Cantor–Bernstein Sextuples for Banach Spaces

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Abstract. Let X and Y be Banach spaces isomorphic to complemented subspaces of each other with supplements A and B. In 1996, W. T. Gowers solved the Schroeder–Bernstein (or Cantor–Bernstein) problem for Banach spaces by showing that X is not necessarily isomorphic to Y. In this paper, we obtain a necessary and sufficient condition on the sextuples (p, q, r, s, u, v) in $\mathbb{N}$ with $p + q \geq 1$, $r + s \geq 1$ and $u, v \in \mathbb{N}^*$, to provide that X is isomorphic to Y, whenever these spaces satisfy the following decomposition scheme

$$A^u \sim X^p \oplus Y^q, \quad B^v \sim X^r \oplus Y^s.$$

Namely, $\Phi = (p - u)(s - v) - (q + u)(r + v)$ is different from zero and Φ divides $p + q$ and $r + s$. These sextuples are called Cantor–Bernstein sextuples for Banach spaces. The simplest case $(1, 0, 0, 1, 1, 1)$ indicates the well-known Pełczyński's decomposition method in Banach space. On the other hand, by interchanging some Banach spaces in the above decomposition scheme, refinements of the Schroeder–Bernstein problem become evident.

1   Introduction.

Let X and Y be Banach spaces. We write $X \sim Y$ if X is isomorphic to Y. If $n \in \mathbb{N}^* = \{1, 2, 3, \cdots\}$, then X^n denotes the sum of n copies of X, $X \oplus X \oplus \cdots \oplus X$. It will be useful to denote $X^0 = \{0\}$. We recall that Y is isomorphic to a complemented subspace of X if there exists a Banach space A such that $X \sim Y \oplus A$. In this case, we say that A is a supplement of Y in X.

Suppose that X and Y are Banach spaces isomorphic to complemented subspaces of each other, that is, there exist Banach spaces A and B such that

$$(1.1) \qquad\qquad X \sim Y \oplus A, \quad Y \sim X \oplus B.$$

In 1996, W. T. Gowers [12] solved the so-called Schroeder–Bernstein (or Cantor–Bernstein) problem for Banach spaces by showing that X is not necessarily isomorphic to Y, (see also [2–7, 13]). So in the Banach spaces theory, when (1.1) holds it is necessary to search for some additional conditions to guarantee that $X \sim Y$. In this direction, the well-known Pełczyński's decomposition method states that $X \sim Y$ whenever X and Y satisfy (1.1) for some Banach spaces A and B, and the following decomposition scheme holds

$$(1.2) \qquad\qquad X \sim X^2, \quad Y \sim Y^2.$$

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See [8, 9] for some extensions of this decomposition method. Although the supplements A and B from (1.1) do not appear in the decomposition scheme (1.2), it is easy to check (see [1, p. 63] that Pełczyński's decomposition method works because the decomposition scheme (1.2) implies the existence of supplements A and B verifying (1.1) and

$$A \sim X, \quad B \sim Y.$$

Thus it follows immediately from (1.1) that $X \sim Y$. In other words, Pełczyński's decomposition method works because under the hypothesis of the decomposition scheme (1.2) it is possible to choose the supplements A and B conveniently close to X and Y. This remark leads naturally to the following problem.

Problem 1 How close to X and Y must be the supplements A and B from (1.1) to guarantee that $X \sim Y$?

For example, $X \sim Y$ whenever there exist supplements A and B satisfying (1.1) and one of the following decomposition schemes:

$$(1.3) \qquad\qquad A^4 \sim X^5, \quad B^5 \sim Y^{24},$$

$$(1.4) \qquad\qquad A \sim X^7 \oplus Y^8, \quad B \sim X^6 \oplus Y^9,$$

$$(1.5) \qquad\qquad A^2 \sim X^{15} \oplus Y, \quad B^3 \sim X^2 \oplus Y^4$$

The case of the decomposition scheme (1.4) follows directly from (1.1). The other cases are consequences of the main result of this paper, (see Theorem 1.2). To be more precise we rewrite Problem 1 as follows.

Problem 2 Is it possible to determine all decomposition schemes that are similar to those of (1.3), (1.4), and (1.5) such that if added to (1.1) yield $X \sim Y$?

In order to present the solution of Problem 2 it is useful to introduce the following definition.

Definition 1.1 A sextuple (p, q, r, s, u, v) in $\mathbb{N}$, with $p + q \geq 1$, $r + s \geq 1$, and $u, v \in \mathbb{N}^*$ is a Cantor–Bernstein sextuple for Banach spaces (in short, CBS) if $X \sim Y$ whenever the Banach spaces X and Y satisfy (1.1) for some Banach spaces A and B and the following decomposition scheme holds:

$$(1.6) \qquad\qquad A^u \sim X^p \oplus Y^q, \quad B^v \sim X^r \oplus Y^s.$$

We also say that $\Phi = (p-u)(s-v) - (q+u)(r+v)$ is the Cantor–Bernstein discriminant of the sextuple (p, q, r, s, u, v).

Our main aim is to show that we know enough solutions of Schroeder–Bernstein problem for Banach spaces to characterize the CBS. Indeed, by using some Banach spaces introduced in [5, 13], (see Remarks 2.1 and 2.2), we will prove the following characterization of CBS in terms of the Cantor–Bernstein discriminant Φ.

Theorem 1.2   *A sextuple* (p, q, r, s, u, v) *in* $\mathbb{N}$ *with* $p + q \geq 1$, $r + s \geq 1$, *and* $u, v \in \mathbb{N}^*$ *is a CBS if and only if its Cantor–Bernstein discriminant* Φ *is different from zero and* Φ *divides* $p + q$ *and* $r + s$.

It is interesting to remark that if we interchange A and X in the first condition of decomposition scheme (1.6) and also B and Y in the second one, then the situation becomes rather complicated. We do not even know how to solve the following problem, (see also [10]).

Problem 3 (**The Square-Cube Cantor-Bernstein Problem for Banach Spaces**)
Give nonisomorphic Banach spaces X and Y satisfying (1.1) for some Banach spaces A and B and the following decomposition scheme

$$X^2 \sim Y^2 \oplus A^3, \quad Y^2 \sim X^2 \oplus B^3.$$

2   Preliminaries

To obtain the characterization of the sextuples in $\mathbb{N}$ that are CBS, we need to recall some recent results on Banach spaces that are isomorphic to complemented subspaces of each other.

Remark 2.1   Gowers and Maurey [13, p. 563] constructed Banach spaces X_t for every $t \in \mathbb{N}$, $t \geq 2$, having the following property: $X_t^m \sim X_t^n$, with $m, n \in \mathbb{N}^*$, if and only if m is equal to n modulo t.

We recall that two Banach spaces X and Y are said to be totally incomparable if no infinite dimensional subspace of X is isomorphic to a subspace of Y.

Remark 2.2   Fix two totally incomparable Banach spaces X and Y from the class of spaces constructed in [11]. Then by [5] there exists a Banach space Z such that
(i) $Z \sim Z^2$ [5, p. 31];
(ii) $Z \sim Z \oplus X^m \oplus Y^m, \forall m \in \mathbb{N}$ [5, p. 31];
(iii) Z is not isomorphic to $Z \oplus X^m, \forall m \in \mathbb{N}^*$ [5, Theorem 3.4].

Remark 2.3   In [8] a quintuple (p, q, r, s, t) in $\mathbb{N}$ with $p + q \geq 2$, $r + s + t \geq 3$, $(r, s) \neq (0, 0)$, and $u \in \mathbb{N}^*$ was said to be a *Schroeder-Bernstein quintuple for Banach spaces* (in short, SBQ) if $X \sim Y$ whenever the Banach spaces X and Y satisfy (1.1) for some Banach spaces A and B and the following decomposition scheme holds:

$$X \sim X^p \oplus Y^q, \quad Y^u \sim X^r \oplus Y^s.$$

The number $\nabla = (p - 1)(s - u) - rq$ was called the discriminant of the quintuple (p, q, r, s, u). The following characterization of SBQ was obtained in [8, Corollary 4.2]. Let (p, q, r, s, v) be a quintuple in $\mathbb{N}$ with $p + q \geq 2$, $r + s + v \geq 3$, $(r, s) \neq (0, 0)$, and $v \in \mathbb{N}^*$. Then (p, q, r, s, v) is a SBQ if and only ∇ is different from zero and ∇ divides $p + q - 1$ and $r + s - v$.

3 On Sextuples in $\mathbb{N}$ with $\Phi = 0$

The purpose of this section is to show that every sextuple in $\mathbb{N}$ with Cantor–Bernstein discriminant Φ equal to zero is not a CBS.

Proposition 3.1 *If a sextuple (p, q, r, s, u, v) in $\mathbb{N}$ with $p + q \geq 1$, $r + s \geq 1$, and $u, v \in \mathbb{N}^*$ has Cantor–Bernstein discriminant Φ equal to zero, then it is not a CBS.*

Proof First notice that $\Phi = -(u - p)(r + s) - (p + q)(r + v)$. Hence if $p \leq u$, then $\Phi \leq -(p + q) \leq -1$. Therefore, by our hypothesis we can assume that $p > u$. Take $m = q + u$ and $n = p - u$. Let X, Y, and Z be the Banach spaces mentioned in Remark 2.2. So by Remark 2.2(i) and (ii)

$$Z \oplus X^m \sim Z \oplus Y^n \oplus Z \oplus X^{m+n}, \quad Z \oplus Y^n \sim Z \oplus X^m \oplus Z \oplus Y^{m+n}.$$

Now observe that $m(p - u) - n(q + u) = 0$ and since $\Phi = 0$, $-m(r + v) + n(s - v) = 0$. Thus again by Remark 2.2(i) and (ii) we see that

$$(Z \oplus X^{m+n})^u \sim (Z \oplus X^m)^p \oplus (Z \oplus Y^n)^q, \quad (Z \oplus Y^{m+n})^v \sim (Z \oplus X^m)^r \oplus (Z \oplus Y^n)^s.$$

Next assume that

$$(3.1) \qquad\qquad Z \oplus X^m \sim Z \oplus Y^n.$$

Thus adding X^n to both sides of (3.1) and using Remark 2.2(ii) we deduce

$$Z \oplus X^{m+n} \sim Z \oplus X^n \oplus Y^n \sim Z.$$

This is absurd by Remark 2.2(iii), because $m + n = p + q \neq 0$. So (p, q, r, s, u, v) is not a CBS. ∎

4 Necessary Conditions for a Sextuple in $\mathbb{N}$ with $\Phi \neq 0$ to be a CBS

In this section we prove that if a sextuple in $\mathbb{N}$ with $\Phi \neq 0$ is a CBS, then Φ divides $p + q$ and $r + s$; see Proposition 4.4. To state this result we need three lemmas.

Lemma 4.1 *Let $p, q, r, s, u, v \in \mathbb{N}$ with $p + q \geq 1$, $r + s \geq 1$ and $u, v \in \mathbb{N}^*$. Suppose that there exist $i, j, t \in \mathbb{N}^*$ with $t \geq 2$ satisfying*

(i)   *t divides $i(p - u) + j(q + u)$,*
(ii) *t divides $i(r + v) + j(s - v)$,*
(iii) *t does not divide $j - i$.*

Then (p, q, r, s, u, v) is not a CBS.

Proof Let $n \in \mathbb{N}^*$ such that $nt - j + i > 0$ and $nt - i + j > 0$. Since $j + (nt - j + i) - i = nt$ and $i + (nt - i + j) - j = nt$, we have by the property of X_t mentioned in Remark 2.1 that

$$X_t^i \sim X_t^j \oplus X_t^{nt-j+i}, \quad X_t^j \sim X_t^i \oplus X_t^{nt-i+j}.$$

Notice that by the conditions (i) and (ii) we deduce

$$X_t^{(nt-j+i)u} \sim X_t^{ip} \oplus X_t^{jq}, \quad X_t^{(nt-i+j)v} \sim X_t^{ir} \oplus X_t^{js}.$$

Furthermore, according to condition (iii) we conclude that X_t^i is not isomorphic to X_t^j. Consequently (p, q, r, s, u, v) is not a CBS. ∎

Lemma 4.2 *Let (p, q, r, s, u, v) be a sextuple in $\mathbb{N}$ with $p+q \geq 1$, $r+s \geq 1$, $u, v \in \mathbb{N}^*$, and Cantor–Bernstein discriminant $\Phi \geq 2$. Suppose that there exist integers α and β satisfying*

(i) $\alpha(s - v) > \beta(q + u)$,
(ii) $\beta(p - u) > \alpha(r + v)$,
(iii) Φ *does not divide* $\beta(p + q) - \alpha(r + s)$.

Then (p, q, r, s, u, v) is not a CBS.

Proof Let $t = \Phi$ and consider the linear system

$$(4.1) \qquad i(p - u) + j(q + u) = \alpha t, \quad i(r + v) + j(s - v) = \beta t.$$

The only solution of (4.1) is $i = \alpha(s - v) - \beta(q + u)$ and $j = \beta(p - u) - \alpha(r + v)$. It follows from (i), (ii), and (iii) that $i > 0$, $j > 0$, and t does not divide $j - i = \beta(p+q) - \alpha(r+s)$. Moreover, clearly t divides $i(p-u) + j(q+u)$ and $i(r+v) + j(s-v)$. Hence Lemma 4.1 implies that (p, q, r, s, u, v) is not a CBS. ∎

Taking $t = -\Phi$ and proceeding as in the proof of Lemma 4.2 we obtain the following lemma.

Lemma 4.3 *Let (p, q, r, s, u, v) be a sextuple in $\mathbb{N}$ with $p+q \geq 1$, $r+s \geq 1$, $u, v \in \mathbb{N}^*$, and Cantor–Bernstein discriminant $\Phi \leq -2$. Suppose that there exist integers α and β such that*

(i) $\alpha(s - v) < \beta(q + u)$,
(ii) $\beta(p - u) < \alpha(r + v)$,
(iii) Φ *does not divide* $\beta(p + q) - \alpha(r + s)$.

Then (p, q, r, s, u, v) is not a CBS.

Proposition 4.4 *If a sextuple $(p, q, , r, s, u, v)$ in $\mathbb{N}$ with $p+q \geq 1$, $r+s \geq 1$, $u, v \in \mathbb{N}^*$, and Cantor–Bernstein discriminant Φ different from zero is a CBS, then Φ divides $p + q$.*

Proof Assume that Φ does not divide $p + q$. We will distinguish two cases: $\Phi \geq 2$ and $\Phi \leq -2$.

Case 1: $\Phi \geq 2$. Therefore

$$(4.2) \qquad (p - u)(s - v) > (q + u)(r + v).$$

Thus $p \neq u$ and there are two subcases: $p < u$ and $p > u$.

Subcase 1.1: $p < u$. By (4.2), $(s - v)/(q + u) < (r + v)/(p - u)$. Take $\alpha = q + u$ and $\beta = s - v - 1$. Hence $\beta/\alpha < (s - v)/(q + u)$ and $\beta(p + q) - \alpha(r + s) = \Phi - (p + q)$. By Lemma 4.2 (p, q, r, s, u, v) is not a CBS.

Subcase 1.2: $p > u$. By (4.2), $(r + v)/(p - u) < (s - v)/(q + u)$. Fix $m \in \mathbb{N}^*$ such that

$$\frac{r + v}{p - u} + \frac{1}{m(p - u)} < \frac{s - v}{q + u},$$

and take $\alpha = m(p - u)$ and $\beta = m(r + v) + 1$. Then we have $(r + v)/(p - u) < \beta/\alpha < (s - v)/(q + u)$ and $\beta(p + q) - \alpha(r + s) = -m\Phi + p + q$. According to Lemma 4.2 (p, q, r, s, u, v) is not a CBS.

Case 2: $\Phi \leq -2$. Consequently

(4.3) $$(p - u)(s - v) < (q + u)(r + v).$$

There are three subcases: $p < u$, $p = u$, and $p > u$.

Subcase 2.1: $p < u$. By (4.3), $(r + v)/(p - u) < (s - v)/(q + u)$. Take $\alpha = q + u$ and $\beta = s - v + 1$. Hence $(s - v)/(q + u) < \beta/\alpha$ and $\beta(p + q) - \alpha(r + s) = \Phi + p + q$. By Lemma 4.3 we conclude that (p, q, r, s, u, v) is not a CBS.

Subcase 2.2: $p = u$. Thus $\Phi = -(q + u)(r + v)$. Take $\alpha = q + u$ and $\beta = s - v + 1$. Since $\alpha > 0$, $(q + u)(s - v) < (s - v + 1)(q + u)$ and $\beta(p + q) - \alpha(r + s) = \Phi + p + q$, it follows from Lemma 4.3 that (p, q, r, s, u, v) is not a CBS.

Subcase 2.3: $p > u$. By (4.3), $(s - v)/(q + u) < (r + v)/(p - u)$. Pick $m \in \mathbb{N}^*$ such that

$$\frac{s - v}{q + u} < \frac{r + v}{p - u} - \frac{1}{m(p - u)},$$

and take $\alpha = m(p - u)$ and $\beta = m(r + v) - 1$. Hence $(s - v)/(q + u) < \beta/\alpha < (r + v)/(p - u)$ and $\beta(p + q) - \alpha(r + s) = -m\Phi - (p + q)$. Once again Lemma 4.3 implies that (p, q, r, s, u, v) is not a CBS.

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5 Sufficient Conditions for a Sextuple (p, q, r, s, u, v) in $\mathbb{N}$ with $\Phi \neq 0$ To Be a CBS

In this last section, we show that the necessary conditions stated in the previous section for a sextuple (p, q, r, s, u, v) in $\mathbb{N}$ with $\Phi \neq 0$ to be a CBS are also sufficient. This completes the proof of Theorem 1.2.

Proposition 5.1 *Let (p, q, r, s, u, v) be a sextuple in $\mathbb{N}$ with $p + q \geq 1$, $r + s \geq 1$, $u, v \in \mathbb{N}^*$. If its Cantor–Bernstein discriminant Φ is different from zero, Φ divides $p + q$ and $r + s$, then (p, q, r, s, u, v) is a CBS.*

Proof Let X and Y be Banach spaces satisfying (1.1) and the decomposition scheme (1.6) for some Banach spaces A and B. We will prove that $X \sim Y$. First observe that $u \leq p + 1$. Otherwise, since

$$\Phi = -(u - p)(r + s) - (p + q)(r + v) < -(p + q),$$

it would follow that Φ does not divide $p + q$. Next notice that by using the second condition of (1.1) in the first one, we obtain

$$(5.1) \qquad\qquad\qquad\qquad X \sim X \oplus A \oplus B.$$

Now adding $A \oplus B$ to both sides of (5.1) we get

$$X \sim X \oplus A \oplus B \sim X \oplus A^2 \oplus B^2.$$

Therefore by induction we conclude $X \sim X \oplus A^u \oplus B^u$. So by the first condition of (1.6) and the second condition of (1.1) we have that

$$X \sim X^{p+1} \oplus Y^q \oplus B^u \sim X^{p-u+1} \oplus X^u \oplus B^u \oplus Y^q \sim X^{p-u+1} \oplus Y^{q+u}.$$

Analogously we infer that $Y \sim X^{r+v} \oplus Y^{s-v+1}$. Hence the following decomposition scheme holds

$$X \sim X^{p-u+1} \oplus Y^{q+u}, \quad Y \sim X^{r+v} \oplus Y^{s-v+1}.$$

Since the discriminant ∇ of the quintuple $(p - u + 1, q + u, r + v, s - v + 1, 1)$ is equal to $(p - u)(s - v) - (r + v)(q + u) = \Phi$, it follows by hypothesis that $\nabla \neq 0$, ∇ divides $(p - u + 1) + (q + u) - 1 = p + q$ and $(r + v) + (s - v + 1) - 1 = r + s$. Thus according to Remark 2.3 $X \sim Y$. ∎

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