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Periodic points of rational area-preserving homeomorphisms

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Abstract. An area-preserving homeomorphism isotopic to the identity is said to have rational rotation direction if its rotation vector is a real multiple of a rational class. We give a short proof that any area-preserving homeomorphism of a compact surface of genus at least two, which is isotopic to the identity and has rational rotation direction, is either the identity or has periodic points of unbounded minimal period. This answers a question of Ginzburg and Seyfaddini and can be regarded as a Conley conjecture-type result for symplectic homeomorphisms of surfaces beyond the Hamiltonian case. We also discuss several variations, such as maps preserving arbitrary Borel probability measures with full support, maps that are not isotopic to the identity and maps on lower genus surfaces. The proofs of the main results combine topological arguments with periodic Floer homology.

Key words: symplectic dynamics, low-dimensional dynamics, Hamiltonian dynamics, smooth dynamics
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1. *Introduction*

1.1. *History and main results.* Questions on the existence and multiplicity of periodic points of area-preserving surface homeomorphisms have a long history, dating back to Poincaré and Birkhoff's work on annulus twist maps and the restricted planar three-body problem, and have attracted significant attention since then. The existence question asks whether any periodic points exist, and the multiplicity question asks how many there are. On a closed surface of genus $g \geq 1$, there exists an area-preserving diffeomorphism that has any prescribed finite number $N \geq 1$ of periodic points. For $N \geq 2g - 2$, as explained in [**LC22**], it suffices to add $N - 2g + 2$ singularities to an irrational translation flow ($g = 1$) or a translation flow in a minimal direction ($g \geq 2$), and then take the time-one map. This map has N periodic points, all of which are fixed, and is isotopic to the identity relative to the fixed point set. More recently, an explicit construction for all $N \geq 1$ was presented by Atallah, Batoréo and Ferreira [ABF24]. The current state of the art on the multiplicity problem for area-preserving surface maps, to our knowledge, is due to Le Calvez [LC22]. He shows, barring some edge cases when $g \leq 1$, that any area-preserving homeomorphism of a closed surface is either periodic, has periodic points of unbounded minimal period or, after passing to an iterate, only has finitely many fixed points and is isotopic to the identity relative to the fixed point set. The last case is the one modeled by the examples from [LC22].

In this article, we discuss a simple and dense homological condition that forces an area-preserving map isotopic to the identity to have infinitely many periodic points. Fix a compact surface Σ and a smooth area form ω. Any area-preserving map $\phi \in \mathrm{Homeo}_0(\Sigma, \omega)$ isotopic to the identity has a *rotation vector*

$$\mathcal{F}(\phi) \in H_1(\Sigma; \mathbb{R}) \,/\, \Gamma_\omega,$$

where $\Gamma_\omega \subseteq H_1(\Sigma; \mathbb{Z})$ is a discrete subgroup. The map ϕ has *rational rotation direction* if $\mathcal{F}(\phi)$ is a real multiple of a rational class, that is, $c \cdot \mathcal{F}(\phi) \in H_1(\Sigma; \mathbb{Q}) \,/\, \Gamma_\omega$ for some real number $c > 0$. Any area-preserving map can be perturbed to an area-preserving map with rational rotation direction by a C^∞-small perturbation, so a dense subset of maps have rational rotation direction. We now state our main result.

THEOREM A. *Fix a compact surface Σ of genus ≥ 2 and a smooth area form ω. Let $\phi \in \mathrm{Homeo}_0(\Sigma, \omega)$ be any area-preserving homeomorphism that is isotopic to the identity and has rational rotation direction. Then ϕ is either the identity or has periodic points of unbounded minimal period.*

Remark 1.1. In the theorem above and all subsequent discussion, we allow compact surfaces to have non-empty boundary, and we say they are *closed* when the boundary is empty. The genus of a compact surface is the genus of the closed surface obtained by attaching disks to each boundary component. All surfaces are assumed to be oriented from now on.

Remark 1.2. Franks and Handel [FH03] and Le Calvez [LC06] proved Theorem A under the assumption that ϕ is Hamiltonian, which is equivalent to the condition that $\mathcal{F}(\phi) = 0$.

Remark 1.3. Similar results for surfaces of genus zero and one are either known or can be shown to hold by combining known results. See §1.3 for a more detailed discussion.

Remark 1.4. The area-preserving maps with finitely many periodic points discussed above do not have rational rotation direction.

Remark 1.5. Fix a compact surface Σ of genus ≥ 2 and a smooth area form ω. Let $\mathrm{Diff}_0(\Sigma, \omega)$ denote the space of all area-preserving diffeomorphisms that are isotopic to the identity. Theorem A and a short Baire category argument show that a C^∞-generic $\phi \in \mathrm{Diff}_0(\Sigma, \omega)$ has periodic points of unbounded minimal period. Any periodic point can be made non-degenerate by a C^∞-small local perturbation, which leaves the rotation vector unchanged. Since maps with rational rotation direction are C^∞-dense, it follows from Theorem A that, for each d, there exists an open and dense subset $\mathcal{U}_d \subset \mathrm{Diff}_0(\Sigma, \omega)$ such that, if $\phi \in \mathcal{U}_d$, then ϕ has a periodic point of minimal period $\geq d$. Each ϕ in $\mathcal{U} := \bigcap_{d \geq 1} \mathcal{U}_d$ has periodic points of unbounded minimal period.

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Theorem A answers a question asked independently by Ginzburg and by Seyfaddini. From a modern perspective, the result contributes to the active stream of research centered around the *Conley conjecture*. The original formulation of the conjecture asserts that any Hamiltonian diffeomorphism of a closed aspherical symplectic manifold has infinitely many periodic points. The Conley conjecture was resolved for surfaces by Franks and Handel [**FH03**] and extended to Hamiltonian homeomorphisms by Le Calvez [**LC06**] before being resolved in full generality by breakthrough work of Hingston [**Hin09**] for higher-dimensional tori and by Ginzburg [**Gin10**] for the general case. The search for extensions of the Conley conjecture to Hamiltonian diffeomorphisms/homeomorphisms of more general symplectic manifolds has attracted a great deal of ongoing activity and progress [**GG09b**, **GG09a**, **GG10**, **GG12**, **Gür3**, **GG14**, **GG15**, **GG16**, **Çin18**, **GG19**]. However, there has been much less progress in establishing Conley conjecture-type results for non-Hamiltonian symplectic maps (some notable results include [**Bat15**, **Bat17**, **Bat18**]). Moreover, to our knowledge, there is no agreed-upon formulation of the Conley conjecture for non-Hamiltonian symplectic maps. Theorem A can be viewed not only as establishing such a 'non-Hamiltonian Conley conjecture' in dimension two, but also as a guidepost towards formulating a 'non-Hamiltonian Conley conjecture' for higher-dimensional symplectic manifolds. We pose the following question.

Question. Let (M, Ω) be a closed and symplectically aspherical symplectic manifold of any dimension. Then, does any symplectic diffeomorphism $\phi \in \mathrm{Diff}_0(M, \Omega)$ such that $\mathcal{F}(\phi) \in H_1(M; \mathbb{Q}) \,/\, \Gamma_\Omega$ have infinitely many periodic points?

The idea that a Conley conjecture-type result may hold for area-preserving homeomorphisms with rational rotation direction was motivated by recent work on the C^∞-closing lemma [**CGPZ21**, **EH21**] for area-preserving surface diffeomorphisms. Herman famously showed [**HZ94**, Ch. 4.5] that a version of the closing lemma using only Hamiltonian perturbations cannot hold for certain irrational maps (Diophantine torus rotations). This issue was avoided by proving a Hamiltonian C^∞-closing lemma for many rational maps (any map with rational asymptotic cycle, see §1.2) and then observing that such maps form a C^∞-dense subset of $\mathrm{Diff}(\Sigma, \omega)$. It seems reasonable to suspect that, given these marked differences in behavior, rationality conditions for rotation vectors have dynamical significance.

Theorem A can be extended, with slightly weaker conclusions, to maps preserving arbitrary Borel probability measures with full support. Given an isotopy Φ from the identity to ϕ and a ϕ-invariant Borel probability measure μ, a rotation vector $\mathcal{F}(\Phi, \mu) \in H_1(\Sigma; \mathbb{R})$ can be defined. When Σ has genus ≥ 2, the rotation vector does not depend on the isotopy Φ, and we write it as $\mathcal{F}(\phi, \mu)$.

THEOREM B. *Fix a compact surface Σ of genus ≥ 2 and a Borel probability measure μ with full support such that $\mu(\partial\Sigma) = 0$. Let $\phi \in \mathrm{Homeo}_0(\Sigma, \mu)$ be any μ-preserving homeomorphism that is isotopic to the identity, such that its rotation vector $\mathcal{F}(\phi, \mu) \in H_1(\Sigma; \mathbb{R})$ is a real multiple of a rational class. Then ϕ has infinitely many periodic points. Moreover, if μ has no atoms, then ϕ is either the identity or has periodic points of unbounded minimal period.*

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Theorem B follows from a short argument combining Theorem A and the Oxtoby–Ulam theorem [OU41], which was suggested to the author by Le Calvez. Theorem A follows from Theorem C, which may be of independent interest, and the results in [LC06, LC22].

THEOREM C. *Fix a closed surface* Σ *of genus* ≥ 2 *and a smooth area form* ω. *Then any* $\phi \in \mathrm{Homeo}_0(\Sigma, \omega)$ *with rational rotation direction is either Hamiltonian or has a non-contractible periodic point.*

Remark 1.6. An interesting problem related to Theorem C, posed by Ginzburg, is to determine whether a C^∞-generic Hamiltonian diffeomorphism of a closed and symplectically aspherical symplectic manifold has a non-contractible periodic point. This was proved for the two-torus by Le Calvez and Tal [LCT18] and was recently extended to all closed surfaces of positive genus by Le Calvez and Sambarino [LCS23].

A recent result from symplectic geometry is central to the proof of Theorem C. In [CGPZ21], a non-vanishing theorem is proved for the periodic Floer homology (PFH) of area-preserving diffeomorphisms of closed surfaces. PFH is a homology theory for area-preserving surface maps built out of their periodic orbits. The non-vanishing theorem relies on a deep result of Lee and Taubes [LT12] which shows that PFH is isomorphic to monopole Floer homology. We explain more about PFH and state the non-vanishing theorem at the beginning of §4.

There are two major issues in the proof of Theorem C that require new arguments to overcome. The first issue is that PFH is only well defined for diffeomorphisms, not homeomorphisms. To get around this, we observe that a quantitative version (Proposition 4.2) of Theorem C holds; there exists a non-contractible periodic point with an upper bound on its minimal period depending only on the rotation vector. This allows us to extend Theorem C to homeomorphisms by an approximation argument. The proof of Proposition 4.2 exploits the homological grading of PFH to extract the required information. To our knowledge, this is a new argument with no analog in previous work.

The second issue is that the non-vanishing of PFH is only known for maps with rotation vector in $H_1(\Sigma; \mathbb{Q})$ and not all maps with rational rotation direction. We cannot hope for too much when $\mathcal{F}(\phi) \notin H_1(\Sigma; \mathbb{Q})$; in this case, there are many examples (irrational torus rotations and translation flows in minimal directions) where PFH essentially vanishes. (More precisely, it vanishes in non-trivial homological gradings, so it only detects null-homologous periodic orbit sets, in which every periodic orbit could be contractible.) This is overcome via a novel blow-up argument.

1.2. *Maps not isotopic to the identity.* Assume that Σ is closed and has genus ≥ 2. It is known [LC22] that either ϕ has periodic points of unbounded minimal period or it has periodic Nielsen–Thurston class. Therefore, if ϕ has periodic Nielsen–Thurston class and some iterate $\phi^q \in \mathrm{Homeo}_0(\Sigma, \omega)$ has rational rotation direction, then Theorem A implies that ϕ is either periodic or has periodic points of unbounded minimal period. We now show that maps with periodic Nielsen–Thurston class and rational asymptotic cycle have an iterate with rational rotation direction.

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Let M_ϕ denote the mapping torus of ϕ. The *asymptotic cycle* $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{R})$ is an analog of the rotation vector for area-preserving maps that are not isotopic to the identity; by rational asymptotic cycle, we mean $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$. It is defined by applying Schwartzman's construction [**Sch57**] to the suspension flow. When ϕ is isotopic to the identity, a choice of identity isotopy Φ defines a diffeomorphism $M_\phi \simeq \mathbb{T} \times \Sigma$, where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denotes the circle. Lemma 3.1 proves that, under this identification,

$$\mathcal{C}(\phi) = [\mathbb{T}] + \mathcal{F}(\Phi).$$

If $\phi \in \mathrm{Homeo}_0(\Sigma, \omega)$ has rational asymptotic cycle, then this computation shows that ϕ has rational rotation direction. The property $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$ is preserved under iteration. Lemma 3.3 below shows that if $\mathcal{C}(\phi)$ is rational, then so is $\mathcal{C}(\phi^k)$ for each $k > 1$. Putting these facts together implies our claim above, that if ϕ has periodic Nielsen–Thurston class and $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$, then it has an iterate $\phi^q \in \mathrm{Homeo}_0(\Sigma, \omega)$ with rational rotation direction. Combining this with Theorem A proves the following theorem.

THEOREM D. *Fix a closed surface Σ of genus ≥ 2 and a smooth area form ω. Let $\phi \in \mathrm{Homeo}(\Sigma, \omega)$ be any area-preserving homeomorphism such that $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$. Then ϕ is either periodic or has periodic points of unbounded minimal period.*

Remark 1.7. Assuming that $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$, for ϕ with periodic Nielsen–Thurston class, is sufficient but not necessary for ϕ to have an iterate with rational rotation direction. It can be weakened, but we were unable to find a sufficiently elegant condition to write down. It is still true that the set of maps with $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$ is dense.

Remark 1.8. Theorem D and a similar Baire category argument extend Remark 1.5 to all area-preserving diffeomorphisms. Fix a closed surface of genus ≥ 2 and a smooth area form ω. Then a C^∞-generic $\phi \in \mathrm{Diff}(\Sigma, \omega)$ is either periodic or has periodic points of unbounded minimal period. We stress that this statement is not new. Previous work [**CGPZ21**, **EH21**] establishes the much stronger statement that a C^∞-generic $\phi \in \mathrm{Diff}(\Sigma, \omega)$ has a dense set of periodic points.

Asymptotic cycles $\mathcal{C}(\phi, \mu) \in H_1(M_\phi; \mathbb{R})$ can be defined for each ϕ-invariant Borel probability measure μ. The same proof as that of Theorem D, after replacing Theorem A with Theorem B, implies the following result.

THEOREM E. *Fix a closed surface Σ of genus ≥ 2 and a smooth area form ω. Let $\phi \in \mathrm{Homeo}(\Sigma, \mu)$ be an area-preserving homeomorphism preserving a Borel probability measure μ of full support. Assume that $\mathcal{C}(\phi, \mu) \in H_1(M_\phi; \mathbb{Q})$. Then ϕ has infinitely many periodic points. Moreover, if μ has no atoms, then ϕ is either periodic or has periodic points of unbounded minimal period.*

1.3. *Lower genus surfaces.* The analogs of the above theorems when Σ has genus zero are already known. Collapsing the boundary components and appealing to [**FH03**, **LC06**] gives a sharp characterization of the existence and multiplicity of periodic points. A genus one version of Theorem A follows from work of Le Calvez.

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PROPOSITION 1.1. [**LC06**] *Fix a compact surface* Σ *of genus one and a smooth area form* ω. *Let* $\phi \in \mathrm{Homeo}_0(\Sigma, \omega)$ *be an area-preserving homeomorphism and assume that* $\mathcal{F}(\phi) \in H_1(\Sigma; \mathbb{Q}) / \Gamma_\omega$. *Then* ϕ *is either periodic or has periodic points of unbounded minimal period.*

The assumptions are stronger than in Theorem A, but the result is sharp. Any translation $(x, y) \mapsto (x + a, y + b)$ on $\mathbb{T}^2 := \mathbb{R}^2/\mathbb{Z}^2$, where at least one of a and b is not rational, has irrational rotation vector and has no periodic points. The proof of Proposition 1.1 is straightforward. Any map satisfying the conditions of Proposition 1.1 has a Hamiltonian iterate, and the main result of [**LC06**] shows that Hamiltonian torus homeomorphisms are either the identity or have periodic points of unbounded minimal period. Our methods do, however, imply something new in the genus one case. The following is a sharp version of Theorem C for smooth torus maps. (An explanation for why we need smoothness is provided below the statement of Proposition 4.3.)

THEOREM F. *Fix* $\phi \in \mathrm{Diff}_0(\mathbb{T}^2, dx \wedge dy)$ *and an identity isotopy* Φ *such that* $\mathcal{F}(\Phi)$ *is a real multiple of a rational class. Then* ϕ *is either Hamiltonian, has no periodic points or has a periodic point that is not* Φ*-contractible.*

The proof is given in §4.

Remark 1.9. The assumptions of Theorem F are satisfied when $\mathcal{F}(\Phi) \in H_1(\mathbb{T}^2; \mathbb{Q})$. In this case, ϕ has a Hamiltonian iterate, so it has a periodic point by Conley–Zehnder's fixed point theorem [**CZ83**].

Theorem D also has an analog for the torus.

PROPOSITION 1.2. *Fix a compact surface* Σ *of genus one and a smooth area form* ω. *Let* $\phi \in \mathrm{Homeo}(\Sigma, \omega)$ *be an area-preserving homeomorphism and assume that* $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$. *Then* ϕ *is either periodic or has periodic points of unbounded minimal period.*

Addas-Zanata and Tal [**AZT07**] proved that an area-preserving torus homeomorphism ϕ either has periodic points of unbounded minimal period, is isotopic to a Dehn twist with no periodic points and vertical rotation set reduced to an irrational number, or has an iterate isotopic to the identity. The assumption that $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$ rules out the second case, so we can assume that ϕ has an iterate ϕ^q isotopic to the identity. The assumption that $\mathcal{C}(\phi) \in H_1(M_\phi; \mathbb{Q})$ implies that ϕ^q has rational rotation direction, and then, by Proposition 1.1, ϕ^q is either the identity or has periodic points of unbounded minimal period.

1.4. *Outline.* Section 2 reviews some important preliminaries. Section 3 contains some computations of asymptotic cycles that seem standard but which we could not find elsewhere. Section 4 presents a brief overview of PFH and the non-vanishing theorem from [**CGPZ21**], and then proves Theorems A–C and F. Theorems D and E were proved above using computations from §3 and Theorems A and B.

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2. *Preliminaries*

2.1. *Area-preserving maps.*

2.1.1. *Diffeomorphisms.* Write Diff(Σ) for the space of diffeomorphisms $\phi : \Sigma \to \Sigma$, equipped with the topology of C^∞-convergence of maps and their inverses, and let Diff(Σ, ω) denote the space of diffeomorphisms such that $\phi^*\omega = \omega$. Let $\text{Diff}_0(\Sigma)$ and $\text{Diff}_0(\Sigma, \omega)$ denote the respective connected components of the identity. The group $\text{Diff}_0(\Sigma, \omega)$ contains a large subgroup Ham(Σ, ω) of *Hamiltonian diffeomorphisms*, the maps with rotation vector 0. An *isotopy* is a continuous path $\Phi : [0, 1] \to \text{Diff}(\Sigma)$. Sometimes, we will write $\Phi = \{\phi_t\}_{t\in[0,1]}$ to emphasize our interpretation of Φ as a one-parameter family of diffeomorphisms. An *identity isotopy* of $\phi \in \text{Diff}(\Sigma)$ is an isotopy Φ with $\Phi(0) = \text{Id}$ and $\Phi(1) = \phi$.

2.1.2. *Homeomorphisms.* Write Homeo(Σ) for the space of homeomorphisms $\phi : \Sigma \to \Sigma$, equipped with the topology of C^0-convergence of maps and their inverses, and let Homeo(Σ, μ) denote the space of homeomorphisms preserving a Borel measure μ. Let $\text{Homeo}_0(\Sigma)$ and $\text{Homeo}_0(\Sigma, \mu)$ denote the respective connected components of the identity. Isotopies of homeomorphisms are defined as above. It is well known that Diff(Σ, ω) is C^0-dense in Homeo(Σ, ω), which is the space of area-preserving homeomorphisms. Write $\overline{\text{Ham}}(\Sigma, \omega) \subset \text{Homeo}_0(\Sigma, \omega)$ for the C^0-closure of Ham(Σ, ω), which is the group of *Hamiltonian homeomorphisms*. Fathi [**Fat80**, §6] showed that these are exactly the area-preserving homeomorphisms with rotation vector 0.

2.1.3. *Periodic points and orbits.* Fix any $\phi \in \text{Homeo}(\Sigma, \omega)$. A *periodic point* of $\phi \in \text{Homeo}(\Sigma, \omega)$ is a point $p \in \Sigma$ such that $\phi^k(p) = p$ for some finite $k \geq 1$, and the *period* of p is the minimal k such that this holds. A *periodic orbit* is a finite set $S = \{x_1, \ldots, x_k\}$ of not necessarily distinct points in Σ that are cyclically permuted by ϕ. A periodic orbit is *simple* if all of the points are distinct.

Fix $\phi \in \text{Homeo}_0(\Sigma, \omega)$ and an identity isotopy Φ. Fix any periodic point p of period $k \geq 1$. The union of arcs

$$\gamma_p := \bigcup_{j=0}^{k-1} \{\phi_t(\phi^j(p))\}_{t\in[0,1]}$$

is a closed loop in Σ. The point p is Φ-*contractible* if γ_p is contractible. Note that if Σ has genus ≥ 2, then $\text{Homeo}_0(\Sigma, \omega)$ is simply connected, so Φ-contractibility is independent of the choice of Φ. We will not specify Φ in this case.

2.2. *Rotation vectors.*

2.2.1. *Definition.* Fix $\phi \in \text{Homeo}_0(\Sigma)$ and an identity isotopy Φ. To any ϕ-invariant Borel probability measure μ we associate a class $\mathcal{F}(\Phi, \mu) \in H_1(\Sigma; \mathbb{R})$ called its *rotation vector*. The rotation vector depends only on the homotopy class of Φ relative to its endpoints. When Σ has genus ≥ 2, the space $\text{Homeo}_0(\Sigma)$ is simply connected, so $\mathcal{F}(\Phi, \mu)$ is independent of the choice of Φ, and we sometimes write it as $\mathcal{F}(\phi, \mu)$ instead.

Let $[\Sigma, \mathbb{T}]$ denote the set of homotopy classes of continuous maps from Σ to the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. This is isomorphic to $H^1(\Sigma; \mathbb{Z})$. The isomorphism sends a class $[f] \in [\Sigma, \mathbb{T}]$

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to the pullback $f^*d\theta$ of the oriented generator $d\theta \in H^1(\mathbb{T};\mathbb{Z})$. The universal coefficient theorem implies that $H_1(\Sigma;\mathbb{R})$ is isomorphic to $\mathrm{Hom}([\Sigma,\mathbb{T}],\mathbb{R})$.

For any continuous circle-valued function $f:\Sigma\to\mathbb{T}$, the isotopy Φ defines a real-valued lift $g:\Sigma\to\mathbb{R}$ of the null-homotopic $\mathbb{T}$-valued function $f\circ\phi - f$. The map

$$f \mapsto \int_\Sigma g\, d\mu$$

defines a real-valued linear functional on $[\Sigma,\mathbb{T}]$, and the rotation vector $\mathcal{F}(\Phi,\mu)\in H_1(\Sigma;\mathbb{R})$ is the associated homology class. Several basic properties follow from this definition. First, observe that $\mathcal{F}(\Phi,\mu)$ is invariant under endpoint-preserving homotopy of Φ. Next, observe that $\mathcal{F}(\Phi,\mu)$ is continuous in μ with respect to the weak topology. Moreover, it is linear with respect to convex combinations: that is,

$$\mathcal{F}(\Phi, t\mu_1 + (1-t)\mu_2) = t\mathcal{F}(\Phi,\mu_1) + (1-t)\mathcal{F}(\Phi,\mu_2).$$

Next, observe that $\mathcal{F}(\Phi,\mu)$ changes with respect to a homeomorphism in the following manner. If $f:\Sigma\to\Sigma'$ is a homeomorphism, then

$$\mathcal{F}(f\Phi f^{-1}, f_*\mu) = f_*\mathcal{F}(\Phi,\mu). \tag{1}$$

Finally, observe that since the space of invariant probability measures is convex and compact with respect to the weak topology, the image of the map $\mathcal{F}(\Phi,-)$ is a convex and compact subset of $H_1(\Sigma;\mathbb{R})$. This subset is called the *rotation set*. The structure of the rotation set has been extensively studied by many authors, and it can have interesting dynamical consequences. See the introduction of [GM22] for a comprehensive survey.

2.2.2. *Rotation vectors of periodic orbits.* Any periodic orbit $S=\{x_1,\ldots,x_k\}$ determines an invariant Borel probability measure: the average of the δ-measures at its points. The rotation vector $\mathcal{F}(\Phi,S)$ is the rotation vector of this measure. This has a nice geometric interpretation. The union of arcs

$$\gamma_S := \bigcup_{j=1}^{k}\{\phi_t(x_j)\}_{t\in[0,1]}$$

is a closed, oriented loop in Σ. It is easy to show that

$$k\cdot\mathcal{F}(\Phi,S) = [\gamma_S]\in H_1(\Sigma;\mathbb{Z}). \tag{2}$$

If $p\in\Sigma$ is a periodic point, then its rotation vector $\mathcal{F}(\Phi,p)$ is defined to be the rotation vector of any simple periodic orbit containing p. The identity (2) shows that if $\mathcal{F}(\Phi,p)\neq 0$, then p is not Φ-contractible.

2.2.3. *Rotation vectors of area-preserving homeomorphisms.* Fix $\phi\in\mathrm{Homeo}_0(\Sigma,\omega)$ and any identity isotopy Φ. We use the following notation for the rotation vector of the normalized area measure: that is,

$$\mathcal{F}(\Phi) := \mathcal{F}\bigg(\Phi, \bigg(\int_\Sigma \omega\bigg)^{-1}\cdot\omega\bigg)\in H_1(\Sigma;\mathbb{R}).$$

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This invariant was introduced by Fathi [**Fat80**] as the *mass flow*. The function $\mathcal{F}$ is C^0-continuous in Φ and additive with respect to pointwise composition of isotopies. Moreover, the image $\Gamma_\omega := \mathcal{F}(\pi_1(\mathrm{Homeo}(\Sigma, \omega))$ of the subgroup of loops based at the identity is a lattice in $H_1(\Sigma; \mathbb{Z})$. If $\Sigma = \mathbb{T}^2$, then $\Gamma_\omega = H_1(\Sigma; \mathbb{Z})$. If Σ is closed and has genus ≥ 2, then $\Gamma_\omega = \{0\}$. We end up with a homomorphism

$$\mathcal{F} : \mathrm{Homeo}_0(\Sigma, \omega) \to H_1(\Sigma; \mathbb{R}) \,/\, \Gamma_\omega.$$

Remark 2.1. For smooth area-preserving maps, the rotation vector is Poincaré dual to the *flux homomorphism*, which is an invariant that might be more familiar to symplectic geometers.

2.3. *Mapping torii.* Fix any $\phi \in \mathrm{Homeo}(\Sigma)$. The *mapping torus* of ϕ is the compact three-manifold M_ϕ defined by quotienting $\mathbb{R}_t \times \Sigma$ by the relationship $(1, p) \sim (0, \phi(p))$. Translation in the t-direction with speed 1 yields a continuous flow $\{\psi_R^t\}_{t\in\mathbb{R}}$ on M_ϕ called the *suspension flow*. Its closed integral curves are in one-to-one correspondence with simple periodic orbits of ϕ. If ϕ preserves a Borel measure μ, then the suspension flow preserves the measure $dt \otimes \mu$ on M_ϕ.

Suppose that $\phi \in \mathrm{Homeo}_0(\Sigma)$. Choose an identity isotopy $\Phi = \{\phi_t\}_{t\in[0,1]}$. This choice defines a homeomorphism

$$\eta : \mathbb{T} \times \Sigma \to M_\phi \tag{3}$$

by the map $[(t, p)] \mapsto [(t, \phi_t^{-1}(p))]$. This homeomorphism provides a useful method for recovering the rotation vector of a periodic orbit. Let $S = \{x_1, \ldots, x_k\} \subset \Sigma$ be a simple periodic orbit, and let $\gamma \subset M_\phi$ be the associated closed integral curve of the suspension flow. Using η to realize γ as a loop in $\mathbb{T} \times \Sigma$, its homology class is easily computed using

$$k^{-1} \cdot [\gamma] = [\mathbb{T}] + \mathcal{F}(\Phi, S) \in H_1(\mathbb{T} \times \Sigma; \mathbb{R}). \tag{4}$$

3. *Asymptotic cycles*

This section discusses the asymptotic cycle construction and carries out several useful computations. Fix a compact surface Σ, a map $\phi \in \mathrm{Homeo}(\Sigma)$ and a ϕ-invariant Borel probability measure μ. The *asymptotic cycle*, which was introduced by Schwartzman [**Sch57**], is a homology class $\mathcal{C}(\phi, \mu) \in H_1(M_\phi; \mathbb{R})$. If ϕ is area-preserving, then $\mathcal{C}(\phi)$ denotes the asymptotic cycle of the normalized area measure.

3.1. *Definition.* We define $\mathcal{C}(\phi, \mu)$ as a real-valued linear functional on $[M_\phi, \mathbb{T}]$. Fix any continuous $f : M_\phi \to \mathbb{T}$. For each $s \in \mathbb{R}$, write $f_s := f \circ \psi_R^s$. The functions $f_s - f$ are a continuous family of null-homotopic circle-valued functions, so they lift to a unique continuous family $\{g_s\}_{s\in\mathbb{R}}$ of functions $M_\phi \to \mathbb{R}$ with $g_0 \equiv 0$. Kingman's subadditive ergodic theorem implies that $G := \lim_{s\to\infty} g_s/s$ is a well-defined $(dt \otimes \mu)$-integrable function. We set $\langle \mathcal{C}(\phi, \mu), f\rangle$ to be the integral of G. This is linear and homotopy-invariant in f (see [**Sch57**]), so it defines a real-valued linear functional on $H^1(M_\phi; \mathbb{Z})$; therefore, it defines a class in $H_1(M_\phi; \mathbb{R})$.

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Remark 3.1. Fix any compact manifold M. An asymptotic cycle, taking values in $H_1(M; \mathbb{R})$, is defined as above for any choice of a continuous flow $\{\psi^t\}_{t\in\mathbb{R}}$ and a ψ-invariant Borel probability measure.

3.2. *Maps isotopic to the identity.* When $\phi \in \mathrm{Homeo}_0(\Sigma)$, the rotation vector of μ can be recovered from $\mathcal{C}(\phi, \mu)$.

LEMMA 3.1. *Fix a compact surface Σ, a ϕ-invariant Borel probability measure μ and $\phi \in \mathrm{Homeo}_0(\Sigma, \mu)$. For any identity isotopy Φ, the pullback of $\mathcal{C}(\phi, \mu)$ by (3) satisfies the identity*

$$\eta^* \cdot \mathcal{C}(\phi, \mu) = [\mathbb{T}] + \mathcal{F}(\Phi, \mu) \in H_1(\mathbb{T} \times \Sigma; \mathbb{R}). \tag{5}$$

Proof. Write $\Phi = \{\phi_t\}_{t\in[0,1]}$ and write $q_t := \eta^{-1} \circ \psi_R^t \circ \eta$, where we recall that $\{\psi_R^t\}_{t\in\mathbb{R}}$ is the suspension flow. Write $\mu_t := (\phi_t)_*(\mu)$ for every t. The pullback $\eta^* \cdot \mathcal{C}(\phi, \mu)$ is the asymptotic cycle of the flow $\{q_t\}_{t\in\mathbb{R}}$ with respect to $dt \otimes \mu_t = \eta^*(dt \otimes \mu)$. We extend the isotopy to a map $\Phi : \mathbb{R} \to \mathrm{Homeo}_0(\Sigma)$ by setting $\phi_t := \phi_{t-\lfloor t\rfloor}\phi^{\lfloor t\rfloor}$. For any $s \in \mathbb{R}$ and $t \in [0, 1)$, we compute $q_s(t, p) = (s + t, \phi_{s+t}\phi_t^{-1}(p)) \in \mathbb{T} \times \Sigma$.

The $\mathbb{T}$-invariant functions and the projection $\pi : \mathbb{T} \times \Sigma \to \mathbb{T}$ define a basis of $H^1(\mathbb{T} \times \Sigma; \mathbb{Z})$. The lemma is proved by showing that

$$\langle \eta^* \cdot \mathcal{C}(\phi, \mu), \pi\rangle = 1, \quad \langle \eta^* \cdot \mathcal{C}(\phi, \mu), f\rangle = \langle \mathcal{F}(\Phi, \mu), f\rangle \tag{6}$$

for any $\mathbb{T}$-invariant $f : \mathbb{T} \times \Sigma \to \mathbb{T}$. The real-valued lift g_s of $\pi \circ q_s - \pi$ is $g_s(t, p) = s$, so the integral of g_s/s is always 1. This proves the first identity in (6).

Fix any $\mathbb{T}$-invariant $f : \mathbb{T} \times \Sigma \to \mathbb{T}$. Write $f_s := f \circ q_s$ and let $\{g_s\}_{s\in\mathbb{R}}$ be the real-valued lift of the family $\{f_s - f\}_{s\in\mathbb{R}}$. Fix any $t \in [0, 1)$ and set $\phi^{(t)} \in \mathrm{Homeo}_0(\Sigma, \mu_t)$ to be the conjugate of ϕ by ϕ_t. The function $(f_\tau - f)(t, -)$ is the displacement of $f(t, -)$ under the identity isotopy $\Phi^t := \{\phi_{\tau+t}\phi_t^{-1}\}_{\tau\in[0,1]}$ ending at $\phi^{(t)}$. It follows that

$$\int_\Sigma g_1(t, -)\mu_t = \mathcal{F}(\Phi^t, \mu_t). \tag{7}$$

Next, we claim that, for any $s \in \mathbb{N}$,

$$\frac{1}{s}\int_\Sigma g_s(t, -)\mu_t = \mathcal{F}(\Phi^t, \mu_t). \tag{8}$$

To see this, observe that $g_s(t, -) = g_{s-1}(t, -) \circ \phi^{(t)} + g_1(t, -)$. Expanding recursively, we find that

$$g_s(t, -) = \sum_{i=0}^{s-1} g_1(t, -) \circ (\phi^{(t)})^i.$$

Integrating both sides with respect to μ_t, dividing by s and applying (7) gives (8). Now, observe that the isotopy Φ^t is homotopic relative to its endpoints to the conjugated isotopy $\phi_t \circ \Phi \circ \phi_t^{-1}$, so

$$\mathcal{F}(\Phi^t, \mu_t) = \mathcal{F}(\Phi, \mu). \tag{9}$$

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Fix $s \in \mathbb{N}$. We integrate (8) over $t \in [0, 1)$ and apply (9) and the fact that f is $\mathbb{T}$-invariant to show that

$$\frac{1}{s}\int_0^1 \bigg(\int_\Sigma g_s(t, -)\, d\mu_t\bigg)\, dt = \int_0^1 \langle \mathcal{F}(\Phi^t, \mu_t), f(t, -)\rangle\, dt = \langle \mathcal{F}(\Phi, \mu), f\rangle$$

for any $s \in \mathbb{N}$. Taking $s \to \infty$ on the left-hand side proves the second identity in (6). □

3.3. *Smooth maps.* When $\phi \in \mathrm{Diff}(\Sigma, \omega)$, the area form ω defines a closed two-form ω_ϕ on the mapping torus M_ϕ, which restricts to 0 on the boundary. Its cohomology class is Poincaré dual to the asymptotic cycle.

LEMMA 3.2. *Assume that* $\phi \in \mathrm{Diff}(\Sigma, \omega)$. *Then* $[\omega_\phi] \in H^2(M_\phi, \partial M_\phi; \mathbb{R})$ *is Poincaré dual to* $(\int_\Sigma \omega) \cdot \mathcal{C}(\phi)$.

Proof. Let $d\theta$ denote the closed one-form on $\mathbb{T}$ with integral 1. If ϕ is smooth, then a circle-valued function $f : M_\phi \to \mathbb{T}$ corresponds to $[f^* d\theta] \in H^1(M_\phi; \mathbb{R})$. The lemma follows from showing that

$$\int_{M_\phi} f^*\, d\theta \wedge \omega_\phi = \bigg(\int_\Sigma \omega\bigg) \cdot \langle \mathcal{C}(\phi), f\rangle \tag{10}$$

for any $f : M_\phi \to \mathbb{T}$. Write $f_s = f \circ \psi_R^s$ for each $s \in \mathbb{R}$, and let $\dot{f}_s = (f_s^* d\theta)(R) : M_\phi \to \mathbb{R}$ denote the time derivative. The associated real-valued lifts are $g_s := \int_0^s \dot{f}_\tau\, d\tau$ for $s > 0$. For any $s > 0$, $g_{2s} = g_s + g_s \circ \psi_R^s$. Since $dt \wedge \omega_\phi$ is R-invariant, this implies that the integral of $g_{2s}/2s$ over M_ϕ is equal to the integral of g_s/s. Repeated division by two shows that

$$\begin{aligned}\int_{M_\phi} \frac{1}{s} g_s\, dt \wedge \omega_\phi &= \lim_{\tau \to 0} \int_{M_\phi} \frac{1}{\tau} g_\tau\, dt \wedge \omega_\phi = \int_{M_\phi} \dot{f}_0\, dt \wedge \omega_\phi \\ &= \int_{M_\phi} (f^*\, d\theta)(R)\, dt \wedge \omega_\phi = \int_{M_\phi} f^*\, d\theta \wedge \omega_\phi\end{aligned}$$

for any $s > 0$. This proves (10). □

3.4. *Behavior under iteration.* We show that rationality of $\mathcal{C}(\phi, \mu)$ is preserved under iteration.

LEMMA 3.3. *Fix any* $k \in \mathbb{N}$. *If* $\mathcal{C}(\phi, \mu) \in H_1(M_\phi; \mathbb{Q})$, *then* $\mathcal{C}(\phi^k, \mu) \in H_1(M_{\phi^k}; \mathbb{Q})$.

Proof. There is a covering map $\pi_k : M_{\phi^k} \to M_\phi$ with deck group $\mathbb{Z}/k\mathbb{Z}$, given by the map $[(t, p)] \mapsto [(kt - \lfloor kt \rfloor, \phi^{\lfloor kt \rfloor}(p))]$. The deck group is generated by the map $T : [(t, p)] \mapsto [(t - 1/k, \phi(p))]$. Denote the suspension flows of ϕ and ϕ^k by $\{\psi^t\}$ and $\{\psi_k^t\}$, respectively. The group $[M_{\phi^k}, \mathbb{T}]$ is spanned by functions of the form $f \circ T - f$ and those which are pulled back by π_k from M_ϕ. The suspension flow of ϕ^k commutes with the covering translations, so if $\{g_s\}_{s \in \mathbb{R}}$ denotes the real-valued lifts of the family $\{f \circ \psi_k^s - f\}_{s \in \mathbb{R}}$, then $\{g_s \circ T\}_{s \in \mathbb{R}}$ are the real-valued lifts of $\{f \circ T \circ \psi_k^s - f \circ T\}_{s \in \mathbb{R}}$. The map T preserves $dt \otimes \mu$, so $g_s \circ T$ and g_s have the same integral. We conclude that

$$\langle \mathcal{C}(\phi^k), f \circ T - f\rangle = 0.$$

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It remains to consider functions pulled back from M_ϕ. Note that, for any $f : M_\phi \to \mathbb{T}$, the pairing $\langle \mathcal{C}(\phi^k, \mu), f \circ \pi_k \rangle$ is equal to $\langle (\pi_k)_* \cdot \mathcal{C}(\phi^k, \mu), f \rangle$. Since we are assuming that $\mathcal{C}(\phi)$ is rational, the rationality of $\mathcal{C}(\phi^k)$ therefore follows from the identity

$$(\pi_k)_* \cdot \mathcal{C}(\phi^k, \mu) = k \cdot \mathcal{C}(\phi, \mu). \tag{11}$$

The key observation here is the commutation relationship $\pi_k \circ \psi_k^{t/k} = \psi^t \circ \pi_k$. This shows that the asymptotic cycle of the flow $\{\psi_k^{t/k}\}_{t\in\mathbb{R}}$ pushes forward to $\mathcal{C}(\phi)$. Rescaling a flow in time by a factor of λ multiplies the asymptotic cycle by λ. The asymptotic cycle of $\{\psi_k^{t/k}\}_{t\in\mathbb{R}}$ is $k^{-1} \cdot \mathcal{C}(\phi^k)$, so this proves (11). □

4. *PFH and proofs of main theorems*

4.1. *Overview of PFH and non-vanishing.* Fix a closed surface Σ and a smooth area form ω. Fix an area-preserving diffeomorphism $\phi \in \mathrm{Diff}(\Sigma, \omega)$. The area form ω defines a closed two-form on the mapping torus M_ϕ, denoted by ω_ϕ. Let t be the coordinate for the interval component of $[0, 1] \times \Sigma$. Then dt pushes forward to a smooth one-form on M_ϕ. The pair (dt, ω_ϕ) forms a stable Hamiltonian structure on M_ϕ, and the Reeb vector field is a smooth vector field R generating the suspension flow $\{\psi_R^t\}_{t\in\mathbb{R}}$. The associated two-plane bundle $\ker(dt)$ is equal to the vertical tangent bundle of the fibration $M_\phi \to \mathbb{T}$, which we denote by V.

Fix some non-zero homology class $\Gamma \in H_1(M_\phi; \mathbb{Z})$. The *PFH generators* are finite sets $\Theta = \{(\gamma_i, m_i)\}$ of pairs of embedded Reeb orbits γ_i and multiplicities $m_i \in \mathbb{N}$ that satisfy the following three conditions: (1) the orbits γ_i are distinct; (2) the multiplicity m_i is 1 whenever γ_i is a hyperbolic orbit; and (3) $\sum_i m_i[\gamma_i] = \Gamma$. The chain complex $\mathrm{PFC}_*(\phi, \Gamma)$ is the free module over a commutative coefficient ring (this can be anything when Γ solves (12) for some d, but, in general, Λ must be a Novikov ring) Λ generated by the set of all PFH generators.

The differential on $\mathrm{PFC}_*(\phi, \Gamma)$ counts 'ECH index 1' J-holomorphic currents between orbit sets. The homology of this chain complex is the *PFH* $\mathrm{PFH}_*(\phi, \Gamma)$. This homology theory was constructed by Hutchings [**Hut02**] (see [**Hut14**] for a detailed exposition of the closely related theory of *embedded contact homology*). The PFH group depends only on the Hamiltonian isotopy class of ϕ; this allows us to define PFH for a degenerate map as the PFH of any sufficiently close non-degenerate Hamiltonian perturbation. We now precisely state the non-vanishing theorem for PFH.

PROPOSITION 4.1. [**CGPZ21**, Theorem 1.4] *Fix a closed surface Σ of any genus g and a smooth area form ω. Fix any area-preserving diffeomorphism $\phi \in \mathrm{Diff}(\Sigma, \omega)$. Then, for any $d > \max(2g - 2, 0)$ and any class $\Gamma \in H_1(M_\phi; \mathbb{Z})$ satisfying*

$$PD(\Gamma) = \left(\int_\Sigma \omega \right)^{-1} (d + 1 - g)[\omega_\phi] - \frac{1}{2} c_1(V), \tag{12}$$

the group $PFH(\phi, \Gamma)$ with $\mathbb{Z}/2$-coefficients is non-zero.

The result as stated in [**CGPZ21**] only asserts non-vanishing for d sufficiently large, but the explicit lower bound is not difficult to extract once the details are understood. We only need d large enough to ensure that PFH is isomorphic to the 'bar' version $\overline{\mathrm{HM}}$ of

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monopole Floer homology. Lee and Taubes [**LT12**, Theorem 1.2, Corollary 1.5] prove this isomorphism assuming that $d > \max(2g-2, 0)$. The non-vanishing theorem is a key ingredient in the following technical result.

PROPOSITION 4.2. *Fix a closed surface Σ of genus $g \geq 2$ and a smooth area form ω. Let $\phi \in \mathrm{Homeo}_0(\Sigma, \omega)$ be such that there exists non-zero $h \in H_1(\Sigma; \mathbb{Z})$ and a positive real number $c > 0$ satisfying $\mathcal{F}(\phi) = c \cdot h$. Then, for any rational number $p/q \in (0, c]$ with p and q coprime and $q > g - 1$, ϕ has a non-contractible periodic point with minimal period $\leq q + g - 1$.*

The same argument proves an analog for smooth torus maps, but we need to rule out maps without fixed points in the statement.

PROPOSITION 4.3. *Fix $\phi \in \mathrm{Diff}_0(\mathbb{T}^2, dx \wedge dy)$. Assume that ϕ has at least one fixed point. Assume further that there exists an identity isotopy Φ, non-zero $h \in H_1(\Sigma; \mathbb{Z})$ and a positive real number $c > 0$ satisfying $\mathcal{F}(\Phi) = c \cdot h$. Then, for any rational number $p/q \in (0, c]$ with p and q coprime, ϕ has a Φ-non-contractible periodic point with minimal period $\leq q$.*

Remark 4.1. We cannot extend Proposition 4.3 to homeomorphisms since the blow-up argument requires the map to be differentiable, and it is not clear that a torus homeomorphism with a fixed point can be approximated by diffeomorphisms with fixed points and the same rotation vector.

Theorems C and F, respectively, from the introduction follow from Propositions 4.2 and 4.3. We now outline the plan for the rest of the section. Section 4.2 proves Theorems A and B assuming Theorem C. Section 4.3 proves Proposition 4.2. Section 4.4 proves Proposition 4.3.

4.2. *Existence of infinitely many periodic points.* We prove Theorems A and B using Theorem C. We assume that Σ is a closed surface of genus ≥ 2, since we can reduce to this case by collapsing the boundary components.

4.2.1. *Proof of Theorem A.* Fix any $\phi \in \mathrm{Homeo}_0(\Sigma, \omega)$ with rational rotation direction. Le Calvez [**LC06**] showed that any Hamiltonian homeomorphism on a surface of genus ≥ 1 is either the identity or has periodic points of unbounded minimal period. Therefore, we consider only the case where ϕ is not Hamiltonian, in which case, it has a non-contractible periodic point, by Theorem C. The arguments from [**LC22**, §4] then show that it has periodic points of unbounded minimal period.

Here is a high-level outline of [**LC22**, §4]. Write $\widetilde{\phi}$ for the lift of ϕ to the universal cover $\widetilde{\Sigma}$ commuting with the covering translations. The non-contractible periodic point p, which we assume to have minimal period k, lifts to a point $\widetilde{p}$ such that $\widetilde{\phi}^k(\widetilde{p}) = T \cdot \widetilde{p}$ for some $T \in \pi_1(\Sigma)$. Pass to the annular cover $\widetilde{\Sigma}/T$ and compactify to produce a homeomorphism $\widehat{\phi}$ of the closed strip $[0, 1] \times \mathbb{R}$ with rotation interval containing $[0, 1/k]$. Le Calvez's refinement of the Poincaré–Birkhoff–Franks theorem [**LC22**, Theorem 2.4] then shows that either ϕ has periodic points of unbounded minimal period or $\widehat{\phi}$ does not satisfy the

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intersection property. In this latter case, a forcing argument is used to produce periodic points of unbounded minimal period regardless.

4.2.2. *Proof of Theorem B.* Fix a map $\phi \in \mathrm{Homeo}_0(\Sigma, \mu)$, where μ is a Borel probability measure of full support with $\mu(\partial\Sigma) = 0$ and $\mathcal{F}(\phi, \mu)$ is a real multiple of a rational class. We may assume, without loss of generality, that Σ is closed by collapsing the boundary.

There exists $t \in [0, 1]$ and a unique decomposition (see [**Joh70**])

$$\mu = t\mu_0 + (1 - t)\mu_1$$

with the following properties. The measures μ_0 and μ_1 are Borel probability measures, μ_0 has no atoms, μ_1 is purely atomic and they are mutually 'S-singular'. This means that, for any Borel set $E \subset \Sigma$,

$$\mu_0(E) = \sup\{\mu_0(E \cap F) \mid \mu_1(F) = 0\}, \quad \mu_1(E) = \sup\{\mu_1(E \cap F) \mid \mu_0(F) = 0\}.$$

The decomposition $\mu = t\phi_*\mu_0 + (1 - t)\phi_*\mu_1$ satisfies the same properties, so, by uniqueness, both μ_0 and μ_1 are ϕ-invariant. Now, we break up the argument into a few cases, depending on the value of t.

First, assume that $t = 0$. This implies that $\mu = \mu_1$. Therefore, μ is atomic and has full support, so it has infinitely many atoms. Since μ is ϕ-invariant, each atom is a periodic point. Therefore, ϕ has infinitely many periodic points.

Second, assume that $t = 1$. This implies that $\mu = \mu_0$. Therefore, μ is atomless and has full support. By a theorem of Oxtoby and Ulam [**OU41**, Theorem 2_1], it is homeomorphic to a smooth area measure, so ϕ is conjugate to an area-preserving homeomorphism. Since $\mathcal{F}(\phi, \mu)$ is proportional to a rational vector, the latter homeomorphism has rational rotation direction (see (1)), so we may apply Theorem A.

Third, assume that $t \in (0, 1)$. We consider two subcases, depending on the rotation vector of μ_1. If the rotation vector of μ_1 is equal to zero, then $\mathcal{F}(\phi, \mu_0) = t\mathcal{F}(\phi, \mu)$. If μ_0 does not have full support, then μ_1 has infinitely many atoms, and we obtain infinitely many periodic points as in the first case. If μ_0 has full support, we apply the Oxtoby–Ulam theorem as in the second case. If the rotation vector of μ_1 is non-zero, then it follows that ϕ must have a periodic point with non-zero rotation vector. This periodic point is non-contractible, which forces the existence of periodic points of unbounded minimal period by the argument of Le Calvez mentioned above.

4.3. *Proof of Proposition 4.2*

4.3.1. *Information from PFH.* The following lemma records the relevant information needed from Proposition 4.1. We only state and prove it for diffeomorphisms, but note that it extends to homeomorphisms by an approximation argument.

LEMMA 4.4. *Fix a closed surface Σ of genus g. Fix $\phi \in \mathrm{Diff}_0(\Sigma, \omega)$ and an identity isotopy Φ, and assume that $\mathcal{F}(\Phi) \in H_1(\Sigma; \mathbb{Q})$. Let d be the smallest integer greater than* $\max(2g - 2, 0)$ *such that*

$$(d + 1 - g) \cdot \mathcal{F}(\Phi) \in H_1(\Sigma; \mathbb{Z}).$$

Then there exists a set of simple periodic orbits $\{S_i\}_{i=1}^N$ *of periods* $\{k_i\}_{i=1}^N$ *and a set of positive integers* $\{m_i\}_{i=1}^N$ *such that*

$$\sum_{i=1}^{N} m_i k_i = d, \quad \sum_{i=1}^{N} m_i k_i \mathcal{F}(\Phi, S_i) = (d+1-g)\cdot\mathcal{F}(\Phi). \tag{13}$$

Proof. Using Lemmas 3.1 and 3.2, we compute

$$\eta^*[\omega_\phi] = \left(\int_\Sigma \omega\right)\cdot \mathrm{PD}([\mathbb{T}] + \mathcal{F}(\Phi)) \in H^2(\mathbb{T}\times\Sigma;\mathbb{R}).$$

The class

$$\Gamma = d[\mathbb{T}] + (d+1-g)\mathcal{F}(\Phi)$$

solves (12). By Proposition 4.1, there exists an orbit set $\Theta = \{(\gamma_i, m_i)\}$ such that $\sum_i m_i[\gamma_i] = \Gamma$. For each i, let S_i be the simple periodic orbit of ϕ corresponding to γ_i. We sum up the homology class computation (4) over all i to conclude that

$$d[\mathbb{T}] + (d+1-g)\mathcal{F}(\Phi) = \sum_i m_i k_i([\mathbb{T}] + \mathcal{F}(\Phi, S_i)).$$

This identity implies (13). □

4.3.2. *Blow-up.* Fix a closed surface Σ of genus ≥ 2, a smooth area form ω of area A and a diffeomorphism $\phi \in \mathrm{Diff}_0(\Sigma, \omega)$. Suppose that ϕ has a contractible fixed point p. Choose an identity isotopy Φ that fixes p (one always exists; see [**HLRS16**, Proposition 9]). We give a precise account here of how to blow up the fixed point p and cap it with a disk of any prescribed area.

Fix polar coordinates (r, θ) on $\mathbb{R}^2$, and, for any $s > 0$, denote by $D_s := \{0 \leq r < s\}$ and $A_s := D_s \backslash \{0\}$ the open disk and punctured disk, respectively, of radius s centered at the origin. Write $\dot{\Sigma} := \Sigma\backslash\{p\}$. For positive $\delta \ll 1$, there is a symplectic embedding $\iota : (A_\delta, r dr \wedge d\theta) \hookrightarrow (\Sigma, \omega)$ that is a diffeomorphism onto a punctured neighborhood of p. Next, fix a parameter $B > A$, which will be the area of the capped surface, and fix $s_1 > s_0 > 0$ such that $B - A = \pi s_0^2$ and $s_1^2 - s_0^2 = \delta^2$. Then the map $(r, \theta) \mapsto (\sqrt{s_0^2 + r^2}, \theta)$ is a symplectic embedding $\tau : (A_\delta, r dr \wedge d\theta) \hookrightarrow (D_{s_1}, dx \wedge dy)$ that identifies A_δ with the annulus $\{s_0 < r < s_1\}$. The surface $\widehat{\Sigma}$ is the surface constructed by gluing $\dot{\Sigma}$ and D_{s_1} along A_δ, using the symplectic embeddings ι and τ. The glued surface $\widehat{\Sigma}$ has a symplectic form $\widehat{\omega}$ restricting to ω on Σ and $dx \wedge dy$ on D_{s_1}. The area of $\widehat{\Sigma}$ is $A + \pi s_0^2 = B$, as desired.

The isotopy $\Phi = \{\phi^t\}_{t\in[0,1]}$ extends to an identity isotopy $\widehat{\Phi} : [0, 1] \to \mathrm{Diff}(\widehat{\Sigma}, \widehat{\omega})$. Since the isotopy fixes p, it coincides with a Hamiltonian isotopy in a neighborhood of p; we extend the generating Hamiltonian to $\widehat{\Sigma}$ to produce the desired extension. The following lemma computes the rotation vector of the extension.

LEMMA 4.5. *Fix a closed surface* Σ*, an area form* ω *and a point* $p \in \Sigma$*. Let* $\Phi : [0, 1] \to \mathrm{Diff}_0(\Sigma, \omega)$ *be an identity isotopy such that* $\Phi(t)$ *fixes* $p \in \Sigma$ *for each* t*.*

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Fix any extension $\widehat{\Phi}$ of Φ to the blown-up surface $(\widehat{\Sigma}, \widehat{\omega})$, and let $\pi : \widehat{\Sigma} \to \Sigma$ be the blow-down map. Then the rotation vector of $\widehat{\Phi}$ satisfies the identity

$$\left(\int_{\widehat{\Sigma}} \widehat{\omega} \right) \cdot \pi_* \mathcal{F}(\widehat{\Phi}) = \left(\int_{\Sigma} \omega \right) \cdot \mathcal{F}(\Phi). \tag{14}$$

Proof. Fix any $f : \Sigma \to \mathbb{T}$ and set $\widehat{f} := f \circ \pi : \widehat{\Sigma} \to \mathbb{T}$. Write $g : \Sigma \to \mathbb{R}$ and $\widehat{g} : \widehat{\Sigma} \to \mathbb{R}$ for the lifts of $f \circ \phi - f$ and $\widehat{f} \circ \widehat{\phi} - \widehat{f}$ induced by the isotopies Φ and $\widehat{\Phi}$. The function $\widehat{f}$ is equal to f on $\dot{\Sigma} \subset \widehat{\Sigma}$ and is constant on its complement. Both sets are $\widehat{\Phi}$-invariant, so $\widehat{g} = g$ on $\dot{\Sigma}$ and $\widehat{g} = 0$ elsewhere. We conclude that

$$\left(\int_{\widehat{\Sigma}} \widehat{\omega} \right) \cdot \langle \mathcal{F}(\widehat{\Phi}), \widehat{f} \rangle = \int_{\widehat{\Sigma}} \widehat{g}\, \widehat{\omega} = \int_{\dot{\Sigma}} g\, \omega = \left(\int_{\Sigma} \omega \right) \cdot \langle \mathcal{F}(\Phi), f \rangle,$$

which implies (14). □

4.3.3. *Proof for diffeomorphisms.* Fix a closed surface Σ of genus ≥ 2, a smooth area form ω of area A and $\phi \in \mathrm{Diff}_0(\Sigma, \omega)$ such that $\mathcal{F}(\phi) = c \cdot h$, where $c > 0$ and $h \in H_1(\Sigma; \mathbb{Z})$. Choose a rational number $p/q \in (0, c]$ with $q > g - 1$. Fix $B > A$ such that $A/B = p/q \cdot c^{-1}$. Blow up the fixed point to get a surface $\widehat{\Sigma}$ of area B, and extend the identity isotopy Φ to an identity isotopy $\widehat{\Phi}$ with endpoint $\widehat{\phi} \in \mathrm{Diff}_0(\widehat{\Sigma}, \widehat{\omega})$. By (14),

$$\pi_* \mathcal{F}(\widehat{\phi}) = A/B \cdot \mathcal{F}(\phi) = p/q \cdot h.$$

The map π_* is an isomorphism $H_1(\widehat{\Sigma}; \mathbb{Z}) \simeq H_1(\Sigma; \mathbb{Z})$, so we conclude that $q\mathcal{F}(\widehat{\phi}) \in H_1(\widehat{\Sigma}; \mathbb{Z})$. By Lemma 4.4, $\widehat{\phi}$ has a non-contractible periodic point z of minimal period $\leq q + g - 1$. This periodic point must lie in $\dot{\Sigma} \subset \widehat{\Sigma}$. This is because the complement $\widehat{\Sigma} \backslash \dot{\Sigma}$ is a $\widehat{\Phi}$-invariant disk, so any periodic point contained in it is contractible. The point z is therefore a non-contractible periodic point of ϕ of minimal period $\leq q + g - 1$.

4.3.4. *Proof for homeomorphisms.* We approximate $\phi \in \mathrm{Homeo}_0(\Sigma, \omega)$ by diffeomorphisms with the same rotation vector. Let $\psi \in \mathrm{Diff}_0(\Sigma, \omega)$ be any diffeomorphism such that $\mathcal{F}(\psi) = \mathcal{F}(\phi)$. It follows that $\mathcal{F}(\psi^{-1} \circ \phi) = 0$, so $\psi^{-1} \circ \phi$ lies in $\overline{\mathrm{Ham}}(\Sigma, \omega)$ [**Fat80**, §6]. Pick any sequence of Hamiltonian diffeomorphisms $h_k \in \mathrm{Ham}(\Sigma, \omega)$ approximating $\psi^{-1} \circ \phi$. The maps $\phi_k := \psi \circ h_k$ converge in the C^0 topology to ϕ and all have $\mathcal{F}(\phi_k) = \mathcal{F}(\phi)$. By the argument above, for any q such that $p/q \in (0, c]$, each diffeomorphism ϕ_k has a non-contractible periodic point z_k with minimal period $\leq q + g - 1$. Any subsequential limit z is a non-contractible periodic point of ϕ with minimal period $\leq q + g - 1$.

4.4. *Proof of Proposition 4.3.* Assume that $\phi \in \mathrm{Diff}_0(\mathbb{T}^2, dx \wedge dy)$ has a fixed point $p \in \mathbb{T}^2$. If it is not Φ-contractible, then the proof is complete. If it is Φ-contractible, then we assume, without loss of generality, that it is fixed by Φ. The same blow-up argument as in the genus ≥ 2 case proves the proposition.

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