SOME CHARACTERIZATIONS OF HEREDITARILY ARTINIAN RINGS

by DINH VAN HUYNH

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Throughout this note, rings will mean associative rings with identity and all modules are unital. A ring R is called *right artinian* if R satisfies the descending chain condition for right ideals. It is known that not every ideal of a right artinian ring is right artinian as a ring, in general. However, if every ideal of a right artinian ring R is right artinian then R is called *hereditarily artinian*. The structure of hereditarily artinian rings was described completely by Kertész and Widiger [**5**] from which, in the case of rings with identity, we get:

A ring R is hereditarily artinian if and only if R is a direct sum $S \oplus F$ of a semiprime right artinian ring S and a finite ring F.

This result plays a basic role in the study of radicals in the class of all hereditarily artinian rings (cf. Widiger and Wiegandt [**9**], Widiger [**8**]). On the other side, this is a generalized form of the Wedderburn–Artin structure theorem. Hence it would be possible to give ring and module characterizations for hereditarily artinian rings as one has done for semiprime artinian rings.

Let M be a right R-module. M is called *completely infinite* if for any distinct submodules $X \supset Y$ of M, X/Y is infinite. Now, the purpose of this note is to prove the following result.

THEOREM 1. *For a ring R the following conditions are equivalent*:

(a) *every ideal of R is right artinian*;

(b) *R is right artinian and every prime ideal of R is right artinian*;

(c) *$R = S \oplus F$, where S is semiprime artinian and F is finite*;

(d) *R is right artinian and the Jacobson radical J of R is right artinian*;

(e) *R is right artinian and J/J^2 is finite*;

(f) *every cyclic right R-module M is a direct sum $E \oplus H$ of a completely infinite injective right R-module E and a finite right R-module H.*

Proof. (a)$\Rightarrow$(b), (c)$\Rightarrow$(e) are obvious and (a)$\Leftrightarrow$(c)$\Leftrightarrow$(d) is proved in [**5**].

(b)$\Rightarrow$(c). Since the prime radical and the Jacobson radical J of a right artinian ring coincide, we get

$$J = P_1 \cap \ldots \cap P_m, \tag{1}$$

where each P_i is a prime ideal of R. If, for every P_i, the factor ring R/P_i is finite then, by (1), R/J is isomorphic to a subring of the finite ring $R/P_1 \oplus \ldots \oplus R/P_m$. By [**3**, Corollary 3, 4°], R is then finite, proving (c).

Assume now that R is infinite. Then there are exactly t prime ideals of R in (1), say $P_1, \ldots, P_t$, such that R/P_j is infinite, $j = 1, \ldots, t$. Hence for each P_j, R/P_j is an infinite prime artinian ring. On the other hand, since $P_j \supseteq J$ and P_j is right artinian as a ring, P_j is contained in a direct summand P_j' of R with finite P_j'/P_j by [7]. Hence $P_j' = P_j$ and $R = e_jRe_j \oplus P_j$ with $e_j^2 = e_j \in R$ for each $j = 1, \ldots, t$. From this we get

$$R = e_1Re_1 \oplus \ldots \oplus e_tRe_t \oplus F,$$

where each e_jRe_j is an infinite prime right (and left) artinian ring and F is a finite ring, i.e. (c) holds.

(e)$\Rightarrow$(c). By [4, Theorem 1], condition (e) implies that R is right and left artinian. Then [3, Theorem 4(a)] allows R to have a direct decomposition $R = S \oplus F$, where F is finite and S is completely infinite as a right (and left) S-module. Denote by J' the Jacobson radical of S. Then the condition (e) yields that J'/J'^2 is infinite; hence $J' = J'^2$, i.e. $J' = (0)$, proving (c).

(c)$\Rightarrow$(f). Suppose $R = S \oplus F$, where S is semiprime artinian and F is finite. Without loss of generality, all simple right S-modules are infinite. Let $X = xR$ be a cyclic right R-module. Then $X = xS + xF$. Clearly xS and xF are submodules of X, xS is completely infinite and xF is finite. Thus $xS \cap xF = (0)$, i.e. $X = xS \oplus xF$. Moreover it is also clear that xS is an injective right R-module.

(f)$\Rightarrow$(c). By hypothesis $R = E \oplus F$, where E is a completely infinite faithful injective right ideal and F is a finite right ideal of R. Let S be the sum of all finite right ideals of R. Then S is an ideal of R with $S \supseteq F$ and $S \cap E = (0)$. It follows that $S = F$. Let $f \in F$ and suppose $fE \neq (0)$. Then $fE \cong E/D$, where $D = \{x \mid x \in E, fx = 0\}$ and E/D is non-zero and finite, a contradiction. Thus $FE = (0)$ and E is an ideal of R. Finally E is a semiprime artinian ring by Osofsky's theorem [6].

The proof of Theorem is complete.

It is known that a ring R is semiprime artinian if and only if every simple right R-module is projective. Let K be a finite field. Then the polynomial ring $R = K[x]$ has the property that every simple right R-module is finite. This shows that there are non-artinian rings whose infinite simple right modules are projective. Hence it would be interesting to determine the class of all rings R satisfying the condition that all infinite simple right R-modules are projective. In the noetherian case we can prove the following proposition.

PROPOSITION 2. *Let R be a right noetherian ring such that every infinite simple right R-module is projective. Then the artinian radical A of R is a hereditarily artinian ring.*

Proof. For the definition and properties of the artinian radical of a noetherian ring, we refer to Chatters and Hajarnavis [2]. Let X be a submodule of the right R-module A such that A/X contains an infinite minimal submodule Y/X. Then, by assumption, $Y = Y_1 \oplus X$, where Y_1 is an infinite minimal right ideal of R, $Y_1 = y_1R \cong R/D$ with $D = \{r \mid r \in R, y_1r = 0\}$. Hence, as a right R-module, $R = Y_1 \oplus D$; therefore Y_1 is idempotent. From this it is not difficult to see that A is a hereditarily artinian ring (not necessarily with identity).

Concerning the question stated before Proposition 2 and Theorem 1, we would like to mention an interesting result of Chatters [1] which says: A ring R is right noetherian if and only if every cyclic right R-module is a direct sum of a projective module and a noetherian module.

For any ring R and right R-module X there is a unique maximal completely infinite submodule C of X by Zorn's lemma. By [**3**, Theorem 4(a)], if R is a right and left artinian ring then, for each right R-module X, $X = C \oplus X'$. In general, it would be interesting to consider the question: when is C a direct summand of X for each right R-module X? In particular, for which rings R is every cyclic right R-module the direct sum of a completely infinite submodule and a finite submodule?

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REFERENCES

1. A. W. Chatters, A characterisation of right noetherian rings, *Quart. J. Math. Oxford Ser.* (2) **33** (1982), 65–69.

2. A. W. Chatters and C. R. Hajarnavis, *Rings with chain conditions* (Pitman, 1980).

3. Dinh van Huynh, A note on artinian rings, *Arch. Math.* (*Basel*) **33** (1979), 546–553.

4. Dinh van Huynh, A note on rings with chain conditions, preprint (Institute of Mathematics, Hanoi, 1984).

5. A. Kertész and A. Widiger, Artinsche Ringe mit artinschem Radikal, *J. Reine Angew. Math.* **242** (1970), 8–15.

6. B. L. Osofsky, Noncommutative rings whose cyclic modules have cyclic injective hulls, *Pacific J. Math.* **25** (1968), 331–340.

7. A. Widiger, Zur Zerlegung artinscher Ringe, *Publ. Math. Debrecen* **21** (1974), 193–196.

8. A. Widiger, Lattice of radicals for hereditarily artinian rings, *Math. Nachr.* **84** (1978), 301–309.

9. A. Widiger and R. Wiegandt, Theory of radicals for hereditarily artinian rings, *Acta Sci. Math.* (*Szeged*) **39** (1977), 303–312.

INSTITUTE OF MATHEMATICS
P.O.BOX 631 BO HÔ
HANOI–VIETNAM