset{h}{\sim} \mu_2$. The set of ergodic measures homoclinically related to a hyperbolic ergodic measure μ is called the *measured homoclinic class* of μ.

Remark 1.11. The homoclinic relation is an equivalence relation, moreover, two atomic measures supported on two periodic orbits are homoclinically related if and only if the two periodic orbits are hyperbolic and homoclinically related.

We have the following theorem. For a discussion on the index of hyperbolic measures, see §§2.3 and 2.4.

THEOREM C. *For any $f \in \mathcal{U}^r$, all the hyperbolic ergodic measures of index 2 are homoclinically related. Let $\phi : \mathbb{T}^4 \to \mathbb{R}$ be any Hölder potential function with* $\max_{x,y\in\mathbb{T}^4} \|\phi(x) - \phi(y)\| < \log \lambda_B$, *where $f \in \mathrm{PH}^0_{A,B}$, then f admits at most one equilibrium state for the potential ϕ. In particular, every $f \in \mathcal{U}^r$ admits at most one measure of maximal entropy.*

A direct consequence of [6, 31] is the following.

COROLLARY D. *Every $f \in \mathcal{U}^\infty \cap \mathrm{PH}^0_{A,B}$ admits a unique equilibrium state for every Hölder potential satisfying* $\max_{x,y\in\mathbb{T}^4} \|\phi(x) - \phi(y)\| < \log \lambda_B$. *In particular, every $f \in \mathcal{U}^\infty$ admits a unique measure of maximal entropy.*

For Shub's example, some similar results were obtained under some extra assumptions. For instance, by Newhouse and Young [21] and Carvalho and Pérez [8], with the extra assumption that within the center foliation there exists a one-dimensional invariant sub-foliation, and by Álvarez [1], assuming that the center bundle admits a further dominated splitting. For other partially hyperbolic diffeomorphisms on $\mathbb{T}^4$, there are results on the uniqueness of u-Gibbs states in [10, 22].

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2. *Preliminaries*

2.1. *Stable and unstable holonomies between center leaves.* As we mentioned before, the condition in equation (1) implies that the holonomies between the center leafs are uniformly C^1. In fact, there exists a continuity of the holonomies in the C^1 topology. If $y \in \mathcal{F}^u_f(x)$, let us denote by $h^u_{f,x,y} : \mathcal{F}^c_f(x) \to \mathcal{F}^c_f(y)$ the unstable holonomy between the two center leaves. Since it is of class C^1, the derivative $Dh^u_{f,x,y}$ induces a continuous map between the unit tangent bundles $Dh^u_{f,x,y*} : T^1\mathcal{F}^c_f(x) \to T^1\mathcal{F}^c_f(y)$.

LEMMA 2.1. *$Dh^u_{f,x,y*}$ is continuous with respect to $f \in \mathcal{SH}^2_b$ (the C^2 topology) and $x, y \in \mathbb{T}^4$, $y \in \mathcal{F}^u_f(x)$. The same holds for the stable holonomy.*

Remark 2.2. The continuity in Lemma 2.1 means that if f_n converges to f in the C^2 topology, x_n converges to x in $\mathbb{T}^4$, and $y_n \in \mathcal{F}^u_{f_n,loc}(x_n)$ converges to y, then $Dh^u_{f_n,x_n,y_n*}$ converges uniformly to $Dh^u_{f,x,y*}$. The proof requires only the weaker global condition in equation (1).

Remark 2.3. Since, in our case, the stable and unstable bundles are one-dimensional, one could approach the continuity question using the classical ordinary differential equation (ODE) theory of the regularity of solutions with respect to the initial conditions and parameters. We prefer to present a different proof which constructs the projectivized holonomies as unstable foliations of the projectivization of f along the center bundle.

Proof. Let $T^1\mathbb{T}^4$ be the unit tangent bundle of $\mathbb{T}^4$ (which can be identified with $\mathbb{T}^4 \times \mathbb{S}^3$) with Df_* being the C^1 diffeomorphism induced by f. We will consider the central unit tangent bundle $S_f := \bigcup_{x\in\mathbb{T}^4} S(f, x)$, where $S(f, x) = T^1_x\mathcal{F}^c_f(x)$ is the unit circle in $E^c_f(x)$. Then, S_f is a Hölder submanifold of $T^1\mathbb{T}^4$ invariant under Df_*, which is also a Hölder bundle over $\mathbb{T}^4$.

We claim that there exists a continuous unstable foliation on S_f and that $Dh^u_{f,x,y*}$ is exactly the unstable holonomy for this foliation between the transversals $T^1\mathcal{F}^c_f(x)$ and $T^1\mathcal{F}^c_f(y)$. We apply the standard construction of the local unstable leaves as the invariant section of a bundle contraction map (see [13] for example), with a minor difficulty arising from the lack of smoothness.

For any $x, y \in \mathbb{T}^4$, we define the $\pi_{f,y,x} : E^c_f(y) \to E^c_f(x)$ as the projection parallel to $E^s_f(x) \oplus E^u_f(x)$. The maps $\pi_{f,y,x}$ depend continuously on $x, y \in \mathbb{T}^4$ and f (in the C^1 topology). For x close to y, this is invertible and close to the identity, and its projectivization $\pi_{f,x,y*}$ is bi-Lipschitz with Lipschitz constant close to 1.

For $\delta > 0$ and $x \in \mathbb{T}^4$, let $\alpha_{f,x} : [-\delta, \delta] \to \mathcal{F}^u_f(x)$ be the length parameterization of the local unstable manifold of f at x. Since the unstable foliation is orientable and depends continuously in the C^1 topology with respect to x and f, we have that $\alpha_{f,x}$ is continuous in x and f (in the C^1 topology).

For any $\delta > 0$, there exists $\epsilon_\delta > 0$ such that for any x, y such that $d(x, y) < \delta$, we have:

- $\|\pi^{\pm 1}_{f,y,x} - \mathrm{Id}\| < \epsilon_\delta$;
- $\pi^{\pm 1}_{f,y,x*}$ is bi-Lipschitz with constant $(1 + \epsilon_\delta)$;

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- $(1+\epsilon_\delta)^{-1}\lambda_c^-(x) < \lambda_c^-(y) \le \lambda_c^+(y) < (1+\epsilon_\delta)\lambda_c^+(x)$.
- If furthermore $y \in \mathcal{F}^u_{f,\delta}(x)$, then $d_u(f(x), f(y)) \ge (1+\epsilon_\delta)^{-1}\lambda_u(x)d_u(x, y)$, where d_u is the distance along the unstable leaves.

We can choose ϵ_δ independent of f in a C^1 neighborhood and $\lim_{\delta\to 0}\epsilon_\delta = 0$.

Now we will construct the bundle with the candidates for the local unstable manifolds in S_f. Consider $\delta > 0$ (small) to be specified later. Let

$$B = \left\{\sigma : [-\delta, \delta]) \to \mathbb{R} : \ \sigma(0) = 0, \ \left|\frac{\sigma(t)}{t}\right| < \infty\right\}$$

be the Banach space of functions σ bounded for the norm

$$\|\sigma\| = \sup_{t\in[-\delta,\delta]} \left|\frac{\sigma(t)}{t}\right|.$$

Then,

$$V(f) := S_f \times B$$

is a continuous (in fact, Hölder) Banach bundle over S_f.

Remark 2.4. The maps σ are candidates for unstable manifolds in S_f in the following sense. For any $(x, v) \in S_f$ and $\sigma \in B$, we can define a section $\tilde{\sigma} : \mathcal{F}^u_{f,\delta}(x) \to S_f$ in the following way:

$$\tilde{\sigma}(y) := \pi^{-1}_{f,y,x*}(v + \sigma(\alpha^{-1}_{f,x}(y))) \in S(f, y).$$

The graph of this section $\tilde{\sigma}$ is a natural candidate for the local unstable manifold in $(x, v) \in S_f$. We construct it as a fixed point of the natural graph transformation.

Let $T : V(f) \to V(f)$ be the bundle map which fibers over Df_* on S_f and is given by

$$\begin{aligned} T\sigma_{(x,v)}(t) &= (\pi_{f,f(y(t)),f(x)} \circ Df(y(t)) \circ \pi^{-1}_{f,y(t),x})_*(v + \sigma(\alpha^{-1}_{f,x}(y(t)))) \\ &\quad - Df(x)_*(v), \\ y(t) &= f^{-1} \circ \alpha_{f,f(x)}(t). \end{aligned}$$

One can check that in fact T is defined in such a way so that we have $\tilde{T\sigma} = Df_*\tilde{\sigma}$. Let us check that T is a continuous bundle map on $V(f)$, which is also a fiber contraction.

Claim 1. *If $\sigma \in B$, then $T\sigma_{(x,v)} \in B$.*

Proof. Remember that $y(t) = f^{-1} \circ \alpha_{f,f(x)}(t)$, and let us denote

$$G(t) := (\pi_{f,f(y(t)),f(x)} \circ Df(y(t)) \circ \pi^{-1}_{f,x,y}).$$

Observe that $G(t)_*$ is Lipschitz with the Lipschitz constant

$$\mathrm{Lip}(G(t)_*) = (1+\epsilon_\delta)^2 \frac{\lambda_c^+(y(t))}{\lambda_c^-(y(t))} \le (1+\epsilon_\delta)^4 \frac{\lambda_c^+(x)}{\lambda_c^-(x)}.$$

Also,

$$|\alpha^{-1}_{f,x}(y(t))| = d_u(x, y(t)) \le (1+\epsilon_\delta)\lambda_u(x)^{-1} d_u(f(x), f(y(t)) = (1+\epsilon_\delta)\lambda_u(x)^{-1}|t|.$$

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Then,

$$
\begin{aligned}
\|T\sigma\| &= \sup_{t\in[-\delta.\delta]}\left|\frac{G(t)_*(v+\sigma(\alpha_{f,x}^{-1}(y(t)))-Df(x)_*(v)}{t}\right| \\
&\le \sup_{t\in[-\delta.\delta]}\left|\frac{G(t)_*(v+\sigma(\alpha_{f,x}^{-1}(y(t)))-G(t)_*(v)}{t}\right| \\
&\quad + \sup_{t\in[-\delta.\delta]}\left|\frac{G(t)_*(v)-Df(x)_*(v)}{t}\right| \\
&\le (1+\epsilon_\delta)^4\frac{\lambda_c^+(x)}{\lambda_c^-(x)}\sup_{t\in[-\delta.\delta]}\left|\frac{\sigma(\alpha_{f,x}^{-1}(y(t))}{t}\right| \\
&\quad + \frac{\pi}{2}\sup_{t\in[-\delta.\delta]}\frac{1}{t}\left\|\frac{G(t)(v)}{\|G(t)(v)\|}-\frac{Df(x)(v)}{\|Df(x)(v)\|}\right\| \\
&\le (1+\epsilon_\delta)^5\frac{\lambda_c^+(x)}{\lambda_c^-(x)\lambda_u(x)}\|\sigma\| + \frac{\pi}{\lambda_c^+(x)}\sup_{t\in[-\delta.\delta]}\left\|\frac{G(t)(v)-Df(x)(v)}{t}\right\|,
\end{aligned}
$$

where in the last line, we used the inequality

$$
\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\| \le \left\|\frac{a}{\|a\|}-\frac{a}{\|b\|}\right\| + \left\|\frac{a}{\|b\|}-\frac{b}{\|b\|}\right\| \le \frac{2}{\|b\|}\|a-b\|.
$$

Let us remark that if $v\in E_f^c(x)$, then $\pi_{f,f(y(t)),f(x)}\circ Df(x)\circ\pi_{f,y,x}^{-1}(v)=Df(x)(v)$, because the partially hyperbolic splitting is invariant under Df. Then,

$$
\begin{aligned}
\|G(t)(v)-Df(x)(v)\| &= \|\pi_{f,f(y(t)),f(x)}\circ(Df(y(t))-Df(x))\circ\pi_{f,y,x}^{-1}(v)\| \\
&\le (1+\epsilon_\delta)^2\mathrm{Lip}(Df)d(y(t),x) \\
&\le \mathrm{Lip}(Df)\frac{(1+\epsilon_\delta)^3}{\lambda_u(x)}|t|.
\end{aligned}
$$

Finally, we obtain the desired bound:

$$
\|T\sigma\| \le \frac{(1+\epsilon_\delta)^5\lambda_c^+(x)}{\lambda_c^-(x)\lambda_u(x)}\|\sigma\| + \frac{\mathrm{Lip}(Df)(1+\epsilon_\delta)^3\pi}{\lambda_c^+(x)\lambda_u(x)}. \qquad \square
$$

CLAIM 2. *T is a fiber contraction.*

Proof. We have

$$
\begin{aligned}
\|T\sigma_1-T\sigma_2\| &= \sup_{t\in[-\delta,\delta]}\left|\frac{T\sigma_1(t)-T\sigma_2(t)}{t}\right| \\
&= \sup_{t\in[-\delta,\delta]}\left|\frac{G(t)_*(v+\sigma_1(\alpha_{f,x}^{-1}(y(t)))-G(t)_*(v+\sigma_2(\alpha_{f,x}^{-1}(y(t)))}{t}\right| \\
&\le (1+\epsilon_\delta)^4\frac{\lambda_c^+(x)}{\lambda_c^-(x)}\sup_{t\in[-\delta,\delta]}\left|\frac{\sigma_1(\alpha_{f,x}^{-1}(y(t))-\sigma_2(\alpha_{f,x}^{-1}(y(t))}{t}\right| \\
&\le (1+\epsilon_\delta)^5\frac{\lambda_c^+(x)}{\lambda_c^-(x)\lambda_u(x)}\|\sigma_1-\sigma_2\|.
\end{aligned}
$$

Now all we have to do is to choose δ small enough so that ϵ_δ is close enough to zero so that we have

$$(1+\epsilon_\delta)^5 \frac{\lambda_c^+(x)}{\lambda_c^-(x)\lambda_u(x)} < 1. \qquad \square$$

Claims 1 and 2 show that we are in the conditions of the invariant section theorem from [13], so there exists a unique bounded continuous invariant section.

From [24], we know that the unstable holonomy along center leaves is uniformly differentiable. The projectivization of the derivative of this local holonomy will then correspond to a bounded invariant section for the transfer operator T, so it has to coincide with the unique continuous invariant section constructed above. This concludes the proof of the continuity of $Dh^u_{f,x,y*}$ with respect to the points x, y (we proved it for $d(x, y) < \delta$, but this can be easily extended to larger distances).

If f_n converges to f, then S_{f_n} converges to S_f (this can be made explicit by projecting $E^c_{f_n}$ to E^c_f parallel to $E^s_f \oplus E^u_f$ for example). One can check that the corresponding transfer operators T_{f_n} also converge to T_f. Since the invariant section is continuous with respect to the bundle map, we obtain the continuity of $Dh^u_{f,x,y*}$ with respect to f. $\square$

Remark 2.5. We gave the proof for our special setting, but the proof can be adapted to general partially hyperbolic diffeomorphisms in higher dimensions. We used that Df is Lipschitz to show that the transfer operator T verifies the conditions of the invariant section theorem. The proof can be adapted for f of class $C^{1+\alpha}$ and the stronger bunching condition $\lambda_s(x)^\alpha < \lambda_c^-(x)/\lambda_c^+(x) \le \lambda_c^+(x)/\lambda_c^-(x) < \lambda_u(x)^\alpha$, using the norm $\|\sigma\| = \sup_{t\in[-\delta,\delta]} |\sigma(t)/t^\alpha|$. Once one obtains the bounded invariant section for the projectivization Df_* on S_f, using the boundness of the Jacobian, one could try to obtain the differentiability of the stable/unstable holonomy along center leaves.

2.2. *Homoclinic holonomies.* Let $f \in \mathcal{SH}^2_b$ and p_f be the fixed point such that $f\mid_{\mathcal{F}^c_f(p_f)}$ is Anosov. We will drop the index f when it is not necessary to specify the dependence on the map f. From [13] and the bunching conditions, we know that $\mathcal{F}^c(p)$, $\mathcal{F}^{cu}(p)$ and $\mathcal{F}^{cs}(p)$ are C^2 submanifolds. Assume that q is a homoclinic point of $W^c(p)$, that is, $q \in \mathcal{F}^{cu}(p) \cap \mathcal{F}^{cs}(p)$, then $W^c(q)$ is also C^2 as a connected component of the intersection of the transverse C^2 submanifolds $\mathcal{F}^{cu}(p)$ and $\mathcal{F}^{cs}(p)$. We can define the stable holonomy $h^s_{p,q} : \mathcal{F}^c(p) \to \mathcal{F}^c(q)$ and the unstable holonomy $h^u_{q,p} : \mathcal{F}^c(q) \to \mathcal{F}^c(p)$, and they are both of class C^1. Then $\tilde{h}_q := h^u_{q,p} \circ h^s_{p,q} : \mathcal{F}^c(p) \to \mathcal{F}^c(p)$ is a C^1 diffeomorphism, so it induces a C^0 map on the unit tangent bundle $T^1\mathcal{F}^c(p)$ which we denote by $D\tilde{h}_{q*}$.

Let $\tilde{v}^s(x)$ be the unit vector tangent in $x \in \mathcal{F}^c(p)$ to the stable bundle of $f\mid_{\mathcal{F}^c(p)}$ (we fix an orientation). Since $f\mid_{\mathcal{F}^c(p)}$ is a C^2 Anosov map on a C^2 surface, we have that $\tilde{v}^s : \mathcal{F}^c(p) \to T^1\mathcal{F}^c(p)$ is C^1. We define the map $\tilde{g}_q : \mathcal{F}^c(p) \to T^1\mathcal{F}^c(p)$,

$$\tilde{g}_q(x) := D\tilde{h}_{q*}(\tilde{v}^s(h_q^{-1}(x))) = \frac{D\tilde{h}_q(h_q^{-1}(x))\tilde{v}^s(h_q^{-1}(x))}{\|D\tilde{h}_q(h_q^{-1}(x))\tilde{v}^s(h_q^{-1}(x))\|}. \tag{6}$$

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Remark 2.6. In fact, we consider the stable foliation of $f\mid_{\mathcal{F}^c(p)}$ inside the leaf $\mathcal{F}^c(p)$, we first push it forward using the stable holonomy $h^s_{p,q}$ to the leaf $\mathcal{F}^c(q)$, and then we push it again using the unstable holonomy $h^u_{q,p}$ back to the leaf $\mathcal{F}^c(p)$. Then $\tilde{g}_q(x)$ is in fact the unit tangent vector in x to this new foliation.

If furthermore $f \in \mathcal{SH}^3_b$, then the stable and unstable holonomies along the fixed center-stable leaf $W^{cs}(p)$ and respectively the fixed center-unstable leaf $W^{cu}(p)$ are C^2, so, in this case, $D\tilde{h}_{q*}$ and $\tilde{g}_q$ are in fact C^1.

If the map f' is C^2 close to f, then the fixed Anosov center leaf $\mathcal{F}^c(p)$ and its homoclinic center leaf $\mathcal{F}^c(q)$ will have continuations $\mathcal{F}^c_{f'}(p(f'))$ and $\mathcal{F}^c_{f'}(q(f'))$. Then we obtain the continuations of the stable holonomy $h^s_{p(f'),q(f'),f'} : \mathcal{F}^c_{f'}(p(f')) \to \mathcal{F}^c_{f'}(q(f'))$ and the unstable holonomy $h^u_{q(f'),p(f'),f'} : \mathcal{F}^c_{f'}(q(f')) \to \mathcal{F}^c_{f'}(p(f'))$; they are C^1 maps and depend continuously in the C^1 topology with respect to f' (in the C^2 topology). We also have a continuation of the homoclinic holonomy $\tilde{h}_{q(f'),f'} : \mathcal{F}^c_{f'}(p(f')) \to \mathcal{F}^c_{f'}(p(f'))$ and also the continuation $\tilde{g}_{q(f'),f'} : \mathcal{F}^c_{f'}(p(f')) \to T^1\mathcal{F}^c_{f'}(p(f'))$, which is continuous both with respect to $x \in \mathcal{F}^c_{f'}(p(f'))$ and with respect to $f' \in \mathcal{SH}^2_b$ (in the C^2 topology).

2.3. *Hyperbolic measures.* Let μ be an ergodic measure of a diffeomorphism *f*, then by the theorem of Oseledets, for μ-almost every point $x \in M$, there exist $k(\mu) \in \mathbb{N}$, real numbers $\lambda_1(\mu) > \cdots \lambda_k(\mu)$, and an invariant splitting $T_xM = E^1(x) \oplus \cdots \oplus E^k(x)$ of the tangent bundle at x, depending measurably on the point, such that $\lim_{n\to\pm\infty}(1/n)\log\|Df^n_x(v)\| = \lambda_j(\mu)$ for all $0 \neq v \in E^j(x)$. The real numbers $\lambda_j(\mu)$ are the *Lyapunov exponents* of μ. We say that the ergodic measure μ is *hyperbolic* if all the Lyapunov exponents of μ are non-zero.

THEOREM 2.7. (Katok's horseshoe theorem [14]) *For any $f \in \mathrm{Diff}^r(M)$, $r > 1$ and any hyperbolic ergodic measure μ, there exists a hyperbolic periodic point p, such that $\mu \overset{h}{\sim} \delta_{\mathrm{Orb}(p)}$, where $\delta_{\mathrm{Orb}(p)}$ is the ergodic measure supported on the orbit* $\mathrm{Orb}(p)$.

If a diffeomorphism *f* admits a dominated splitting, then the Oseledet's splitting must be subordinated to the dominated splitting. In particular, since every $f \in \mathcal{SH}$ is partially hyperbolic on $\mathbb{T}^4$, then for any ergodic measure μ of *f*, its biggest Lyapunov exponent is positive ($\lambda^u > 0$) and its associated Oseledet's bundle is tangent to the strong unstable bundle E^u of *f*. A similar result holds for the minimal Lyapunov exponent $\lambda^s < 0$ with its associated Oseledet's bundle tangent to the strong stable bundle E^s. There are also two center Lyapunov exponents (counted with multiplicity) $\lambda^c_1 \geq \lambda^c_2$ whose associated Oseledet's bundles are tangent to the center bundle E^c of *f*.

2.4. *Criterion of uniqueness of equilibrium state*

Definition 2.8. Let μ be an ergodic hyperbolic measure of a diffeomorphism *f*. The *stable index of* μ is the number of negative Lyapunov exponents, counted with multiplicity.

PROPOSITION 2.9. *Let $f : M \to M$ be a C^r diffeomorphism $r > 1$, $\phi : M \to \mathbb{R}$ be a Hölder potential, and p a hyperbolic periodic point. Then there is at most one equilibrium* *state for ϕ which is homoclinically related to $\delta_{Orb(p)}$, and its support coincides with* $\mathrm{HC}(Orb(p))$.

Proof. This is explained in [3, Theorem 2.4] and [7, §1.6]. See also [2] and [7, Corollary 3.3]. □

2.5. *Hyperbolicity of equilibrium states.* If $f \in \mathrm{PH}^0_{A,B}$, then standard results of Franks and Manning [12, 20] imply that *f* is semi-conjugate to $f_{A,B}$, that is, there exists a continuous surjection $h : \mathbb{T}^4 \to \mathbb{T}^4$ homotopic to the identity such that $f_{A,B} \circ h = h \circ f$. By the Ledrappier–Walters variational principle [16], we have

$$h_{\mathrm{top}}(f) \geq h_{\mathrm{top}}(f_{A,B}) = \log \lambda_A + \log \lambda_B. \tag{7}$$

For any invariant probability measure μ of *f*, we say that a measurable partition ξ is μ *adapted (sub-ordinated) to* $\mathcal{F}^u$ if the following conditions are satisfied:

- there is $r_0 > 0$ such that $\xi(x) \subset B^{\mathcal{F}^u}_{r_0}(x)$ for μ almost every *x*, where $B^{\mathcal{F}^u}_{r_0}(x)$ is a ball of $\mathcal{F}^u(x)$ with radius r_0;
- $\xi(x)$ contains an open neighborhood of *x* inside $\mathcal{F}^u(x)$;
- ξ is increasing, that is, for μ almost every *x*, $\xi(x) \subset f(\xi(f^{-1}(x)))$.

The existence of such a partition was provided by [15] (see also [17, 30]). The *partial entropy of μ along the expanding foliation $\mathcal{F}^u$* is defined by

$$h_\mu(f, \mathcal{F}^u) = H_\mu(f^{-1}\xi \mid \xi).$$

The definition of the partial entropy does not depend on the choice of the partition.

The following two lemmas are important for our further discussion.

LEMMA 2.10. *If $f \in \mathrm{PH}^0_{A,B}$, then $h_\mu(f, \mathcal{F}^u) \leq \log \lambda_A$.*

Proof. Denote by $\mathcal{F}^c_f$ the center foliation of *f*. By Proposition 1.3, the projection map π^c_f along the center foliation induces a topological Anosov homeomorphism $\overline{f}$ on the quotient space $\overline{\mathbb{T}}^2_f = \mathbb{T}^4/\mathcal{F}^c_f$, which is topological conjugate to *A*, so we may in fact identify $\overline{\mathbb{T}}^2_f$ with $\mathbb{T}^2$ and $\overline{f}$ with *A*.

Denote by $\mathcal{F}^s_A$ (respectively $\mathcal{F}^u_A$) the stable (respectively unstable) foliation of *A*. The projection map π^c_f maps each center unstable leaf $\mathcal{F}^{cu}$ of *f* to an unstable leaf $\mathcal{F}^u_A$ of *A*. In particular, π^c_f maps every unstable leaf $\mathcal{F}^u$ of *f* to an unstable leaf $\mathcal{F}^u_A$ of *A*. Proposition 1.3 implies that all the hypotheses of Tahzibi and Yang [28, Theorem A] are satisfied (see also [1, §2.5]), and this implies that $h_\mu(f, \mathcal{F}^u) \leq h_{\mathrm{top}}(A) = \log \lambda_A$. □

The following lemma is a generalization of [1, Theorem A].

LEMMA 2.11. *Let $f \in \mathrm{PH}^0_{A,B}$ be a C^r diffeomorphism, $r > 1$, and μ an ergodic invariant measure of f with $h_\mu(f) > \log \lambda_A$. Then μ is a hyperbolic ergodic measure of f with stable index 2, that is, $\lambda^c_1 > 0 > \lambda^c_2$.*

Proof. We will show that $\lambda^c_1 > 0$. To prove that $\lambda^c_2 < 0$, one only needs to consider the diffeomorphism f^{-1} instead of diffeomorphism *f*.

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Suppose by contradiction that $\lambda_1^c \le 0$. The entropy formula of Ledrappier and Young (see [5, 15]) implies that $h_\mu(f) = h_\mu(f, \mathcal{F}^u)$.

Combining with the previous lemma, we obtain that $h_\mu(f) \le \log \lambda_A$, which is a contradiction with the hypothesis that $h_\mu(f) > \log \lambda_A$. The proof is complete. $\square$

3. *Proof of Theorem A*

3.1. *Definition of $\mathcal{U}^r$ and plan of the proof.* Let us define the open set $\mathcal{U}^r$ which is the candidate for the set $\mathcal{U}^r$ in Theorem A. We recall that p is a fixed point for $f \in \mathcal{SH}_b^r$, and the restriction of f to the center leaf $\mathcal{F}^c(p)$ is Anosov. Here, q is a homoclinic point of $\mathcal{F}^c(p)$ if $q \in \mathcal{F}^{cu}(p) \cap \mathcal{F}^{cs}(p)$. The map $\tilde{g}_q : \mathcal{F}^c(p) \to T^1\mathcal{F}^c(p)$ is defined by equation (6), and represents in fact the unit tangent vector to the foliation obtained by pushing the stable foliation of $f\,|_{\mathcal{F}^c(p)}$ along the homoclinic loop corresponding to q.

Definition 3.1. Let $\mathcal{U}_s^r \subset \mathcal{SH}_b^r$,

$$\mathcal{U}_s^r = \{f \in \mathcal{SH}_b^r : \text{ for all } x \in \mathcal{F}^c(p),\ \text{there exists } q \text{ homoclinic to } \mathcal{F}^c(p) \\ \text{such that } \tilde{g}_q(x) \ne \pm\tilde{v}^s(x)\}. \tag{8}$$

In a similar way, we define $\mathcal{U}_u^r$. Let $\mathcal{U}^r = \mathcal{U}_s^r \cap \mathcal{U}_u^r$.

The definition of $\mathcal{U}_s^r$ is given, in fact, by a transversality condition. What we ask is that the stable foliation of $f\,|_{\mathcal{F}^c(p)}$ and its pushed forward by holonomies along homoclinic loops are transverse.

To prove Theorem A, we will have to show the following three facts:

(1) the set $\mathcal{U}^r$ is C^2 open;
(2) the set $\mathcal{U}^r$ is C^r dense;
(3) the set $\mathcal{U}^r$ verifies the conclusion of Theorem A, in other words, if $f \in \mathcal{U}^r$, then any two hyperbolic periodic points of f of index 2 are homoclinically related.

Consequently, the proof of Theorem A is divided into the following three propositions.

PROPOSITION 3.2. *$\mathcal{U}^r$ is C^2 open.*

Proof. An immediate consequence of the compactness of $\mathcal{F}^c(p)$ and of the fact that the stable and unstable holonomies depend continuously in the C^1 topology with respect to the points (see Remark 1.7) is the following lemma.

LEMMA 3.3. *Let $f \in \mathcal{SH}_b^r$. Then $f \in \mathcal{U}_s^r$ if and only if there exist $q_1, q_2, \ldots q_k$ homoclinic points of $\mathcal{F}^c(p)$ such that the image of $\tilde{g}_{q_1} \times \tilde{g}_{q_2} \times \cdots \times \tilde{g}_{q_k}$ is disjoint from the image of $\pm\tilde{v}^{s^k}$.*

However, the holonomies along the center leaves depend continuously in the C^1 topology with respect to the map f (in C^2 topology), so the images of $\tilde{g}_{q_1} \times \tilde{g}_{q_2} \times \cdots \times \tilde{g}_{q_k}$ and $\pm\tilde{v}^{s^k}$ depend continuously on the map f. Since these images are compact, this concludes the C^2 openness of $\mathcal{U}_s^r$. The proof for $\mathcal{U}_u^r$ is similar, so $\mathcal{U}^r = \mathcal{U}_s^r \cap \mathcal{U}_u^r$ is C^2 open. $\square$

PROPOSITION 3.4. *$\mathcal{U}^r$ is C^r dense.*

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We will give the proof of the proposition in §3.2.

PROPOSITION 3.5. *If $f \in \mathcal{U}^r$, then any two hyperbolic periodic points of f of index 2 are homoclinically related.*

We will give the proof of the proposition in §3.3. As we mentioned before, the proof of these three propositions will imply Theorem A.

3.2. *Proof of C^r density.* We will show that $\mathcal{U}_s^r$ is C^r dense in $\mathcal{SH}_b^r$, the proof for $\mathcal{U}_u^r$ is similar. Then $\mathcal{U}^r$ will be C^r dense as the intersection of two C^r open dense sets.

The main perturbation result which we will use is the following lemma.

LEMMA 3.6. *Let $f \in \mathcal{SH}_b^3$. Let q be a homoclinic point of the fixed Anosov leaf $\mathcal{F}^c(p)$. Then we have the following.*

(1) *For any C^2 family (ϕ_T), where $T \in \mathbb{R}^n$ is a parameter, of perturbations of the identity on $\mathbb{T}^4$ (in other words, $\phi_{0^n} = \mathrm{Id}_{\mathbb{T}^4}$), supported in a neighborhood of $\mathcal{F}^c(q)$ disjoint from all the other iterates $f^k(\mathcal{F}^c(q))$, $k \in \mathbb{Z} \setminus \{0\}$, the map $(T, x) \mapsto \tilde{g}_{q(f\circ\phi_T), f\circ\phi_T}(x)$ is C^1 on $[-\delta, \delta]^n \times \mathcal{F}^c(p)$ for some $\delta > 0$.*
(2) *For any $x_0 \in \mathcal{F}^c(p)$ and any $r_0 > 0$, there exists a C^∞ family $(\phi_t)_{t\in[-\delta,\delta]}$ of (volume-preserving) perturbations of the identity on $\mathbb{T}^4$, supported in $B(y_0, r_0)$ where $y_0 := h^u_{p,q}(x_0) \in \mathcal{F}^c(q)$, such that*

$$\frac{\partial}{\partial t}\tilde{g}_{q(f\circ\phi_t), f\circ\phi_t}(x) \mid_{(x,t)=(x_0,0)} \neq 0. \tag{9}$$

Proof. Part (1). Since $\mathcal{F}^c(p)$ is compact, it is enough to prove that $\tilde{g}_{q(f\circ\phi_T), f\circ\phi_T}(x)$ is C^1 in (x, T) in a small neighborhood of every point $(x_0, 0^n) \in \mathcal{F}^c(p) \times \mathbb{R}^n$.

Let $x_0 \in \mathcal{F}^c(p)$ and denote $y_0 = h^u_{p,q}(x_0) \in \mathcal{F}^c(q)$, $y_1 = f(y_0) \in \mathcal{F}^c(f(q))$, $z_1 = h^s_{f(q),p}(y_1) \in \mathcal{F}^c(p)$, and $z_0 = f^{-1}(z_1) \in \mathcal{F}^c(p)$. The f-invariance of the stable holonomy implies that $h^s_{q,p}(y_0) = z_0$, or $\tilde{h}_q(z_0) = x_0$.

Let $\psi_p : \mathbb{T}^2 \to \mathcal{F}^c(p)$ be a C^2 embedding, $a_0 = \psi_p^{-1}(x_0)$, $b_0 = \psi_p^{-1}(z_0)$. Let $I_\delta = [-\delta, \delta]$. There exist C^2 foliations charts of $\mathcal{F}^s$ (respectively $\mathcal{F}^u$) on a small neighborhood of y_1 (respectively y_0) inside $\mathcal{F}^{cs}(p)$ (respectively $\mathcal{F}^{cu}(p)$):

$$\alpha^s : B^c_{y_1} \times I_\delta \to B^{cs}_{y_1}, \alpha^s(\cdot, 0) = \mathrm{Id}_{\mathcal{F}^c_{\mathrm{loc}}(y_1)}, \alpha^s(\{y\} \times I_\delta) = \mathcal{F}^s_{\mathrm{loc}}(y) \quad \text{for all } y \in B^c_{y_1};$$
$$\alpha^u : B^c_{y_0} \times I_\delta \to B^{cu}_{y_0}, \alpha^u(\cdot, 0) = \mathrm{Id}_{\mathcal{F}^c_{\mathrm{loc}}(y_0)}, \alpha^u(\{y\} \times I_\delta) = \mathcal{F}^u_{\mathrm{loc}}(y) \quad \text{for all } y \in B^c_{y_0},$$

where B^*_x denotes a small ball centered in x inside $\mathcal{F}^*(x)$. Define the C^2 maps

$$\beta^s : B_{b_0} \times I_\delta \to B^{cs}_{y_1},\ \beta^s(b, s) = \alpha^s(h^s_{p,f(q)}(f(\psi_p(b))), s);$$
$$\beta^u : B_{a_0} \times I_\delta \to B^{cu}_{y_0},\ \beta^u(a, r) = \alpha^u(h^u_{p,q}(\psi_p(a)), r),$$

where B_x is a small ball centered at x in $\mathbb{T}^2$.

We know that the support of ϕ_T does not intersect $f^k(\mathcal{F}^{cs}_{\mathrm{loc}}(y_1))$ for all $k \geq 0$ and $f^l(\mathcal{F}^{cu}_{\mathrm{loc}}(y_0))$ for all $l < 0$. This implies that $\mathcal{F}^{cs}_{\mathrm{loc}}(y_1)$ remains a local center-stable leaf for $f_T := f \circ \phi_T$ for all T, $\beta^s(a, \cdot)$ remain parameterizations of the strong stable manifolds

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inside $\mathcal{F}^{cs}_{\rm loc}(y_i)$, $\mathcal{F}^{cs}_{\rm loc}(y_1)$ remains inside $\mathcal{F}^{cs}(p)$ and the stable holonomy between $\mathcal{F}^c(p)$ and $\mathcal{F}^{cs}_{\rm loc}(y_1)$ is unchanged. A similar statement holds for $\mathcal{F}^{cu}_{\rm loc}(y_0)$.

The maps f_T do change the center leaves $\mathcal{F}^c(q)$, we have that $\mathcal{F}^c(q(f_T), f_T) = f^{-1}(\mathcal{F}^s_{\rm loc}(\mathcal{F}^c(y_1))) \cap \mathcal{F}^u_{\rm loc}(\mathcal{F}^c(y_0))$. We can in fact compute implicitly the homoclinic stable–unstable holonomy $\tilde{h}_{q(f_T),f_T}$ in a neighborhood of z_0 in the following way:

$$\begin{aligned}\psi_p(a) = \tilde{h}_{q(f_T),f_T}(\psi_p(b)) &\iff h^u_{p,q(f_T),f_T}(\psi_p(a)) = h^s_{p,q(f_T),f_T}(\psi_p(b))\\ &\iff h^u_{p,q(f_T),f_T}(\psi_p(a)) = f_T^{-1}(h^s_{p,q(f_T),f_T}(f(\psi_p(b)))\\ &\iff \beta^u(a,r) = f_T^{-1}(\beta^s(b,s)) \quad \text{for some } r, s \in I_\delta\\ &\iff \phi_T(\beta^u(a,r)) = f^{-1}(\beta^s(b,s)) \quad \text{for some } r, s \in I_\delta.\end{aligned}$$

In conclusion, denoting $h_T = \psi_p^{-1} \circ \tilde{h}_{q(f_T),f_T} \circ \psi_p$ (the map $\tilde{h}_{q(f_T),f_T}$ in the chart ψ_p), we have

$$a = h_T(b) \iff \phi_T(\beta^u(a,r)) = f^{-1}(\beta^s(b,s)) \quad \text{for some } r, s \in I_\delta. \tag{10}$$

We choose a C^∞ chart $\psi_q : B_{y_0} \to \mathbb{R}^4$ (can also be volume preserving) such that:

- $\psi_q(y_0) = 0^4$;
- $D\psi_q(y_0)(E^c(q_0)) = \operatorname{span}\{e_1, e_2\}$;
- $D\psi_q(y_0)(E^u(q_0)) = \operatorname{span}\{e_3\}$;
- $D\psi_q(y_0)(E^s(q_0)) = \operatorname{span}\{e_4\}$.

Let $E : B_{a_0} \times B_{b_0} \times I_\delta^{n+2} \to \mathbb{R}^4$,

$$E(a,b,r,s,T) = \psi_q(\phi_T(\beta^u(a,r))) - \psi_q(f^{-1}(\beta^s(b,s))). \tag{11}$$

We have that E is C^2 and $E(a_0, b_0, 0, 0, 0^n) = \psi_q(y_0) - \psi_q(f^{-1}(y_1)) = 0$.

CLAIM. $\partial E/\partial(b,r,s)(a_0, b_0, 0, 0, 0^n)$ *is invertible.*

Proof. We observe that since α^s is a diffeomorphism such that $\alpha^s(\{y\} \times I_\delta) = \mathcal{F}^s_{\rm loc}(y)$, we have that $D\alpha^s(y, 0) \cdot \partial/\partial s$ is a non-zero vector in $E^s(y)$. Since Df preserves E^s, and $D\psi_q(y_0)$ takes $E^s(q_0)$ to the line generated by e_4, we have that $DE(a_0, b_0, 0, 0, 0^n)$ takes the line generated by $\partial/\partial s$ isomorphically to the line generated by e_4.

A similar argument shows that $DE(a_0, b_0, 0, 0, 0^n)$ takes the line generated by $\partial/\partial r$ isomorphically to the line generated by e_3 (remember that $\phi_{0^n} = \mathrm{Id}_{\mathbb{T}^4}$).

Now let us analyze the action of $DE(a_0, b_0, 0, 0, 0^n)$ on the two-dimensional space $T_{b_0}B_{b_0}$. It is not hard to see that $D\beta^s(b_0, 0)$ takes $T_{b_0}B_{b_0}$ isomorphically to $E^c(y_1)$. Since Df preserves E^c and $D\psi_q(y_0)$ takes $E^c(q_0)$ to the plane generated by e_1 and e_2, we have that $DE(a_0, b_0, 0, 0, 0^n)$ takes $T_{b_0}B_{b_0}$ isomorphically to the plane generated by e_1 and e_2. This concludes the proof of the claim. $\square$

Now let us finish the proof of the first part of the lemma. The implicit function theorem gives us the existence of a C^2 function $H : B_{a_0} \times I_\delta^n \to B_{b_0} \times I_\delta^2$, $H(a,T) = (h(a,T), r(a,T), s(a,T))$ such that $E(a, h(a,T), r(a,T), s(a,T), T) = 0$ (eventually by making smaller the balls and the intervals). Then the map $h_T(a) = h(a,T)$ is C^2 in both variables, which means that $\tilde{h}_{q(f_T),f_T}(x)$ is C^2 in both variables, and then $\tilde{g}_{q(f\circ\phi_T),f\circ\phi_T}(x)$ is C^1 in both variables. This finishes the proof of the first part.

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Part (2). We will use the same notation from part (1). Let $\rho : \mathbb{R}^4 \to [0, \infty)$ be a smooth bump function supported on a small ball centered at the origin, and constantly equal to one near the origin. The family $\phi_t : \mathbb{T}^4 \to \mathbb{T}^4$ is defined as

$$\phi_t := \psi_q^{-1} \circ (R_{\rho t} \times \mathrm{Id}_{\mathbb{R}^2}) \circ \psi_q,$$

where R_t is the rotation of angle t in $\mathbb{R}^2$. Assume that the support of ρ is small enough so that the support of ϕ_t is inside $B(y_0, r_0)$ and disjoint of all the other iterates of $W^c(q)$. From part (1), we have

$$E(a, b, r, s, t) = (R_{\rho t} \times \mathrm{Id}|\mathbb{R}^2)(\psi_q(\beta^u(a, r))) - \psi_q(f^{-1}(\beta^s(b, s))). \tag{12}$$

We will compute $DE(a_0, b_0, 0, 0, t)$. Observe that $L_b := DE(a_0, b_0, 0, 0, t) \mid_{T_{b_0} B_{b_0}}: T_{b_0} B_{B_0} \to \mathrm{span}\{e_1, e_2\}$ and $L_s := DE(a_0, b_0, 0, 0, t) \mid_{\mathrm{span}\{\partial/\partial s\}}: \mathrm{span}\{\partial/\partial s\} \to \mathrm{span}\{e_4\}$ are isomorphisms independent of t. Since $D(R_{\rho t} \times \mathrm{Id}_{\mathbb{R}^2})$ keeps e_3 invariant, we have that also $L_r := DE(a_0, b_0, 0, 0, t) \mid_{\mathrm{span}\{\partial/\partial r\}}: \mathrm{span}\{\partial/\partial r\} \to \mathrm{span}\{e_3\}$ is also an isomorphism independent of t, and

$$\frac{\partial E}{\partial(b, r, s)}(a_0, b_0, 0, 0, t) = L_b \times L_r \times L_s.$$

From equation (12), we can compute

$$DE(a_0, b_0, 0, 0, t) \mid_{T_{a_0} B_{a_0}} = R_t \circ L_a : T_{a_0} B_{a_0} \to \mathrm{span}\{e_1, e_2\},$$

where $L_a := DE(a_0, b_0, 0, 0, 0) \mid_{T_{a_0} B_{a_0}}: T_{a_0} B_{a_0} \to \mathrm{span}\{e_1, e_2\}$ is an isomorphism. From the implicit function theorem, we deduce that

$$Dh_t(a_0) = L_b^{-1} \circ R_t \circ L_a. \tag{13}$$

Define $g : B_{a_0} \times I_\delta \to \mathbb{T}^1$,

$$g(a, t) := D\psi_p^{-1}(a)_* \tilde{g}_{q(f \circ \phi_t), f \circ \phi_t}(\psi(a)) = \frac{Dh_t(a)(v^s(a))}{\|Dh_t(a)(v^s(a))\|},$$

where $D\psi_p^{-1}(a)_*$ is the diffeomorphism induced by $D\psi_p^{-1}(a)$ on the unit tangent bundles and $v^s(a) = D\psi_p^{-1}(a)(\tilde{v}^s(\psi_p(a)))$. In other words, $g(\cdot, t)$ is in fact the map $\tilde{g}_{q(f \circ \phi_t), f \circ \phi_t}$ seen in the chart ψ_p which identifies $W^c(p)$ with $\mathbb{T}^2$ and the unit tangent spaces to $W^c(p)$ with $\mathbb{T}^1$. To prove equation (9), it is enough to show that

$$\frac{\partial}{\partial t} g(a, t) \mid_{(a,t)=(a_0,0)} \neq 0,$$

which in turns is equivalent to the fact that $Dh_0(a_0)(v^s(a_0))$ and $\partial/\partial t Dh_t(a_0)(v^s(a_0))|_{t=0}$ are not collinear. Using equation (13), we obtain $Dh_0(a_0)(v^s(a_0)) = L_b^{-1} \circ L_a(v^s(a_0))$ and $\partial/\partial t Dh_t(a_0)(v^s(a_0)) \mid_{t=0} = L_b^{-1} \circ R_{\pi/2} \circ L_a(v^s(a_0))$, which are clearly non-collinear since L_a and L_b are isomorphisms while $v^s(a_0)$ is non-zero. This finishes the proof of part (2). □

Now let us prove Proposition 3.4.

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Proof of Proposition 3.4. Let $f \in \mathcal{SH}_b^r$. We need to find maps in $\mathcal{U}_s^r$ arbitrarily C^r close to f. Since the C^∞ maps are dense in the C^r maps in the C^r topology (even inside the volume preserving class), we can assume that f is C^∞.

Choose q_1, q_2, q_3 homoclinic points of $\mathcal{F}^c(p)$ such that the orbits of the homoclinic leaves $\mathcal{F}^c(q_i)$ are mutually disjoint, $i \in \{1, 2, 3\}$.

For any $x \in \mathcal{F}^c(p)$ and any $1 \leq i \leq 3$, there exists $r_{x,i} > 0$ such that if $y_i := h^u_{p,q_i}(x) \in \mathcal{F}^c(q_i)$, then the ball $B(y_i, r_{x,i})$ is disjoint from $\mathcal{F}^c(p)$, from all the iterates $f^k(\mathcal{F}^c(q_i))$ for all $k \neq 0$, and from all the iterates of $\mathcal{F}^c(q_j)$, $j \neq i$. Applying Lemma 3.6 part (2), we obtain the family of perturbations $\phi_{t,x,i}$ such that the derivative of $\tilde{g}_{q_i, f\circ\phi_{t,x,i}}$ with respect to t in $(x, 0)$ does not vanish. By the continuity of the derivative, there exists a neighborhood $U_{x,i}$ of x such that $(\partial/\partial t)\tilde{g}_{q_i, f\circ\phi_{t,x,i}}$ is non-zero on $\overline{U}_{x,i} \times \{0\}$.

Let $U_x = \bigcap_{i=1}^3 U_{x,i}$. By compactness of $\mathcal{F}^c(p)$, there exist finitely many $x^1, x^2, \ldots$ $x^K \in \mathcal{F}^c(p)$ such that $\mathcal{F}^c(p) = \bigcup_{j=1}^K U_{x^j}$.

Let us fix some notation. Denote

$$T = (t_i^j)_{1\leq i\leq 3, 1\leq j\leq K} = (T_i)_{1\leq i\leq 3} = (T^j)_{1\leq j\leq K} \in I^{3K} := [-\delta, \delta]^{3K},$$

with $T_i = (t_i^j)_{1\leq j\leq K} \in I^K$, $i \in \{1, 2, 3\}$ and $T^j = (t_i^j)_{1\leq i\leq 3} \in I^3$, $j \in \{1, 2, \ldots K\}$.

For every $1 \leq i \leq 3$, we let $\phi_i : \mathbb{T}^4 \times I^K \to \mathbb{T}^4$ given by

$$\phi_i(\cdot, T_i) = \phi_{t_i^1, x^1, i} \circ \phi_{t_i^2, x^2, i} \circ \cdots \circ \phi_{t_i^K, x^K, i} \quad \text{for all } T_i \in I^K. \tag{14}$$

We define $\phi,\ F : \mathbb{T}^4 \times I^{3K} \to \mathbb{T}^4$,

$$\begin{aligned} \phi(\cdot, T) &= \phi_T(\cdot) := \phi_1(\cdot, T_1) \circ \phi_2(\cdot, T_2) \circ \phi_3(\cdot, T_3), \\ F(\cdot, T) &= F_T(\cdot) := f \circ \phi_T. \end{aligned}$$

The maps ϕ_i, ϕ, and F have the following properties:

(1) ϕ_i, ϕ, and F are of class C^∞ on (x, T);

(2) ϕ_i is a small perturbation of the identity on a small neighborhood of $\mathcal{F}^c(q_i)$, in particular, it leaves the other homoclinic orbits of $\mathcal{F}^c(q_j)$ unchanged for $j \neq i$;

(3) F_T is equal to f on a neighborhood of $\mathcal{F}^c(p)$, so it does not change $\mathcal{F}^c(p)$ and the function $\tilde{v}^s$.

Let $V_j = \psi_p^{-1}(U_{x^j})$, where $\psi_p : \mathbb{T}^2 \to \mathcal{F}^c(p)$ is the C^2 embedding. For every $1 \leq i \leq 3$, define $g_i : \mathbb{T}^2 \times I^{3K} \to \mathbb{T}^1$,

$$g_i(x, T) = D\psi_p^{-1}(x)_* \tilde{g}_{q_i(f_T), f_T}(\psi_p(x)).$$

In other words, $g_i(\cdot, T)$ is again the map $\tilde{g}_{q_i(f_T), f_T}$ seen in the chart ψ_p which identifies $\mathcal{F}^c(p)$ with $\mathbb{T}^2$ and the unit tangent spaces $T^1\mathcal{F}^c(p)$ with $\mathbb{T}^1$. Lemma 3.6 part (1) tells us that g_i is C^1 with respect to $(x, T) \in \mathbb{T}^2 \times I^{3K}$ (maybe for a smaller interval I). Furthermore,

$$\frac{\partial g_i}{\partial t_i^j}(x, T) \neq 0 \quad \text{for all } (x, T) \in \overline{V}_j \times \{0\}^{3K}, \text{ for all } 1 \leq j \leq K. \tag{15}$$

However, because for $l \neq i$, the perturbation ϕ_l does not touch the orbit of $\mathcal{F}^c(q_i)$, we have

$$\frac{\partial g_i}{\partial t_l^j}(x, T) = 0 \quad \text{for all } (x, T), \text{ for all } l \neq i, \text{ for all } 1 \leq j \leq K. \tag{16}$$

Define $G : \mathbb{T}^2 \times I^{3K} \to \mathbb{T}^3$

$$G(x, T) = (g_1(x, T), g_2(x, T), g_3(x, T)). \tag{17}$$

Again, G is C^1 in $(x, T) \in \mathbb{T}^2 \times I^{3K}$. Equations (15) and (16) tell us that for every $1 \leq j \leq K$, we have

$$\det\left(\frac{\partial G}{\partial T^j}(x, T)\right) = \det\left(\frac{\partial g_i}{\partial t_i^j}(x, T)\right) = \prod_{i=1}^{3} \frac{\partial g_i}{\partial t_i^j}(x, T) \neq 0 \quad \text{for all } (x, T) \in \overline{V}_j \times \{0\}^{3K}.$$

From the compactness of $\overline{V}_j$ and the C^1 continuity of G with respect to T, there exists $J \subset I$ with $0 \in J$ such that, for all $1 \leq j \leq K$, we have

$$\det\left(\frac{\partial G}{\partial T^j}(x, T)\right) \neq 0 \quad \text{for all } (x, T) \in V_j \times J^{3K}, \tag{18}$$

and since every point from $\mathbb{T}^2$ is inside some V_j, we conclude that G has maximal rank at every point in $\mathbb{T}^2 \times J^{3K}$.

Remember that $v^s : \mathbb{T}^2 \to \mathbb{T}^1$ is the C^1 map given by $v^s(x) = D\psi_p^{-1}(x)_* \tilde{v}^s(\psi(x))$. Let $A := \{(x, T) \in \mathbb{T}^2 \times J^{3K} : G(x, T) \in \{-v^s(x), v^s(x)\}^3\}$ and $B = \pi_2(A)$, where π_2 is the projection from $\mathbb{T}^2 \times J^{3K}$ on the T component in J^{3K}.

A simple consequence of the above definitions is the fact that if $T \notin B$, then $f_T \in \mathcal{U}_s^r$. To finish the proof of the density of $\mathcal{U}_s^r$, we have to find T arbitrarily close to 0^{NK} such that $T \notin B$. We will prove in fact that B has Lebesgue measure zero in J^{3K}.

It is enough to show this for $B_1 = \pi_2(A_1)$, where $A_1 = \{(x, T) \in \mathbb{T}^2 \times J^{3K} : G(x, T) = v^s(x)^3\}$, the other combinations of $\pm v^s$ work similarly. Let $H(x, T) = G(x, T) - v^s(x)^3$, this is a C^1 map from $\mathbb{T}^2 \times J^{3K}$ to $\mathbb{T}^3$. Equation (18) tells us that H has maximal rank equal to 3 at every point (v^s is independent of T), so $H^{-1}(0^3)$ is a C^1 submanifold of codimension 3 (or dimension $3K - 1$) inside $\mathbb{T}^2 \times J^{3K}$. Since $\pi_2 |_{H^{-1}(0^3)}: H^{-1}(0^3) \to J^{3K}$ is a C^1 map, Sard's theorem tells us that the image B_1 has Lebesgue measure zero.

This implies that we can find arbitrarily small $T \notin B$, which finishes the proof of the C^r density of $\mathcal{U}_s^r$. □

3.3. *Proof of Proposition 3.5*

Proof. We first remark that, because of the transitivity of the homoclinic relation, it is enough to show that every hyperbolic periodic point of index 2 of $f \in \mathcal{U}^r$ is homoclinically related to the fixed point p of the hyperbolic fixed leaf $\mathcal{F}^c(p)$.

Let x be a hyperbolic point of $f \in \mathcal{U}^r$ of index 2. Let $\tilde{v}^u(x)$ be the unit tangent vector to the weak unstable direction inside $T_x\mathcal{F}^c(x)$. The strong unstable manifold $\mathcal{F}^u(x)$ must accumulate on the fixed hyperbolic leaf $\mathcal{F}^c(p)$, so there exists a sequence of homoclinic points $p_n \in \mathcal{F}^u(x) \cap \mathcal{F}^s_{\mathrm{loc}}(\mathcal{F}^c(p))$ such that $\lim_{n\to\infty} p_n = p_0 \in \mathcal{F}^c(p)$. If for some p_n

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we have that $Dh^u_{x,p_n*}(\tilde{v}^u(x)) \neq \pm Dh^s_{p,p_n*}(\tilde{v}^s(h^{s^{-1}}_{p,p_n}(p_n)))$, then the two-dimensional unstable manifold of x, $W^u(x)$, intersects $\mathcal{F}^c(p_n)$ in a C^1 curve locally transverse to the weak stable foliation inside $\mathcal{F}^c(p_n)$ (which is then pushed forward by the stable holonomy of the weak stable foliation in $\mathcal{F}^c(p)$). Since the two-dimensional global stable manifold of p, $W^s(p)$, is dense inside the weak stable foliation of $\mathcal{F}^c(p_n)$, we obtain a transverse homoclinic intersection from x to p.

Suppose that $W^u(x) \cap W^s(p) = \emptyset$. The above argument implies that

$$Dh^u_{x,p_n*}(\tilde{v}^u) = \pm Dh^s_{p,p_n*}(\tilde{v}^s(h^{s^{-1}}_{p,p_n}(p_n))) \quad \text{for all } n \in \mathbb{N}.$$

Since $f \in \mathcal{U}^r$, there exists a homoclinic point q of $\mathcal{F}^c(p)$ such that $\tilde{g}_q(p_0) \neq \pm\tilde{v}^s(p_0) \in T^1\mathcal{F}^c(p)$. Let $q_0 := h^u_{p,q}(p_0)$, consider the strong unstable holonomy $h^u_{\text{loc}} : \mathcal{F}^{cs}_{\text{loc}}(p_0) \to \mathcal{F}^{cs}_{\text{loc}}(q_0)$, and let $q_n := h^u_{\text{loc}}(p_n) \in \mathcal{F}^{cs}_{\text{loc}}(q_0)$. Then, $q_n \to q_0$. The lack of homoclinic relations between x and p implies that also

$$Dh^u_{x,q_n*}(\tilde{v}^u) = \pm Dh^s_{p,q_n*}(\tilde{v}^s(h^{s^{-1}}_{p,q_n}(q_n))) \quad \text{for all } n \in \mathbb{N}.$$

Since $h^u_{x,q_n} = h^u_{p_n,q_n} \circ h^u_{x,p_n}$, we obtain that

$$Dh^s_{p,q_n*}(\tilde{v}^s(h^{s^{-1}}_{p,q_n}(q_n))) = \pm Dh^u_{p_n,q_n*} \circ Dh^s_{p,p_n*}(\tilde{v}^s(h^{s^{-1}}_{p,p_n}(p_n))).$$

Using the continuity of $\tilde{v}^s$ and of $Dh^{s,u}$, we can pass to the limit and obtain that

$$Dh^s_{p,q*}(\tilde{v}^s(h^{s^{-1}}_{p,q}(q_0))) = \pm Dh^u_{p,q*}(\tilde{v}^s(p_0)),$$

or $\tilde{g}_q(p_0) = \pm\tilde{v}^s(p_0)$, which is a contradiction.

The proof of the intersection of the global stable manifold of x with the global unstable manifold of p is similar. This concludes the proof. □

Now, as we explained in §3.1, the proof of Theorem A is concluded by this last proposition.

4. *Proof of Corollary B*

We have to show that for any $f \in \mathcal{U}^r$, the transverse homoclinic intersections of the invariant manifolds of the fixed hyperbolic point p_f are dense in $\mathbb{T}^4$. The proof uses the same ideas from the proof of Proposition 3.5.

Let $f \in \mathcal{U}^r$, and p be the hyperbolic fixed point of f (for simplicity, we will drop the index f in the following arguments). Let U be an open set in $\mathbb{T}^4$. Since $W^u(p) \cap W^s(\mathcal{F}^c(p))$ is dense in $\mathbb{T}^4$, choose $x \in W^u(p) \cap W^s(\mathcal{F}^c(p))$ such that $B(x, \delta) \subset U$ for some $\delta > 0$. If $Dh^s_{p,x*}(\tilde{v}^s(h^s_{x,p}(x))) \notin T_x W^u(p)$, then clearly there is a transverse homoclinic intersection between $W^s(p)$ and $W^u(p)$ arbitrarily close to x.

Suppose that $v := Dh^s_{p,x*}(\tilde{v}^s(h^s_{x,p}(x))) \in T_x W^u(p)$. Then there exists a subsequence $n_k \to \infty$ and $(p_0, v_0) \in T^1\mathcal{F}^c(p)$ such that $Df^{n_k}_*(x, v) \to (p_0, \tilde{v}^s(p_0))$. There exists a homoclinic point q of $\mathcal{F}^c(p)$ such that $\tilde{g}_q(p_0) \neq \pm\tilde{v}^s(p_0) \in T^1\mathcal{F}^c(p)$. We consider again

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the strong unstable holonomy $h^u_{\mathrm{loc}} : \mathcal{F}^{cs}_{\mathrm{loc}}(p_0) \to \mathcal{F}^{cs}_{\mathrm{loc}}(q_0)$ and let $q_k := h^u_{\mathrm{loc}}(f^{n_k}(x)) \in \mathcal{F}^{cs}_{\mathrm{loc}}(q_0)$, where $q_0 := h^u_{p,q}(p_0)$. We have that

$$Dh^s_{p,q*}(\tilde{v}^s(h^{s^{-1}}_{p,q}(q_0))) \neq \pm Dh^u_{p,q*}(\tilde{v}^s(p_0)),$$

and by continuity, for all k large enough, we have

$$Dh^s_{p,q_k*}(\tilde{v}^s(h^{s^{-1}}_{p,q_k}(q_k))) \neq \pm Dh^u_{f^{n_k}(x),q_k*}(\tilde{v}^s(f^{n_k}(x))).$$

Iterating by f^{-n_k} and denoting $f^{-n_k}(q_k) = x_k \to x$, we obtain

$$Dh^s_{p,x_k*}(\tilde{v}^s(h^{s^{-1}}_{p,x_k}(x_k))) \neq \pm Dh^u_{x,x_k*}(\tilde{v}^s(x)).$$

Since Dh^u preserves $TW^u(p)$, we obtain that $Dh^s_{p,x_k*}(\tilde{v}^s(h^s_{x_k,p}(x_k))) \notin T_{x_k}W^u(p)$, with $x_k \in W^u(p) \cap W^s(\mathcal{F}^c(p))$, and this implies again that arbitrarily close to x_k (and thus close to x), there are transverse homoclinic intersection between $W^s(p)$ and $W^u(p)$. This finishes the proof.

5. *Proof of Theorem C*

Remember that $\phi : \mathbb{T}^4 \to \mathbb{R}$ is a Hölder potential satisfying $\sup(\phi) - \inf(\phi) < \log \lambda_B$. For simplicity, we may assume

$$0 < \inf \phi \leq \sup \phi \leq \log \lambda_B. \tag{19}$$

First, by the variation principle, there is a sequence of ergodic measures μ_n of f such that

$$\limsup h_{\mu_n} = h_{\mathrm{top}}(f) \geq \log \lambda_A + \log \lambda_B,$$

the last inequality comes from equation (7).

Again, by the variation principle, the pressure of the function ϕ is

$$P_{\mathrm{top}}(\phi) \geq \limsup \left(h_{\mu_n} + \int \phi \, d\mu_n \right) \geq \limsup h_{\mu_n} \geq \log \lambda_A + \log \lambda_B,$$

where the last inequality comes from the assumption that $\phi > 0$ in equation (19).

Thus, for any ergodic measure μ with pressure sufficiently large, that is,

$$h_\mu + \int \phi \, d\mu > P_{\mathrm{top}}(\phi) + \left(\int \phi \, d\mu - \log \lambda_B \right), \tag{20}$$

we have

$$h_\mu > P_{\mathrm{top}}(\phi) - \log \lambda_B \geq \log \lambda_A.$$

As a consequence of Lemma 2.11, μ is a hyperbolic measure with stable index 2. By Lemma 2.7, μ is homoclinically related to the atomic measure supported on a hyperbolic periodic point O. Since $f \in \mathcal{U}$, by Theorem A, all the hyperbolic periodic orbits with stable index 2 are homoclinically related, and as a consequence of Remark 1.11, all the hyperbolic ergodic measures with stable index 2 are homoclinically related. In particular, all the ergodic measures satisfying equation (20) are homoclinically related.

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Thus, all equilibrium states for the Hölder potential ϕ are homoclinically related, if they do exist. By Proposition 2.9, there exists at most one equilibrium state. The proof is complete.

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