11

Many charges

There is little effort in extending the Abraham model to several particles. We label their positions and velocities as $\boldsymbol{q}_j(t)$, $\boldsymbol{v}_j(t)$, $j = 1, \ldots, N$. The j-th particle has bare mass $m_{\mathrm{b}j}$ and charge e_j, where for simplicity the form factor $\widehat{\varphi}$ is assumed to be the same for all particles. The motion of each particle is governed by the Lorentz force as before, and the current in the Maxwell equations now becomes the sum over the single-particle currents. Therefore the equations of motion read

$$\begin{aligned}
&c^{-1}\partial_t \boldsymbol{B}(\boldsymbol{x}, t) = -\nabla \times \boldsymbol{E}(\boldsymbol{x}, t)\,,\\
&c^{-1}\partial_t \boldsymbol{E}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{B}(\boldsymbol{x}, t) - \sum_{j=1}^{N} e_j \varphi(\boldsymbol{x} - \boldsymbol{q}_j(t)) c^{-1} \boldsymbol{v}_j(t)\,,\\
&\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) = \sum_{j=1}^{N} e_j \varphi(\boldsymbol{x} - \boldsymbol{q}_j(t))\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0\,,
\end{aligned} \tag{11.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_{\mathrm{b}i}\, \gamma_i \boldsymbol{v}_i(t)\big) = e_i \big(\boldsymbol{E}_\varphi(\boldsymbol{q}_i(t), t) + c^{-1}\boldsymbol{v}_i(t) \times \boldsymbol{B}_\varphi(\boldsymbol{q}_i(t), t)\big)\,, \tag{11.2}$$

where $i = 1, \ldots, N$ with $\gamma_i = (1 - (\boldsymbol{v}_i/c)^2)^{-1/2}$.

There are no external forces. The force acting on a given particle is due to the other particles, as mediated through the Maxwell field, and to the self-force, which we have discussed already at length. If two particles are at a distance of only a few times R_φ, then they interact strongly with forces which depend on the details of the phenomenological and unknown charge distribution. Thus physically we trust our model only if particles are far apart on the scale set by R_φ.

11.1 Retarded interaction

Let us take as a starting point the condition that initially particles are far apart, thus $|\boldsymbol{q}_i^0 - \boldsymbol{q}_j^0| = \mathcal{O}(\varepsilon^{-1} R_\varphi)$. The velocities are less than c, not necessarily

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small, and the initial fields are the linear superposition of N charge soliton fields corresponding to the initial conditions $\boldsymbol{q}_i^0, \boldsymbol{v}_i^0$, $i = 1, \ldots, N$. To understand the scales involved it is convenient to switch to macroscopic coordinates, which simply amounts to replacing in (11.1), (11.2) e_j by $\sqrt{\varepsilon} e_j$ and φ by φ_ε with $\varphi_\varepsilon(\boldsymbol{x}) = \varepsilon^{-3}\varphi(\varepsilon^{-1}\boldsymbol{x})$; compare with the second half of section 6.1. Then $|\boldsymbol{q}_i^0 - \boldsymbol{q}_j^0| = \mathcal{O}(1)$.

We insert the solution of the inhomogeneous Maxwell–Lorentz equations (11.1) into the Lorentz force of (11.2). The forces are additive and the force on particle i naturally splits into a self-force ($i = j$) and a mutual force ($i \neq j$). For the self-force one uses the Taylor expansion of chapter 7. Thereby the mass is renormalized and the next order is the radiation reaction. For the mutual force we recall that in section 7.2 it was shown already that, to leading order, the field generated by charge j is the Liénard–Wiechert field. Thus, one obtains as retarded equations of motion

$$m_i(\boldsymbol{v}_i)\dot{\boldsymbol{v}}_i = \sum_{\substack{j=1\\ j\neq i}}^{N} \varepsilon e_i \big(\boldsymbol{E}_{\mathrm{ret}\,j}(\boldsymbol{q}_i, t) + \boldsymbol{v}_i \times \boldsymbol{B}_{\mathrm{ret}\,j}(\boldsymbol{q}_i, t)\big)$$
$$+ \varepsilon (e_i^2/6\pi)\big[\gamma_i^4(\boldsymbol{v}_i \cdot \ddot{\boldsymbol{v}}_i)\boldsymbol{v}_i + 3\gamma_i^6(\boldsymbol{v}_i \cdot \dot{\boldsymbol{v}}_i)^2\boldsymbol{v}_i + 3\gamma_i^4(\boldsymbol{v}_i \cdot \dot{\boldsymbol{v}}_i)\dot{\boldsymbol{v}}_i + \gamma^2\ddot{\boldsymbol{v}}_i\big], \tag{11.3}$$

$t \geq 0$, which accounts for the effective mass m_i and the radiation reaction of the i-th particle; compare with Eq. (8.1). $\boldsymbol{E}_{\mathrm{ret}\,j}(\boldsymbol{x}, t)$ equals (2.24) with e replaced by e_j, $\boldsymbol{q}$ replaced by $\boldsymbol{q}_j$, and t_{ret} replaced by $t_{\mathrm{ret}\,j}$ which is implicitly defined through

$$t_{\mathrm{ret}\,j} = t - |\boldsymbol{x} - \boldsymbol{q}_j(t_{\mathrm{ret}\,j})|\,. \tag{11.4}$$

For $\boldsymbol{x} = \boldsymbol{q}_i$ the retarded time is of order 1. Similarly $\boldsymbol{B}_{\mathrm{ret}\,j}(\boldsymbol{x}, t)$ equals (2.25) with $\boldsymbol{q}$ replaced by $\boldsymbol{q}_j$ and t_{ret} replaced by $t_{\mathrm{ret}\,j}$. The strength ε results from the charge, $\sqrt{\varepsilon}e_i$, and the scale factor $\sqrt{\varepsilon}$ in (8.47). Viewed differently, on the microscopic scale the force is of order (distance)$^{-2} = \varepsilon^2$ and thus of order ε when accumulated over a time span ε^{-1}. To solve (11.3) one needs the trajectories for the whole past. Our assumption of no initial slip is equivalent to

$$\boldsymbol{q}_i(t) + \boldsymbol{q}_i^0 + t\boldsymbol{v}_i^0\,,\ i = 1, \ldots, N\,,\ t \leq 0\,, \tag{11.5}$$

which must be added to (11.3).

Using (11.3) one can estimate the size of the various contributions. The near fields of $\boldsymbol{E}_{\mathrm{ret}\,j}$ and $\boldsymbol{B}_{\mathrm{ret}\,j}$ are of order 1. Therefore the acceleration is of order ε,

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which implies that the far field of $\boldsymbol{E}_{\mathrm{ret}\,j}$ and $\boldsymbol{B}_{\mathrm{ret}\,j}$ is $\mathcal{O}(\varepsilon^2)$. The radiation reaction term involves $\ddot{\boldsymbol{v}}_i$ and is therefore $\mathcal{O}(\varepsilon^3)$.

We see that the various contributions are well ordered in powers of ε. The forces are weak, however, and therefore over longer times the particles will move apart, which is of somewhat reduced interest. There are two limiting situations of physical relevance, which will be discussed in the following sections. One possibility is to take the initial velocity $|\boldsymbol{v}_i/c| \ll 1$. Then to lowest order the particles interact through the static Coulomb potential and post-Coulombic corrections can be studied meaningfully. The other option is to let $N \to \infty$, which yields a kinetic description for charge densities as commonly used in plasma physics.

11.2 Limit of small velocities

We impose the condition that initially $|\boldsymbol{v}_j/c| \ll 1$. Then retardation effects should be negligible and the particles interact through the static Coulomb potential. According to the standard textbook recipe, $|\boldsymbol{v}_j/c| \ll 1$ is to be interpreted as $c \to \infty$. Indeed, from (11.1) one concludes $\boldsymbol{B} = 0$ and

$$\nabla \times \boldsymbol{E}(\boldsymbol{x}, t) = 0\,, \quad \nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) = \sum_{j=1}^{N} e_j \varphi(\boldsymbol{x} - \boldsymbol{q}_j(t))\,, \tag{11.6}$$

which together with Newton's equations of motion yields the desired result. Unfortunately, our argument fails on two counts. First, the interaction is obtained as the smeared Coulomb potential. More severely, in Newton's equations of motion only the bare mass of charge i appears, whereas physically it should respond to forces with its renormalized mass. Of course, the reason is that $c \to \infty$ does not ensure charges to be far apart on the scale of R_φ.

To improve we require, as in the previous section, that the initial positions satisfy

$$|\boldsymbol{q}_i^0 - \boldsymbol{q}_j^0| = \mathcal{O}(\varepsilon^{-1} R_\varphi)\,, \quad i \neq j\,. \tag{11.7}$$

Then the force is of order ε^2. Under rescaling the dynamical variables should be of order 1 as $\varepsilon \to 0$. If in addition we demand the relation $\dot{\boldsymbol{q}} = \boldsymbol{v}$ to be preserved, the only choice remaining is

$$|\boldsymbol{v}_j| = \mathcal{O}(\sqrt{\varepsilon} c) \quad \text{and} \quad t = \varepsilon^{-3/2} R_\varphi / c\,. \tag{11.8}$$

Indeed, the accumulated force is of order $\sqrt{\varepsilon}$, which means that the magnitude of the velocity is preserved. We have arrived at the following scale transformation

$$t = \varepsilon^{-3/2} t' , \quad \boldsymbol{q}_j = \varepsilon^{-1} \boldsymbol{q}'_j , \quad \boldsymbol{v}_j = \sqrt{\varepsilon} \boldsymbol{v}'_j ,$$
$$\boldsymbol{x} = \varepsilon^{-1} \boldsymbol{x}' , \quad \boldsymbol{E} = \varepsilon^{3/2} \boldsymbol{E}' , \quad \boldsymbol{B} = \varepsilon^{3/2} \boldsymbol{B}' , \tag{11.9}$$

where the primed quantities are considered to be of $\mathcal{O}(1)$. The field amplitudes are scaled by $\varepsilon^{3/2}$ so as to preserve the field energy. There is little risk in omitting the primes below. We set

$$\boldsymbol{q}_j^\varepsilon(t) = \varepsilon \boldsymbol{q}_j\big(\varepsilon^{-3/2} t\big) , \quad \boldsymbol{v}_j^\varepsilon(t) = \varepsilon^{-1/2} \boldsymbol{v}_j(\varepsilon^{-3/2} t) . \tag{11.10}$$

Then the rescaled Maxwell's and Newton's equations of motion are

$$\sqrt{\varepsilon}\, \partial_t \boldsymbol{B}(\boldsymbol{x}, t) = -\nabla \times \boldsymbol{E}(\boldsymbol{x}, t) ,$$
$$\sqrt{\varepsilon}\, \partial_t \boldsymbol{E}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{B}(\boldsymbol{x}, t) - \sum_{j=1}^{N} \sqrt{\varepsilon} e_j \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{q}_j^\varepsilon(t)) \sqrt{\varepsilon} \boldsymbol{v}_j^\varepsilon(t) ,$$
$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) = \sum_{j=1}^{N} \sqrt{\varepsilon} e_j \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{q}_j^\varepsilon(t)) , \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0 , \tag{11.11}$$

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \big(m_{\mathrm{b}i} (1 - \varepsilon \boldsymbol{v}_i^\varepsilon(t)^2)^{-1/2} \boldsymbol{v}_i^\varepsilon(t) \big) = \sqrt{\varepsilon} e_i \big(\boldsymbol{E}_{\varphi_\varepsilon}(\boldsymbol{q}_i^\varepsilon(t), t) + \sqrt{\varepsilon} \boldsymbol{v}_i^\varepsilon(t) \times \boldsymbol{B}_{\varphi_\varepsilon}(\boldsymbol{q}_i^\varepsilon(t), t) \big) . \tag{11.12}$$

On the new scale the velocity of light tends to infinity as $c/\sqrt{\varepsilon}$ and the charge distribution has total charge $\sqrt{\varepsilon} e_j$, finite electrostatic energy m_{f}, and shrinks to a δ-function as φ_ε. Recall that the scale parameter ε is just a convenient way to order the magnitudes of the various contributions.

Before entering into more specific computations, it is useful first to sort out what should be expected. We follow our practice from before and denote positions and velocities of the comparison dynamics by $\boldsymbol{r}_j, \boldsymbol{u}_j$, $j = 1, \dots, N$, i.e. $\boldsymbol{q}_j^\varepsilon(t) \cong \boldsymbol{r}_j(t)$, $\boldsymbol{v}_j^\varepsilon(t) \cong \boldsymbol{u}_j(t)$. Since the velocities are small, the kinetic energy takes its nonrelativistic limit

$$T_0(\boldsymbol{u}_j) = \frac{1}{2} \Big(m_{\mathrm{b}j} + \frac{4}{3} m_{\mathrm{f}j} \Big) \boldsymbol{u}_j^2 , \tag{11.13}$$

up to a constant; compare with (4.24). Note that the mass of the particle is renormalized through the interaction with the field. For small velocities, magnetic fields are small and retardation effects can be neglected. Thus the potential energy of the

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effective dynamics should be purely Coulombic and be given by

$$V_{\text{coul}}(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N) = \frac{1}{2} \sum_{i \neq j = 1}^{N} \frac{e_i e_j}{4\pi |\boldsymbol{r}_i - \boldsymbol{r}_j|} \, . \tag{11.14}$$

To obtain post-Coulombic corrections, one has to expand properly the self- and retarded forces, which we will carry to order $\varepsilon^{5/2}$ where the radiation reaction appears first. Since, as can be seen from (11.1), (11.2), the forces are additive, it suffices to consider two particles only. For initial conditions we choose the linear superposition of the two charge solitons corresponding to the initial data $\boldsymbol{q}_i^0$, $\boldsymbol{v}_i^0$, $i = 1, 2$. One solves the Maxwell equations and inserts them in the Lorentz force. As already explained, in the self-interaction the contribution from the initial fields vanishes for $t \geq \varepsilon \bar{t}_\varphi$. In the mutual interaction the initial fields take a time of order $\sqrt{\varepsilon}$ to reach the other particle and their contribution vanishes for $t \geq \sqrt{\varepsilon}|\boldsymbol{q}_1^0 - \boldsymbol{q}_2^0|$. Thus for larger times one is allowed to insert in (11.2) the retarded fields only, which yields

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\big(m_{\mathrm{b}1}\, \gamma_1 \boldsymbol{v}_1^\varepsilon(t)\big) = \boldsymbol{F}_{\text{ret},11}(t) + \boldsymbol{F}_{\text{ret},12}(t) \, , \tag{11.15}$$

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\big(m_{\mathrm{b}2}\, \gamma_2 \boldsymbol{v}_2^\varepsilon(t)\big) = \boldsymbol{F}_{\text{ret},21}(t) + \boldsymbol{F}_{\text{ret},22}(t) \, , \tag{11.16}$$

where

$$\begin{aligned} \boldsymbol{F}_{\text{ret},ij}(t) = e_i e_j \int_0^t \mathrm{d}s \int \mathrm{d}^3k |\widehat{\varphi}(\varepsilon \boldsymbol{k})|^2 \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}_i^\varepsilon(t) - \boldsymbol{q}_j^\varepsilon(s))} \\ \times \Big(- \varepsilon^{1/2}(|\boldsymbol{k}|^{-1} \sin(|\boldsymbol{k}|(t-s)/\sqrt{\varepsilon}))\mathrm{i}\boldsymbol{k} - \varepsilon(\cos(|\boldsymbol{k}|(t-s)/\sqrt{\varepsilon}))\boldsymbol{v}_j^\varepsilon(s) \\ + \varepsilon^{3/2}(|\boldsymbol{k}|^{-1} \sin(|\boldsymbol{k}|(t-s)/\sqrt{\varepsilon}))\boldsymbol{v}_i^\varepsilon(t) \times (\mathrm{i}\boldsymbol{k} \times \boldsymbol{v}_j^\varepsilon(s))\Big) \, , \end{aligned} \tag{11.17}$$

$i, j = 1, 2$.

For the self-interaction we set $\varepsilon \boldsymbol{k} = \boldsymbol{k}'$, $\varepsilon^{-3/2} t = t'$. Then

$$\begin{aligned} \boldsymbol{F}_{\text{ret},11}(t) = \varepsilon^{-3/2}(e_1)^2 \int_0^\infty \mathrm{d}\tau \int \mathrm{d}^3k |\widehat{\varphi}(\boldsymbol{k})|^2 \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}_1^\varepsilon(t) - \boldsymbol{q}_1^\varepsilon(t - \varepsilon^{3/2}\tau))/\varepsilon} \\ \times \Big(- \varepsilon^{1/2}(|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau)\mathrm{i}\boldsymbol{k} - \varepsilon(\cos |\boldsymbol{k}|\tau)\boldsymbol{v}_1^\varepsilon(t - \varepsilon^{3/2}\tau) \\ + \varepsilon^{3/2}(|\boldsymbol{k}|^{-1} \sin |\boldsymbol{k}|\tau)\boldsymbol{v}_1^\varepsilon(t) \times \big(\mathrm{i}\boldsymbol{k} \times \boldsymbol{v}_1^\varepsilon(t - \varepsilon^{3/2}\tau)\big)\Big) \, . \end{aligned} \tag{11.18}$$

One Taylor-expands as

$$\varepsilon^{-1}\big(\boldsymbol{q}_1^\varepsilon(t)-\boldsymbol{q}_1^\varepsilon(t-\varepsilon^{3/2}\tau)\big)=\varepsilon^{1/2}\tau\boldsymbol{v}-\frac{1}{2}\varepsilon^2\tau^2\dot{\boldsymbol{v}}+\frac{1}{6}\varepsilon^{7/2}\tau^3\ddot{\boldsymbol{v}}\,,$$
$$\boldsymbol{v}_1^\varepsilon(t-\varepsilon^{3/2}\tau)=\boldsymbol{v}-\varepsilon^{3/2}\tau\dot{\boldsymbol{v}}+\frac{1}{2}\varepsilon^3\tau^2\ddot{\boldsymbol{v}}\,. \tag{11.19}$$

Then, up to errors of order ε^3,

$$\begin{aligned}\boldsymbol{F}_{\mathrm{ret},11}(t)=(e_1)^2\int_0^\infty\mathrm{d}\tau\int\mathrm{d}^3k|\widehat{\varphi}(\boldsymbol{k})|^2\Big(\varepsilon\big[-(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\,\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})\boldsymbol{k}\\+(\cos|\boldsymbol{k}|\tau)\tau\dot{\boldsymbol{v}}\big]+\varepsilon^2\big[\big(-(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})\boldsymbol{k}\\+(\cos|\boldsymbol{k}|\tau)\tau\dot{\boldsymbol{v}}\big)\Big(-\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\boldsymbol{v})^2\Big)-(\cos|\boldsymbol{k}|\tau)\frac{1}{2}\tau^3(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{v}\\+(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\big(\tau^2(\boldsymbol{k}\cdot\boldsymbol{v})\boldsymbol{v}\times(\boldsymbol{k}\times\dot{\boldsymbol{v}})+\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}})(\boldsymbol{v}\times(\boldsymbol{k}\times\boldsymbol{v}))\big)\big]\\+\varepsilon^{5/2}\big[(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\frac{1}{6}\tau^3(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}})\boldsymbol{k}-(\cos|\boldsymbol{k}|\tau)\frac{1}{2}\tau^2\ddot{\boldsymbol{v}}\big]\Big)\,.\end{aligned} \tag{11.20}$$

For the mutual interaction we leave the k-integration and set $\varepsilon^{1/2}t=t'$. Then

$$\begin{aligned}\boldsymbol{F}_{\mathrm{ret},12}(t)=\sqrt{\varepsilon}e_1e_2\int_0^\infty\mathrm{d}\tau\int\mathrm{d}^3k|\widehat{\varphi}(\varepsilon\boldsymbol{k})|^2\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}_1^\varepsilon(t)-\boldsymbol{q}_2^\varepsilon(t-\sqrt{\varepsilon}\tau))}\\\times\Big(-\varepsilon^{1/2}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\mathrm{i}\boldsymbol{k}-\varepsilon(\cos|\boldsymbol{k}|\tau)\boldsymbol{v}_2^\varepsilon(t-\sqrt{\varepsilon}\tau)\\+\varepsilon^{3/2}(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\boldsymbol{v}_1^\varepsilon(t)\times(\mathrm{i}\boldsymbol{k}\times\boldsymbol{v}_2^\varepsilon(t-\sqrt{\varepsilon}\tau))\Big)\,.\end{aligned} \tag{11.21}$$

One Taylor-expands as

$$\boldsymbol{q}_1^\varepsilon(t)-\boldsymbol{q}_2^\varepsilon(t-\sqrt{\varepsilon}\tau)=\boldsymbol{r}+\varepsilon^{1/2}\tau\boldsymbol{v}_2-\frac{1}{2}\varepsilon\tau^2\dot{\boldsymbol{v}}_2+\frac{1}{6}\varepsilon^{3/2}\tau^3\ddot{\boldsymbol{v}}_2\,,$$
$$\boldsymbol{v}_1^\varepsilon(t)=\boldsymbol{v}_1\,,\quad\boldsymbol{v}_2^\varepsilon(t-\sqrt{\varepsilon}\tau)=\boldsymbol{v}_2-\varepsilon^{1/2}\tau\dot{\boldsymbol{v}}_2+\frac{1}{2}\varepsilon\tau^2\ddot{\boldsymbol{v}}_2 \tag{11.22}$$

with $\boldsymbol{r} = \boldsymbol{q}_1^\varepsilon(t) - \boldsymbol{q}_2^\varepsilon(t)$. Then, up to errors of order ε^3,

$$\begin{aligned}
\boldsymbol{F}_{\mathrm{ret},12} = e_1 e_2 \int_0^\infty \mathrm{d}\tau \int \mathrm{d}^3k |\widehat{\varphi}(\varepsilon \boldsymbol{k})|^2 \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \Big(&- \varepsilon(|\boldsymbol{k}|^{-1} \sin|\boldsymbol{k}|\tau)\, \mathrm{i}\boldsymbol{k} + \varepsilon^2 \big[(|\boldsymbol{k}|^{-1} \sin|\boldsymbol{k}|\tau) \\
&\times \big(-\frac{1}{2}\tau^2(\boldsymbol{k}\cdot\dot{\boldsymbol{v}}_2)\boldsymbol{k} + \frac{1}{2}\tau^2(\boldsymbol{k}\cdot\boldsymbol{v}_2)^2 \mathrm{i}\boldsymbol{k} + \boldsymbol{v}_1 \times (\mathrm{i}\boldsymbol{k}\times\boldsymbol{v}_2) \big) \\
&+ (\cos|\boldsymbol{k}|\tau)(\tau\dot{\boldsymbol{v}}_2 - \mathrm{i}\tau(\boldsymbol{k}\cdot\boldsymbol{v}_2)\boldsymbol{v}_2) \big] \\
&+ \varepsilon^{5/2} \big[(|\boldsymbol{k}|^{-1}\sin|\boldsymbol{k}|\tau)\big(\frac{1}{6}\tau^3(\boldsymbol{k}\cdot\ddot{\boldsymbol{v}}_2)\boldsymbol{k} - \frac{1}{2}\tau^3(\boldsymbol{k}\cdot\boldsymbol{v}_2)(\boldsymbol{k}\cdot\dot{\boldsymbol{v}}_2)\mathrm{i}\boldsymbol{k} \\
&- \frac{1}{6}\tau^3(\boldsymbol{k}\cdot\boldsymbol{v}_2)^3\boldsymbol{k}\big) - (\cos|\boldsymbol{k}|\tau)\frac{1}{2}\tau^2\ddot{\boldsymbol{v}} \big] \Big) \\
= (e_1 e_2/4\pi) \Big(- \varepsilon \nabla_{\boldsymbol{r}} |\boldsymbol{r}|^{-1} &+ \varepsilon^2 \big[\big(\frac{1}{2}\nabla_{\boldsymbol{r}}(\dot{\boldsymbol{v}}_2\cdot\nabla_{\boldsymbol{r}}) - \frac{1}{2}\nabla_{\boldsymbol{r}}(\boldsymbol{v}_2\cdot\nabla_{\boldsymbol{r}})^2 \big)|\boldsymbol{r}| \\
&- (\dot{\boldsymbol{v}}_2 - \boldsymbol{v}_2(\boldsymbol{v}_2\cdot\nabla_{\boldsymbol{r}}))|\boldsymbol{r}|^{-1} + (\boldsymbol{v}_1 \times (\nabla_{\boldsymbol{r}}\times\boldsymbol{v}_2))|\boldsymbol{r}|^{-1} \big] \\
&+ \frac{2}{3}\frac{1}{4\pi}\varepsilon^{5/2} e_1 e_2 \ddot{\boldsymbol{v}}_2 \Big) .
\end{aligned} \tag{11.23}$$

We discuss each order separately, where we recall that in (11.15), (11.16) the acceleration is multiplied by ε. As anticipated, to order 1 one obtains the Coulomb dynamics with renormalized mass from $\boldsymbol{F}_{jj}(t)$. Let us define the Coulomb Lagrangian

$$L_{\mathrm{coul}} = \sum_{j=1}^N \frac{1}{2}\Big(m_{\mathrm{b}j} + \frac{4}{3}m_{\mathrm{f}j}\Big)\boldsymbol{u}_j^2 - \frac{1}{2}\sum_{i\neq j=1}^N \frac{e_i e_j}{4\pi|\boldsymbol{r}_i - \boldsymbol{r}_j|} . \tag{11.24}$$

Then the comparison dynamics is

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\nabla_{\boldsymbol{u}_j} L_{\mathrm{coul}}\big) - \nabla_{\boldsymbol{r}_j} L_{\mathrm{coul}} = 0 , \quad j = 1, \dots, N , \tag{11.25}$$

with the error bounds

$$|\boldsymbol{q}_j^\varepsilon(t) - \boldsymbol{r}_j(t)| = \mathcal{O}(\varepsilon) , \quad |\boldsymbol{v}_j^\varepsilon(t) - \boldsymbol{u}_j(t)| = \mathcal{O}(\varepsilon) . \tag{11.26}$$

The first-order correction is $\mathcal{O}(\varepsilon)$. More conventionally the error is counted in powers of $|\boldsymbol{v}/c|$ relative to the zeroth-order Coulomb dynamics. To convert, one only has to set $\varepsilon = 1$. The first correction is then of order $|\boldsymbol{v}/c|^2$ ($= \mathcal{O}(\varepsilon)$, compare with (11.8)), and the next-order corrections $|\boldsymbol{v}/c|^3$. The order ε^2 terms in (11.20), (11.23) combine in a simple fashion and yield the Darwin correction. Let us define

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the Darwin Lagrangian

$$L_{\rm darw} = \sum_{j=1}^{N}\Big(\Big(m_{\mathrm{b}j}+\frac{4}{3}m_{\mathrm{f}j}\Big)\frac{1}{2}\boldsymbol{u}_j^2+\varepsilon\Big(\frac{1}{8}m_{\mathrm{b}j}+\frac{2}{15}m_{\mathrm{f}j}\Big)c^{-2}\boldsymbol{u}_j^4\Big)$$
$$-\frac{1}{2}\sum_{i\neq j=1}^{N}\frac{e_ie_j}{4\pi|\boldsymbol{r}_i-\boldsymbol{r}_j|}\Big[1-\varepsilon\frac{1}{2c^2}\big(\boldsymbol{u}_i\cdot\boldsymbol{u}_j+(\boldsymbol{u}_i\cdot\widehat{\boldsymbol{r}}_{ij})(\boldsymbol{u}_j\cdot\widehat{\boldsymbol{r}}_{ij})\big)\Big] \tag{11.27}$$

with $\widehat{\boldsymbol{r}}_{ij}=(\boldsymbol{r}_i-\boldsymbol{r}_j)/|\boldsymbol{r}_i-\boldsymbol{r}_j|$. In the first sum one recognizes the correction to the kinetic energy, while in the second-term corrections due to retardation and the magnetic field combine into a velocity-dependent potential. The comparison dynamics is governed by the improved Lagrangian,

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\nabla_{\boldsymbol{u}_j}L_{\rm darw}\big)-\nabla_{\boldsymbol{r}_j}L_{\rm darw}=0\,,\quad j=1,\dots,N\,, \tag{11.28}$$

with the error bounds

$$|\boldsymbol{q}_j^{\varepsilon}(t)-\boldsymbol{r}_j(t)|=\mathcal{O}(\varepsilon^{3/2})\,,\quad |\boldsymbol{v}_j^{\varepsilon}(t)-\boldsymbol{u}_j(t)|=\mathcal{O}(\varepsilon^{3/2})\,. \tag{11.29}$$

At order $|\boldsymbol{v}/c|^3$ one picks up terms proportional to $\ddot{\boldsymbol{v}}_j$. Remarkably, the prefactors in $\boldsymbol{F}_{\mathrm{ret}jj}$ and $\boldsymbol{F}_{\mathrm{ret}ji}$ are identical, and one obtains the comparison dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\nabla_{\boldsymbol{u}_j}L_{\rm darw}\big)-\nabla_{\boldsymbol{r}_j}L_{\rm darw}=\varepsilon^{3/2}\frac{e_j}{6\pi c^3}\sum_{i=1}^{N}e_i\ddot{\boldsymbol{u}}_i,\quad j=1,\dots,N\,. \tag{11.30}$$

The physical solutions have to be on the center manifold of (11.30). At the present level of precision it suffices to substitute the Lagrangian dynamics to lowest order, which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\nabla_{\boldsymbol{u}_j}L_{\rm darw}\big)-\nabla_{\boldsymbol{r}_j}L_{\rm darw}=\varepsilon^{3/2}\frac{e_j}{6\pi c^3}\frac{1}{2}\sum_{\substack{i,i'=1\\ i\neq i'}}^{N}\Big(\frac{e_i}{m_i}-\frac{e_{i'}}{m_{i'}}\Big)\frac{e_ie_{i'}}{4\pi|\boldsymbol{r}_i-\boldsymbol{r}_{i'}|^3}$$
$$\times\big((\boldsymbol{u}_i-\boldsymbol{u}_{i'})-3(\widehat{\boldsymbol{r}}_{ii'}\cdot(\boldsymbol{u}_i-\boldsymbol{u}_{i'}))\widehat{\boldsymbol{r}}_{ii'}\big)\,. \tag{11.31}$$

If the charge–mass ratio e_j/m_j does not depend on j, the damping term is suppressed. The collection of charges has vanishing dipole moment. This can be seen also directly by considering the dipole moment $\boldsymbol{d}=\sum_{j=1}^{N}e_j\boldsymbol{q}_j=\sum_{j=1}^{N}(e_j/m_j)m_j\boldsymbol{q}_j$. If $(e_j/m_j)=$ const., then $\boldsymbol{d}$ equals the center of mass and $\ddot{\boldsymbol{d}}=0$. Thus there is no dipole radiation. Only quadrupole radiation is allowed and radiation damping would appear at the scale $|\boldsymbol{v}/c|^5$.

We briefly return to the limit $c\to\infty$ from the beginning of this subsection. In fact, the expansion for computing the effective dynamics turns out to be not so drastically different as one might have anticipated. To lowest order the kinetic energy is $m_{\mathrm{b}j}\boldsymbol{u}_j^2/2$ and is modified to $(m_{\mathrm{b}j} + (4m_{\mathrm{f}j}/3c^2))\boldsymbol{u}_j^2/2$ at the Darwin order $|\boldsymbol{v}/c|^2$. The correction to the quadratic behavior is visible only at order $|\boldsymbol{v}/c|^4$. The friction term is identical to that of (11.30). Only in (11.27) must the Coulomb potential be smeared by the charge distribution φ.

11.3 The Vlasov–Maxwell equations

If N is large, it is impractical to follow the trajectory of individual particles and, as widely used for example in plasma physics, a kinetic description is more appropriate. The basic object describing matter is now the distribution function $f_\alpha(\boldsymbol{x}, \boldsymbol{v}, t)$. For each component α it is a function on the one-particle phase space and defined through

$$f_\alpha(\boldsymbol{x}, \boldsymbol{v}, t)\mathrm{d}^3x\mathrm{d}^3v = \tfrac{1}{N}\big(\text{number of particles with charge } e_\alpha \text{ in the volume element } \mathrm{d}^3x\mathrm{d}^3v \text{ at time } t\big)\,.$$

The charge density of the α-th component is then

$$\rho_\alpha(\boldsymbol{x}, t) = e_\alpha \int \mathrm{d}^3v f_\alpha(\boldsymbol{x}, \boldsymbol{v}, t) \tag{11.32}$$

and the total charge density

$$\rho(\boldsymbol{x}, t) = \sum_\alpha \rho_\alpha(\boldsymbol{x}, t)\,. \tag{11.33}$$

Similarly, the current density is

$$\boldsymbol{j}_\alpha(\boldsymbol{x}, t) = e_\alpha \int \mathrm{d}^3v \boldsymbol{v} f_\alpha(\boldsymbol{x}, \boldsymbol{v}, t), \quad \boldsymbol{j}(\boldsymbol{x}, t) = \sum_\alpha \boldsymbol{j}_\alpha(\boldsymbol{x}, t)\,. \tag{11.34}$$

The Maxwell field is governed by (2.2), (2.3) with ρ from (11.33) and $\boldsymbol{j}$ from (11.34) as source terms. As densities on the one-particle phase space the distribution functions evolve according to

$$\partial_t f_\alpha(\boldsymbol{x}, \boldsymbol{v}, t) + \nabla_{\boldsymbol{x}} \cdot \big(\boldsymbol{v} f_\alpha(\boldsymbol{x}, \boldsymbol{v}, t)\big) + \big(\nabla_{\boldsymbol{v}} \cdot (m_\alpha \gamma)^{-1} \times (\boldsymbol{F}_\alpha - (\boldsymbol{v} \cdot \boldsymbol{F}_\alpha)\boldsymbol{v}) f_\alpha(\boldsymbol{x}, \boldsymbol{v}, t)\big) = 0 \tag{11.35}$$

with the Lorentz force

$$\boldsymbol{F}_\alpha = e_\alpha\big(\boldsymbol{E}(\boldsymbol{x}, t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{x}, t)\big)\,. \tag{11.36}$$

The system of equations (2.2), (2.3), and (11.32)–(11.36) are called the Vlasov–Maxwell system. They were written down first by Vlasov in 1938 in the more

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conventional form where the velocity $\boldsymbol{v}$ is replaced by the kinetic momentum $\boldsymbol{u} = m_\alpha \boldsymbol{v}/\sqrt{1-\boldsymbol{v}^2}$. Then in (11.35), (11.36) $\boldsymbol{v}$ is to be replaced by $\boldsymbol{u}/\sqrt{m_\alpha^2 + \boldsymbol{u}^2}$ and the Vlasov equation for the distribution function $f_\alpha(\boldsymbol{x}, \boldsymbol{u}, t)\mathrm{d}^3x\mathrm{d}^3u$ reads

$$\partial_t f_\alpha(\boldsymbol{x}, \boldsymbol{u}, t) + (m_\alpha^2 + \boldsymbol{u}^2)^{-1/2}\boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} f_\alpha(\boldsymbol{x}, \boldsymbol{u}, t) + \boldsymbol{F}_\alpha \cdot \nabla_{\boldsymbol{u}} f_\alpha(\boldsymbol{x}, \boldsymbol{u}, t) = 0\,. \tag{11.37}$$

The static limit of the Vlasov–Maxwell system, namely $c \to \infty$ yielding $\boldsymbol{B} = 0$, $\nabla \times \boldsymbol{E} = 0$, $\nabla \cdot \boldsymbol{E} = \rho$, is the Vlasov equation.

To establish the link to the Abraham model with N charges it is convenient to start on the macroscopic scale, for simplicity for a single component, where

$$\begin{aligned}
&\partial_t \boldsymbol{B}(\boldsymbol{x}, t) = -\nabla \times \boldsymbol{E}(\boldsymbol{x}, t)\,,\\
&\partial_t \boldsymbol{E}(\boldsymbol{x}, t) = \nabla \times \boldsymbol{B}(\boldsymbol{x}, t) - \varepsilon \sum_{j=1}^{N} e\varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{q}_j(t))\boldsymbol{v}_j(t)\,,\\
&\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t) = \varepsilon \sum_{j=1}^{N} e\varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{q}_j(t))\,, \quad \nabla \cdot \boldsymbol{B}(\boldsymbol{x}, t) = 0\,,
\end{aligned} \tag{11.38}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(m_\mathrm{b}\gamma_i \boldsymbol{v}_i(t)\big) = e\big(\boldsymbol{E}_{\varphi_\varepsilon}(\boldsymbol{q}_i(t), t) + \boldsymbol{v}_i(t) \times \boldsymbol{B}_{\varphi_\varepsilon}(\boldsymbol{q}_i(t), t)\big)\,. \tag{11.39}$$

We used here the freedom in the scale factor for the amplitude of the electromagnetic fields which accounts for an extra $\sqrt{\varepsilon}$ as compared to (6.11). On a formal level, the step to the Vlasov–Maxwell equation is immediate. We set $N = \varepsilon^{-1}$. The typical distance between particles is then $\varepsilon^{1/3}R_\varphi$ while the charge diameter is $\varepsilon R_\varphi \ll \varepsilon^{1/3}R_\varphi$. Thus particles are still very far apart. If we assume that at time t the particle configuration is well approximated by a distribution function, then the source term of the Maxwell equations is of the form claimed in (11.33), (11.34). For (11.39) we have again to split into the self- and mutual parts. The self-part renormalizes the mass to $m(\boldsymbol{v})$ from (8.2) and the mutual part yields the force of (11.36) for the considered component. Put differently, in (11.39) the Maxwell fields $\boldsymbol{E}$, $\boldsymbol{B}$, smeared by φ_ε and evaluated at $\boldsymbol{q}_i(t)$, have a singular part which renormalizes the mass and a smooth part from all the other charges which is governed by (11.38). To carry out this program and to thereby derive the Vlasov–Maxwell equations along the lines indicated remains as a task for the future.

11.4 Statistical mechanics

For a system of many particles the first impetus is to investigate its equilibrium statistical mechanics. Although this means venturing into the domain of nonzero temperatures, let us see how much will be captured by our oversimplified model of matter. Statistical mechanics starts with a Hamiltonian defined on phase space.

Since this is also the starting point for canonical quantization, in Chapter 13, our discussion of the Pauli–Fierz model necessarily deals with the Lagrangian and Hamiltonian structure of the Abraham model. We preview the result (13.24) for a system of N particles. The canonical coordinates for the particles are $(\boldsymbol{q}_j, \boldsymbol{p}_j)$, $j = 1, \dots, N$. For the Maxwell field we adopt the Coulomb gauge, $\nabla \cdot \boldsymbol{A} = 0$. The canonical field variables are then $(\boldsymbol{A}(\boldsymbol{x}), -\boldsymbol{E}_\perp(\boldsymbol{x}))$, $\boldsymbol{x} \in \mathbb{R}^3$. Both fields are purely transverse, $\nabla \cdot \boldsymbol{A} = 0 = \nabla \cdot \boldsymbol{E}_\perp$. In terms of these variables the Hamiltonian for the Abraham model reads

$$H = \sum_{j=1}^{N} \frac{1}{2m_{\mathrm{b}j}} \big(\boldsymbol{p}_j - e_j \boldsymbol{A}_\varphi(\boldsymbol{q}_j)\big)^2 + \frac{1}{2} \int \mathrm{d}^3x \big(\boldsymbol{E}_\perp(\boldsymbol{x})^2 + (\nabla \times \boldsymbol{A}(\boldsymbol{x}))^2\big)$$
$$+ \frac{1}{2} \sum_{i,j=1}^{N} e_i e_j V_{\varphi\mathrm{coul}}(\boldsymbol{q}_i - \boldsymbol{q}_j)\,. \tag{11.40}$$

For simplicity we adopt the nonrelativistic kinetic energy, $\boldsymbol{p}^2/2m$. The potential $V_{\varphi\mathrm{coul}}$ originates from the longitudinal part of $\boldsymbol{E}$ and is defined through

$$V_{\varphi\mathrm{coul}}(\boldsymbol{q}) = \int \mathrm{d}^3k |\widehat{\varphi}(\boldsymbol{k})|^2 |\boldsymbol{k}|^{-2} \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{q}}\,. \tag{11.41}$$

$V_{\varphi\mathrm{coul}}$ is the Coulomb potential smeared by the charge distribution φ, which appears twice, since both the i-th and the j-th particles carry a charge distribution.

The particles are confined to the box $\Lambda \subset \mathbb{R}^3$. We should also restrict the fields to the box Λ, but it will be somewhat simpler to regard them as filling all space. Then, formally, the equilibrium distribution at inverse temperature $\beta = 1/k_{\mathrm{B}}T$ is given by

$$\frac{1}{Z} \mathrm{e}^{-\beta H} \prod_{j=1}^{N} \chi_\Lambda(\boldsymbol{q}_j) \mathrm{d}^3 q_j \mathrm{d}^3 p_j \prod_{\boldsymbol{x} \in \mathbb{R}^3} \mathrm{d}^2 A(\boldsymbol{x}) \mathrm{d}^2 E_\perp(\boldsymbol{x})\,, \tag{11.42}$$

where Z is the normalizing partition function and χ_Λ is the indicator function for the box Λ. Since the field energy is quadratic in $\boldsymbol{E}_\perp$ and $\boldsymbol{A}$, combined with the a priori measure and the normalization, it follows that $\boldsymbol{E}_\perp(\boldsymbol{x})$ and $\boldsymbol{A}(\boldsymbol{x})$ are Gaussian fields. We will only need $\boldsymbol{A}(\boldsymbol{x})$. It has mean zero and covariance

$$\langle \boldsymbol{A}(\boldsymbol{x}) \boldsymbol{A}(\boldsymbol{x}') \rangle_0 = \frac{1}{\beta} \int \mathrm{d}^3k |\boldsymbol{k}|^{-2} (\mathbb{1} - \widehat{\boldsymbol{k}} \otimes \widehat{\boldsymbol{k}}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\,. \tag{11.43}$$

From the experience with black-body radiation we have little trust in the statistics of the Maxwell field at large wave numbers and therefore concentrate on the particle degrees of freedom, only.

According to (11.40), (11.42) for fixed positions $\boldsymbol{q}_j$, $j = 1, \dots, N$, the momenta are Gaussian distributed with mean zero and covariance

$$\begin{aligned}\langle \boldsymbol{p}_i \boldsymbol{p}_j \rangle_{(\boldsymbol{q}_1,\dots,\boldsymbol{q}_N)} &= \langle (\boldsymbol{p}_i + e_i \boldsymbol{A}_\varphi(\boldsymbol{q}_i))(\boldsymbol{p}_j + e_j \boldsymbol{A}_\varphi(\boldsymbol{q}_j)) \rangle \\ &= \langle \boldsymbol{p}_i \boldsymbol{p}_j \rangle + e_i e_j \langle \boldsymbol{A}_\varphi(\boldsymbol{q}_i) \boldsymbol{A}_\varphi(\boldsymbol{q}_j) \rangle \\ &= \frac{1}{\beta}\Big(m_{\mathrm{b}i} \delta_{ij} \mathbb{1} + e_i e_j \int \mathrm{d}^3 k |\widehat{\varphi}(\boldsymbol{k})|^2 |\boldsymbol{k}|^{-2} (\mathbb{1} - \widehat{\boldsymbol{k}} \otimes \widehat{\boldsymbol{k}}) \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{q}_i - \boldsymbol{q}_j)} \Big) .\end{aligned} \tag{11.44}$$

Here in the first equality we shifted $\boldsymbol{p}_j$ by $e_j \boldsymbol{A}_\varphi(\boldsymbol{q}_j)$ which transforms $\langle \cdot \rangle$ to a Gaussian averaging factorized with respect to the $\boldsymbol{p}$'s and $\boldsymbol{A}$'s. For $i = j$ we recover the renormalized mass $m_{\mathrm{b}i} + m_{\mathrm{f}i}$. For $i \neq j$, there are momentum correlations which decay as $|\boldsymbol{q}_i - \boldsymbol{q}_j|^{-1}$ in the distance of the two particles.

For the distribution of the positions, we integrate first over $\boldsymbol{p}$ and then over $\boldsymbol{A}$ with the result

$$\frac{1}{Z} \mathrm{e}^{-\beta V} \prod_{j=1}^{N} \chi_\Lambda(\boldsymbol{q}_j) \mathrm{d}^3 q_j \, , \quad V = \frac{1}{2} \sum_{i,j=1}^{N} e_i e_j V_{\varphi\mathrm{coul}}(\boldsymbol{q}_i - \boldsymbol{q}_j) \, , \tag{11.45}$$

which is the standard Gibbs distribution for a Coulombic system of charges. The equilibrium statistics decouples into a positional part and, when conditioned on the positions, a Gaussian velocity part.

The equilibrium properties of Coulomb systems have been studied very extensively. To be specific, let us consider a two-component charge-symmetric plasma, which is neutral in the sense that both components have the same chemical potential. Since the system is very large, the natural quantities are the free energy and the correlation functions in the limit where the volume tends to infinity, $\Lambda \uparrow \mathbb{R}^3$. Indeed this limit has been established together with one major qualitative result, namely the validity of the Debye–Hückel theory at sufficiently low density. One inserts an extra charge at the origin into the system at thermal equilibrium. Then the charges of opposite sign screen in a statistical sense and the average charge density decays on the scale of the Debye length $l_{\mathrm{D}} = (4\pi e^2 \beta \rho)^{-1/2}$.

While we cannot enter into details, it might be useful to understand how the smearing of the charge distribution is needed even on the level of equilibrium statistical mechanics. Let us assume that the two components have equal charge of opposite sign, which means either $e_j = e$ or $e_j = -e$. Since $V_{\varphi\mathrm{coul}}$ is of positive type (the Fourier transform of a positive measure), one has

$$\begin{aligned}\frac{1}{2} \sum_{i \neq j=1}^{N} e_i e_j V_{\varphi\mathrm{coul}}(\boldsymbol{q}_i - \boldsymbol{q}_j) &\geq -\frac{1}{2} \sum_{j=1}^{N} e_j^2 V_{\varphi\mathrm{coul}}(0) \\ &= -\Big(\frac{1}{2} e^2 \int \mathrm{d}^3 k |\widehat{\varphi}(\boldsymbol{k})|^2 |\boldsymbol{k}|^{-2} \Big) N \, .\end{aligned} \tag{11.46}$$

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The energy is bounded from below by a constant proportional to N, which means that $V_{\varphi\text{coul}}$ defines a thermodynamically stable interaction. To control the behavior for large Λ one uses again the positive definiteness of $V_{\varphi\text{coul}}$ and introduces the auxiliary Gaussian field $\phi(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$, with mean zero and covariance

$$\langle \phi(\boldsymbol{x})\phi(\boldsymbol{y})\rangle_G = V_{\varphi\text{coul}}(\boldsymbol{x}-\boldsymbol{y})\,, \tag{11.47}$$

which is well defined since $\widehat{V}_{\varphi\text{coul}}(\boldsymbol{k}) \geq 0$. Then

$$\mathrm{e}^{-\beta V(\boldsymbol{q}_1,\dots,\boldsymbol{q}_N)} = \Big\langle \exp\Big[\mathrm{i}\sqrt{\beta}\sum_{j=1}^{N} e_j \phi(\boldsymbol{q}_j)\Big]\Big\rangle_G \tag{11.48}$$

and the grand canonical partition function becomes

$$\begin{aligned} Z_\Lambda &= \sum_{N=0}^{\infty} \frac{z^N}{N!} \sum_{\sigma_1,\dots,\sigma_N=\pm 1} \int_\Lambda \mathrm{d}^3 q_1 \dots \int_\Lambda \mathrm{d}^3 q_N \\ &\quad \times \exp\Big[-\beta e^2 \frac{1}{2}\sum_{i,j=1}^{N} \sigma_i\sigma_j V_{\varphi\text{coul}}(\boldsymbol{q}_i - \boldsymbol{q}_j)\Big] \\ &= \sum_{N=0}^{\infty} \frac{z^N}{N!} \Big\langle\Big(\int_\Lambda \mathrm{d}^3 q \sum_{\sigma=\pm 1} \exp\big[\mathrm{i}\sqrt{\beta} e\sigma\phi(\boldsymbol{q})\big]\Big)^N\Big\rangle_G \\ &= \Big\langle \exp\Big[2z\int_\Lambda \mathrm{d}^3 q \cos(\sqrt{\beta} e\phi(\boldsymbol{q}))\Big]\Big\rangle_G\,. \end{aligned} \tag{11.49}$$

Thus our system of charges has been converted into a field theory. The a priori measure $\langle\cdot\rangle_G$ is known as the Gaussian massless free field. In (11.49) it is perturbed by the interaction $\int_\Lambda \mathrm{d}^3 q \cos(\sqrt{\beta} e\phi(\boldsymbol{q}))$, which is clearly proportional to $|\Lambda|$. Thus we conclude that the pressure is extensive,

$$\log Z_\Lambda \cong |\Lambda|\,. \tag{11.50}$$

Despite the long-range forces, a neutral Coulomb system has extensive (volume-proportional) thermodynamics, provided the charges are somewhat smeared.

Notes and references

Section 11.1

On the quantized level the retarded interaction between neutral atoms shows as an attractive R^{-7} decay of the interaction potential in contrast to the nonretarded, attractive van der Waals R^{-6} law.

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Section 11.2

The Darwin Lagrangian is discussed in Jackson (1999). In Kunze and Spohn (2000c) the errors in (11.29) are estimated. Kunze and Spohn (2001) extend their analysis to include radiation reaction. The major novel difficulty is to properly match the initial conditions of the comparison dynamics (11.31). The next post-Coulombic correction, of order $|v/c|^4$, is computed formally by Landau and Lifshitz (1959), Barker and O'Connell (1980a, 1980b), and Damour and Schäfer (1991). It contains quadrupole corrections to the Coulomb interaction and terms proportional to $\ddot{v}$. It would be of interest to compare these results with the systematic expansion presented here.

A qualitatively rather similar problem arises in general relativity. The object of interest is a binary pulsar, like the famous Hulse–Taylor pulsar PSR 1913 + 16. It consists of two neutron stars, each with a mass of roughly 1.4 solar mass and a diameter of 10 km. They rotate around their common center of mass with a period of 7 h 45 min. The neutron stars move slowly with $|v/c| \cong 10^{-3}$. Since one of the neutron stars is rotating, it emits radio waves through which the orbit can be tracked with very high precision, in fact so precise that damping through the emission of gravitational waves can be verified quantitatively. I refer to Hulse (1994) and Taylor (1994). As in the case of charges, the theoretical challenge is to obtain the orbits of the two neutron stars in an expansion in $|v/c|$. For gravitation there is no dipole radiation and damping appears only at order $|v/c|^5$, with $|v/c|^0$ being the Newtonian orbit. Since experimental accuracy is expected to increase further (Will 1999) various groups have taken up the challenge with the present order at $|v/c|^7$ (Jaranowski and Schäfer 1998).

Section 11.3

The relativistic Vlasov–Maxwell equations already appear in the original 1938 paper of Vlasov, see Vlasov (1961). The existence of solutions is studied at increasing level of generality in Glassey and Schaeffer (1991, 1997, 2000). In the nonretarded Vlasov–Poisson approximation the existence of solutions is now well understood (Pfaffelmoser 1992; Schaeffer 1991) and the link to the N-particle system has been established for a mollified potential (Neunzert 1975; Braun and Hepp 1977), a review being Spohn (1991). Physically the natural requirement is to have the charge diameter much smaller than the interparticle distance. Since this case is somewhat singular, a satisfactory derivation of the Vlasov–Poisson approximation is open, with a partial step towards its solution in Batt (2001).

As in the case of N charges, the solution to the Vlasov–Maxwell system can be expanded in powers of $1/c$. The leading order is then Vlasov–Poisson, as established by Schaeffer (1986), which is corrected à la Darwin at order c^{-2}, as proved by Bauer and Kunze (2003). A one-component system can dissipate energy only through quadrupole radiation, which first appears at order c^{-5}. A two-component system emits dipole radiation at order c^{-3}. Properties of the formally derived Vlasov equation including radiative friction are studied by Kunze and Rendall (2001).

Section 11.4

The statistical mechanics of charges plus Maxwell field is usually treated only on the level of thermodynamics (Alastuey and Appel 2000). Lebowitz and Lieb (1969) and Lieb and Lebowitz (1972) prove the existence of the, in fact shape-dependent, thermodynamic limit for Coulomb systems. A very readable review is Lieb and Lebowitz (1973). The existence of the infinite-volume limit of the correlation functions in the case of charge-symmetric systems is proved by Fröhlich and Park (1978). For the Debye–Hückel theory I recommend the excellent survey by Brydges and Martin (1999).