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EXTREME MEASURES WITH GIVEN MOMENTS OR MARGINALS

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ABSTRACT. We study the generalized moment problem and the marginal constraints problem. We connect them when the measures have a finite support.

The extreme points of the convex set of solutions with a finite support are determined in both problems.

For the moment problem, they are shown to span in the weak topology the set of all the solutions.

RÉSUMÉ. Nous étudions le problème des moments généralisés et le problème des constraintes de marginales. Nous les relions lorsque les mesures considérées sont à support fini.

Nous déterminons les points extrémaux du convexe des solutions à support discret dans les deux cas.

Nous montrons pour le problème des moments qu'ils engendrent pour la topologie de la convergence étroite toutes les solutions.

1. **Introduction.** This paper is devoted to the study of extreme positive measures of which moments or marginal measures are fixed.

MARGINAL CONSTRAINTS PROBLEM. Let d finite measures ρ_j be given on d sets I_j. The problem consists in determining positive measures defined on $\prod_{j=1}^{d} I_j$ such that for all $f \in L^1_{\rho_j}(I_j)$

CONDITION 1.

$$\int_{\prod I_j} f(\lambda_j)\, d\mu(\lambda_1, \ldots, \lambda_d) = \int_{I_j} f(\lambda_j)\, d\rho_j(\lambda_j) \quad j = 1, \ldots, d.$$

Note that the marginal constraints problem is also known as the problem of *doubly stochastics measures*, see Lindenstrauss [12]. When the given measures have a finite support, it is known in statistics as the problem of *contingency tables* with fixed marginal counts. See Lauritzen [11] for details on this matter.

MOMENT PROBLEM. Let I be a compact subset of a Polish space.

Let $\phi_K = (\phi_k)_{k \in K}$, for K a finite subset of $\mathbb{N}^d$ (or $\mathbf{Z}^d$), be a family of linearly independent measurable functions defined on I.

Let $\mathcal{M}^\phi$ be the space of measures on I such that

$$\int_I \phi_k(\lambda)\, d\mu(\lambda) < +\infty, \quad \text{for } k \in K.$$

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Let Φ_K be the linear functional defined on $\mathcal{M}^\phi$ by

$$\Phi_K\mu = \Big(\int_I \phi_k(\lambda)\, d\mu(\lambda)\Big)_{k\in K}.$$

We denote by

$\mathcal{M}_+^\phi$ the subset of $\mathcal{M}^\phi$ of nonnegative measures on I,

$\mathcal{M}_+^d$ the subset of $\mathcal{M}_+^\phi$ of measures having a discrete support,

$\mathcal{M}_+^f$ the subset of $\mathcal{M}_+^d$ of measures having a finite support,

S_μ the support of any measure μ.

A sequence $c = (c_k)_{k\in K} \in \mathbb{R}^{|K|}$ being given, the moment problem consists in determining measures belonging to $\mathcal{M}_+^\phi$ such that

CONDITION 2. $\Phi_K\mu = c$.

Throughout this paper, c will be supposed such that $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi \neq \emptyset$; for conditions ensuring it, see Krein and Nudelman [8], Berg and Christensen and Ressel [1], Cassier [2].

The moment problem and the marginal constraints problem are studied in measure theory, transportation theory and linear programming theory, see Kemperman [7], Dantzig [3].

These problems are different in data, functions and a truncated sequence in the former case and positive measures in the latter. There always exists a solution in the latter case (the product of the given measures) while conditions are required in the former case. If conditions of support are added, problems of existence are introduced in the latter case, see Shortt [14].

On the other hand, for measures having a discrete support, the studies of the extreme points of the convex sets of solutions are similar. Moreover, we prove that these problems can be connected for measures with a finite support.

In both cases, algebraic arguments enable us to characterize the discrete extreme measures of the general moment problem and of the marginal constraints problem, see Section 2.1 and Section 4.1. They are determined uniquely by their supports.

We prove that the extreme measures of $\Phi_K^{-1}(c)\cap\mathcal{M}_+^f$ span $\Phi_K^{-1}(c)\cap\mathcal{M}_+^d$ by finite convex combinations and $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$ by weak convergence, under suitable assumptions on the family $(\phi_k)_{k\in\mathbb{N}^d}$, see Section 2.3.

Similar results ensue for restrictive moment problems such as the Markov problem, see Section 2.2.

Note that the density results cannot be directly deduced from the classical density theorems; see Fuchssteiner and Lusky [4] for further detail about convex sets.

As regards the truncated moment problem, these results seem to be the first about extreme measures in a general context. Studies have been made for $I \subset \mathbb{R}$, *e.g.* by Karlin and Shapley [6] for the power moment problem on $[0, 1]$. More precise results have been obtained for the Tchebicheff families by Landau [9] and [10], and Krein and Nudelman [8]. But these are based on the factorization of polynomials and on the properties of their zeros so cannot be extended to higher dimensions or to other sets.

As regards the marginal constraints problem, the extreme doubly stochastics measures are characterized by density properties in Lindenstrauss [12]; this yields a support characterization for the extreme measures with fixed marginals in the finite symmetric case, $I_1 = \cdots = I_d$ and $|I_1| < \infty$. No results for the infinite case seem to exist.

2. Moment problem.

2.1. *Discrete extreme measures.* A measure having a discrete support can be written

$$\mu = \sum_{\lambda_h \in \mathcal{S}_\mu} \mu_h \delta_{\lambda_h}.$$

Hence the moment problem is reduced to determine positive sequences of reals satisfying Condition 2.

Let us define the following matrix, for any finite subset S of I,

$$M_S^K = \left(\phi_k(\lambda_h)\right)_{\lambda_h \in S}^{k \in K}. \tag{1}$$

THEOREM 2.1. *Let* $\mu \in \Phi_K^{-1}(c) \cap \mathcal{M}_+^d$.

Then μ *is an extreme point of* $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$ *if and only if* μ *has a finite support* $\mathcal{S}_\mu$ *such that* $|\mathcal{S}_\mu| = \mathrm{Rank}[M_{\mathcal{S}_\mu}^K]$.

Note that

1) if the cardinal of the support of a discrete measure is greater than the number of fixed moments, this measure cannot be an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$.

2) for the power moment problem over $I \subset \mathbb{R}$, $|\mathcal{S}_\mu| = \mathrm{Rank}[M_{\mathcal{S}_\mu}^K]$ is equivalent to $|\mathcal{S}_\mu| \leq |K|$.

PROOF. If $\mu \in \Phi_K^{-1}(c) \cap \mathcal{M}_+^f$, then (μ_h), for $h = 1, \ldots, |\mathcal{S}_\mu|$ is a solution of the following system with $|\mathcal{S}_\mu|$ unknowns and $|K|$ equations

$$\sum_{\lambda_h \in \mathcal{S}_\mu} \phi_k(\lambda_h) m_h = c_k, \qquad \text{for } k \in K. \tag{2}$$

The rank of this system is $\mathrm{Rank}[M_{\mathcal{S}_\mu}^K]$ and the dimension of the affine space of solutions $\mathcal{A}$ is $|\mathcal{S}_\mu| - \mathrm{Rank}[M_{\mathcal{S}_\mu}^K]$. The measure μ is associated to a positive solution. If μ is a convex combination of measures belonging to $\Phi_K^{-1}(c) \cap \mathcal{M}_+^f$, their supports are included in $\mathcal{S}_\mu$. They are associated to nonnegative solutions of System (2). It is not possible if the system has only one solution *i.e.*, if $\mathrm{Rank}[M_{\mathcal{S}_\mu}^K] = |\mathcal{S}_\mu|$. μ is then an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$.

Conversely, if $|\mathcal{S}_\mu| > \mathrm{Rank}[M_{\mathcal{S}_\mu}^K]$, System (2) has solutions distinct from (μ_h). Then this point belongs to a line Δ included in $\mathcal{A}$. Since $(\mu_h) \in \mathbb{R}_{+*}^{|\mathcal{S}_\mu|}$, the set $\Delta \cap \mathbb{R}_+^{|\mathcal{S}_\mu|}$ is not reduced to a single point. Hence (μ_h) can be written as a convex combination of points (η_h) and (ν_h) belonging to $\Delta \cap \mathbb{R}_+^{|\mathcal{S}_\mu|}$ and μ as a convex combination of the associated measures η and ν which obviously belong to $\Phi_K^{-1}(c) \cap \mathcal{M}_+^f$. So μ is not an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$.

If $\mu \in \Phi_K^{-1}(c) \cap (\mathcal{M}_+^d \setminus \mathcal{M}_+^f)$, let us take $S \subset \mathcal{S}_\mu$ such that $|K| < |S| < +\infty$. Set

$$\sum_{\lambda_h \in S} \phi_k(\lambda_h)\alpha_h = 0, \quad k \in K.$$

This system has $|K|$ equations and $|S|$ unknowns.

Its rank is $\mathrm{Rank}[M_S^K] < |S|$, hence it has solutions distinct from 0.

Let $(\alpha_i)_{i=1,\ldots,|S|}$ be one of them. Define a measure on I by

$$\tilde{\alpha}_{\lambda_i} = \begin{cases} M\alpha_i & \text{if } \lambda_i \in S, \text{ and} \\ 0 & \text{if not.} \end{cases}$$

where $M = [\min(\mu_i / i = 1, \ldots, |S|)][\max(|\alpha_i| / i = 1, \ldots, |S|)]^{-1}$.

Then $\mu - \tilde{\alpha}$ and $\mu + \tilde{\alpha}$ belong to $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$, and μ is not an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$. ■

THEOREM 2.2. *Let $\mu \in \Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$. If $\phi_k \in L^2(\mu)$ for $k \in K$, then μ is an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$ if and only if μ has a finite support $\mathcal{S}_\mu$ such that* $|\mathcal{S}_\mu| = \mathrm{Rank}[M_{\mathcal{S}_\mu}^K]$.

PROOF. Let μ be an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$.

Let $\mathcal{F}_\phi$ be the subspace of $L^2(\mu)$ spanned by $(\phi_k)_{k \in K}$ and let Π be the orthogonal projection on $\mathcal{F}_\phi$. Let $\mathcal{S}$ denote the space of step functions on I.

If $\mathcal{F}_\phi \neq L^2(\mu)$, then a subspace $\mathcal{E}$ of $\mathcal{S}$ exists such that

$$\dim \mathcal{F}_\phi \leq \dim \mathcal{E} < +\infty.$$

Hence Π restricted to $\mathcal{E}$ is not an injective function. Let $f \in \mathcal{E}$ be orthogonal to $\mathcal{F}_\phi$, and set $d\nu = (\mu - f)\, d\mu$ and $d\eta = (\mu + f)\, d\mu$. Since ν and η belong to $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$, μ is not an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$.

If $\mathcal{F}_\phi = L^2(\mu)$, then μ has a finite support and Theorem 2.1 yields the result. ■

2.2. *Bounded extreme measures.* If we add to the moment problem the extra condition $\mu \in \mathcal{M}_+^1$ *i.e.*,

CONDITION 3. $\mu \in \mathcal{M}_+^d$ and $\mu_\lambda \leq 1$ for all $\lambda \in \mathcal{S}_\mu$,
the characterization of the extreme measures ensues from Theorem 2.1.

COROLLARY 2.1. *Let $\mu \in \mathcal{M}_+^d$.*

μ is an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^d$ satisfying Condition 3 if and only if one of the two following conditions is satisfied:

a) $\mu \equiv 1$ over $\mathcal{S}_\mu$.

b) $|\tilde{\mathcal{S}}_\mu| < +\infty$ and $|\tilde{\mathcal{S}}_\mu| = \mathrm{Rank}[M_{\tilde{\mathcal{S}}_\mu}^K]$, with $\tilde{\mathcal{S}}_\mu = \{\lambda \in \mathcal{S}_\mu / \mu_\lambda \neq 1\}$ and $M_{\tilde{\mathcal{S}}_\mu}^K$ as defined by Relation (1).

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PROOF. If μ satisfies a), the result is obvious.

If μ satisfies b) and if $2\mu = \nu + \eta$, firstly, $\nu \equiv \eta \equiv \mu \equiv 1$ over $\mathcal{S}_\mu \setminus \tilde{\mathcal{S}}_\mu$; secondly, $(\nu_\lambda)_{\lambda \in \tilde{\mathcal{S}}_\mu}$ and $(\eta_\lambda)_{\lambda \in \tilde{\mathcal{S}}_\mu}$ satisfy the following system

$$\sum_{\tilde{\mathcal{S}}_\mu} m_\lambda \phi_k(\lambda) = \sum_{\tilde{\mathcal{S}}_\mu} \mu_\lambda \phi_k(\lambda).$$

Since this system has a unique solution (see the proof of Theorem 2.1), we get $\nu \equiv \eta \equiv \mu$ over $\tilde{\mathcal{S}}_\mu$. Thus, μ is an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$.

Conversely, if μ satisfies neither a) nor b), let

$$S = \mathcal{S}_\mu \quad \text{if } |\tilde{\mathcal{S}}_\mu| < +\infty \quad \text{or}$$
$$S \subset \mathcal{S}_\mu \quad \text{with } |K| < |S| < +\infty, \quad \text{if not.}$$

We have $|S| > \operatorname{Rank}[M_S^K]$, and the homogeneous system

$$\sum_{\lambda \in S} m_\lambda \phi_k(\lambda) = 0$$

has a solution $(\nu_\lambda) \not\equiv 0$. Set

$$\tilde{\nu}_\lambda = \begin{cases} M'\nu_\lambda, & \text{if } \lambda \in S, \\ 0, & \text{if } \lambda \in \mathcal{S}_\mu \setminus S, \end{cases}$$

where $M' = [\min_S(\mu_\lambda, 1 - \mu_\lambda)][\max_S(|\nu_\lambda|)]^{-1}$.

Both the associated measures $\mu + \nu$ and $\mu - \nu$ belong to $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$. Thus μ is not an extreme point of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$. ∎

Note that all the extreme measures of the Markov problem can be determined in the same way. Their densities are characteristic functions of Borel subsets of I, see Girardin [5].

The Markov problem is the moment problem restricted to measures absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, with a bounded density. See Krein and Nudelman [8] for more on the matter.

2.3. *Spanning by extreme measures.* If the family ϕ_K satisfies suitable hypothesis, the extreme measures determined in Section 2.1 span all the measures having the same moments, by convex combination for the measures having a finite support, and in the weak topology for the others.

THEOREM 2.3. *Let $\mu \in \Phi_K^{-1}(c) \cap \mathcal{M}_+^f$. If $\phi_0 \equiv 1$ over I, then μ is a convex combination of less than $|\mathcal{S}_\mu| + 2$ extreme measures of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^f$, supported within $\mathcal{S}_\mu$.*

PROOF. Let $\mathcal{M}_+^{\mathcal{S}_\mu}$ be the subspace of $\mathcal{M}_+^\phi$ of measures supported within $\mathcal{S}_\mu$.

Since $\Phi_K^{-1}(c) \cap \mathcal{M}_+^{\mathcal{S}_\mu}$ is a bounded convex set in $\mathbb{R}^{|\mathcal{S}_\mu|}$, Caratheodory's theorem can be applied (see Krein and Nudelman [8]). It yields that μ is a convex combination of less than $|\mathcal{S}_\mu| + 2$ extreme measures of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^{\mathcal{S}_\mu}$.

If these measures are convex combinations of other measures belonging to $\Phi_K^{-1}(c)\cap \mathcal{M}_+^f$, the latter measures are supported within $\mathcal{S}_\mu$, hence they belong to $\Phi_K^{-1}(c)\cap \mathcal{M}_+^{\mathcal{S}_\mu}$ too, which is not possible.

And the result follows. ∎

THEOREM 2.4. *If $(\phi_k)_{k\in\mathbb{N}^d}$ is a dense subset in $\mathcal{C}(I)$, and if $\phi_0 \equiv 1$ over I, then the extreme measures of $\Phi_K^{-1}(c)\cap\mathcal{M}_+^f$ span $\Phi_K^{-1}(c)\cap\mathcal{M}_+^\phi$ in the weak topology.*

PROOF. Let $\mu \in \Phi_K^{-1}(c)\cap\mathcal{M}_+^\phi$.

If $f \in \mathcal{C}(I)$ (*i.e.*, continuous on I), then f is the limit of a sequence of generalized polynomials, $P_H = \sum_{k\in H} a_k^H \phi_k$.

Set for every H,

$$\bar\mu = (\overline{\mu_k})_{k\in K\cup H}, \quad \text{where } \overline{\mu_k} = \int_I \phi_k(\lambda)\,d\mu(\lambda).$$

By Lemma 2.1 (see below), there exists a measure $\mu^H \in \Phi_{K\cup H}^{-1}(\bar\mu)\cap\mathcal{M}_+^f$ such that

$$\sum_{\lambda\in S_{\mu^H}} \mu_\lambda^H\phi_k(\lambda) = \overline{\mu_k} \quad \text{for } k\in K\cup H.$$

Then

$$\int_I f(\lambda)\,d\mu(\lambda) - \int_I f(\lambda)\,d\mu^H(\lambda) = \int_I (f-P_H)(\lambda)\,d\mu(\lambda) - \int_I (f-P_H)(\lambda)\,d\mu^H(\lambda)$$

and

$$\left|\int_I f(\lambda)\,d\mu(\lambda) - \int_I f(\lambda)\,d\mu^H(\lambda)\right| \le \int_I |f-P_H|(\lambda)\,d\mu(\lambda) + \int_I |f-P_H|(\lambda)\,d\mu^H(\lambda).$$

Hence

$$\left|\int_I f(\lambda)\,d\mu^H(\lambda) - \int_I f(\lambda)\,d\mu(\lambda)\right| \longrightarrow 0,$$

and the result ensues by Theorem 2.3. ∎

To achieve Theorem 2.4 proof, we shall describe $\Phi_K(\mathcal{M}_+^\phi)$ by use of convex analysis. Note that it also gives a necessary and sufficient condition of existence of solutions for the moment problem. It holds true for any family ϕ_K of measurable functions, even if they are not continuous.

LEMMA 2.1. *If $\phi_0\equiv 1$ over I, then $\Phi_K(\mathcal{M}_+^\phi) = \Phi_K(\mathcal{M}_+^f)$.*

PROOF. Obviously,

$$\Phi_K(\mathcal{M}_+^f) = \left\{ \left(\sum_{\lambda\in S}\rho_\lambda\phi_k(\lambda)\right)_{k\in K} \in \mathbb{R}^{|K|} / \rho_\lambda \ge 0, S\subset I, |S| < +\infty \right\}, \tag{3}$$

i.e., $\Phi_K(\mathcal{M}_+^f)$ is the hull cone of the set

$$\mathcal{U}_K^\phi = \left\{ \left(\phi_k(\lambda)\right)_{k\in K} \in \mathbb{R}^{|K|} / \lambda \in I \right\}.$$

We shall prove that if $c \in \Phi_K(\mathcal{M}_+^\phi)$ then $c \in \Phi_K(\mathcal{M}_+^f)$, separately for interior and border points.

(a) $\Phi_K(\mathcal{M}_+^f) \subset \Phi_K(\mathcal{M}_+^\phi) \subset \bar{\Phi}_K(\mathcal{M}_+^f)$,
where $\bar{\Phi}_K(\mathcal{M}_+^f)$ denotes the closure of $\Phi_K(\mathcal{M}_+^f)$ in $\mathbb{R}^{|K|}$. We have

$$\bar{\Phi}_K(\mathcal{M}_+^f) = \Big\{ c \in \mathbb{R}^{|K|} / \forall (a_k)_{k\in K} \in \mathbb{R}^{|K|},\ \text{if} \sum_K a_k\phi_k \geq 0 \text{ on } I,\ \text{then} \sum_K a_k c_k \geq 0 \Big\}.$$

If $c \in \Phi_K(\mathcal{M}_+^\phi)$, and if $\mu \in \Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$, then

$$\int_I \sum_K a_k\phi_k(\lambda)\, d\mu(\lambda) = \sum_K a_k c_k,$$

and if $(a_k)_{k\in K}$ is such that $\sum_K a_k\phi_k$ is nonnegative on I, then

$$\int_I \sum_K a_k\phi_k(\lambda)\, d\mu(\lambda) \geq 0.$$

Hence $c \in \bar{\Phi}_K(\mathcal{M}_+^f)$.

(b) $\Phi_K(\mathcal{M}_+^\phi) \cap \partial\,\Phi_K(\mathcal{M}_+^f) \subset \Phi_K(\mathcal{M}_+^f)$.
We shall prove it by induction on the number of moments.

If $|K| = 1$, or if $|K| > 1$ and $\Phi_K(\mathcal{M}_+^f) = \mathbb{R}^{|K|}$, the result is obvious.

If $c \in \Phi_K(\mathcal{M}_+^\phi) \cap \partial\,\Phi_K(\mathcal{M}_+^f)$, and if $\Phi_K(\mathcal{M}_+^f) \neq \mathbb{R}^{|K|}$, then c belongs to a hyperplane of support $\mathcal{H}$ for $\Phi_K(\mathcal{M}_+^f)$.

We shall prove that

$$\Phi_K(\mathcal{M}_+^\phi) \cap \mathcal{H} \subset \Phi_K(\mathcal{M}_+^f) \cap \mathcal{H}, \quad \text{for any } \mathcal{H}, \tag{4}$$

by two steps.

(I) $\Phi_K(\mathcal{M}_+^\phi) \cap \mathcal{H} \subset C_{\mathcal{H}}^\phi$, where $C_{\mathcal{H}}^\phi$ is the hull cone of $\mathcal{U}_K^\phi \cap \mathcal{H}$.
The equation of $\mathcal{H}$ is

$$\sum_K h_k b_k = 0, \tag{5}$$

with $\sum_K h_k^2 = 1$ and $\sum_K h_k\phi_k$ nonnegative on I.

If $c \in \Phi_K(\mathcal{M}_+^\phi) \cap \mathcal{H}$ and if $\mu \in \Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$, then

$$\int_I \sum_K h_k\phi_k(\lambda)\, d\mu(\lambda) = \sum_K h_k c_k = 0.$$

Thus $\sum_K h_k\phi_k(\lambda) = 0$, on $I \setminus E$, where $\mu(E) = 0$. Set

$$\phi_k^* = \begin{cases} \phi_k & \text{on } I \setminus E \\ 0 & \text{on } E. \end{cases}$$

Then $c_k = \int_I \phi_k^*(\lambda)\, d\mu(\lambda)$, for $k \in K$ and $\sum_K h_k\phi_k^* \equiv 0$ on I. There exists $j \in K$ such that $h_j \neq 0$; we have

$$\phi_j^* = \sum_{k\neq j} \frac{h_k}{h_j}\phi_k^* \quad \text{and} \quad c_j = \sum_{k\neq j} \frac{h_k}{h_j} c_k. \tag{6}$$

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So $(c_k)_{K\setminus\{j\}} \in \Phi^*_{K\setminus\{j\}}(\mathcal{M}_+^\phi)$, and by induction, $(c_k)_{K\setminus\{j\}} \in \Phi^*_{K\setminus\{j\}}(\mathcal{M}_+^f)$, *i.e.*, by Relation (3),

$$c_k = \sum_{\lambda\in S} \rho_\lambda \phi_k^*(\lambda), \quad \text{for } k \in K \setminus \{j\}.$$

Finally, Relations (6) imply

$$c_j = \sum_{k\neq j} \frac{h_k}{h_j} \sum_S \rho_\lambda \phi_k^*(\lambda) = \sum_S \rho_\lambda \phi_j^*(\lambda).$$

Hence $(c_k)_{k\in K} \in \Phi^*_{K\setminus\{j\}}(\mathcal{M}_+^f) \cap \mathcal{H}$.

We prove in (II) that

$$\Phi_K(\mathcal{M}_+^f) \cap \mathcal{H} = C^\phi_{\mathcal{H}}, \quad \text{so that} \quad \Phi^*_{K\setminus\{j\}}(\mathcal{M}_+^f) \cap \mathcal{H} = C^{\phi^*}_{\mathcal{H}}.$$

Since obviously $C^{\phi^*}_H \subset C^\phi_{\mathcal{H}}$, it yields Relation (4).

(II) $C^\phi_{\mathcal{H}} = \Phi_K(\mathcal{M}_+^f) \cap \mathcal{H}$.

Obviously, $C^\phi_{\mathcal{H}} \subset \Phi_K(\mathcal{M}_+^f) \cap \mathcal{H}$.

Conversely, if $c \in \Phi_K(\mathcal{M}_+^f) \cap \mathcal{H}$, then by Relations (3) and (5),

$$c_k = \sum_S \rho_\lambda \phi_k(\lambda), \quad \rho_\lambda > 0 \text{ and } \sum_K h_k c_k = 0,$$

thus

$$\sum_S \rho_\lambda \sum_K h_k \phi_k(\lambda) = 0, \quad \text{for all } \lambda \in S,$$

and since $\sum_K h_k\phi_k$ is nonnegative on I and ρ_λ is positive for all $\lambda \in S$

$$\sum_K h_k\phi_k(\lambda) = 0, \quad \text{for all } \lambda \in S.$$

Hence $\big(\phi_k(\lambda)\big)_{k\in K} \in \mathcal{H}$ for all $\lambda \in S$, and the result follows. ∎

Note that if $\phi_k \in L^2(\mu)$ for all $k \in K$, (*e.g.* if functions ϕ_k are bounded functions) then Theorem 2.2 yields a shorter proof of this lemma: if c is an extreme point of $\Phi_K(\mathcal{M}_+^\phi)$, any extreme measure of $\Phi_K^{-1}(c) \cap \mathcal{M}_+^\phi$ has a finite support. Hence $c \in \Phi_K(\mathcal{M}_+^f)$ and Lemma 2.1 ensues since $\Phi_K(\mathcal{M}_+^\phi)$ and $\Phi_K(\mathcal{M}_+^f)$ are compact convex sets.

3. **Connection between moments and marginals.** Here we show the connection between the moment problem and the marginals constraints problem for measures having a finite support.

THEOREM 3.1. *Let* $A = (x_i)_{i\in I}$ *and* $B = (y_j)_{j\in J}$, *where* $I = \{0,\ldots,n\}$ *and* $J = \{0,\ldots,m\}$.

Let $(\phi_k)_{k\in I}$ *and* $(\psi_l)_{l\in J}$ *be families of functions defined respectively on A and on B, such that*

- $\phi_0 \equiv \psi_0 \equiv 1$

- *the ranks of the matrices M_E^I and M_F^J (as defined by Relation (1)) are maximal for all $E \subset A$ and $F \subset B$.*

Let $(\Phi, \Psi)_{I,J}$ denote the linear functional associated to the family of functions $\big((\phi_k)_{k\in I}, (\psi_l)_{l\in J}\big)$.

Let ρ and τ be two positive measures on A and B.

Let r_k denote the ϕ-moment of order k for ρ and let t_l denote the ψ-moment of order l for τ.

Let $c=(c_{kl})_{(k,l)\in I\times J}$ be such that $c_{k0} = r_k$ for $k \in I$ and $c_{0l} = t_l$ for $l \in J$.

Then $(\Phi, \Psi)_{I,J}^{-1}(c) \cap \mathcal{M}_+^f = M_{\rho\tau}(I, J)$.

Note that this theorem holds true for power moments. It holds true too for the functions $\phi_k(x_i) = \mathbf{1}_{i=k}$, $\psi_l(y_j) = \mathbf{1}_{j=l}$, which give a canonical map of the marginal constraints problem onto moment problems.

PROOF. For $k \in I$ and $l \in J$,

$$r_k = \sum_{i\in I} \rho_i \phi_k(x_i) \quad \text{and} \quad t_l = \sum_{j\in J} \tau_j \psi_l(y_j). \tag{7}$$

So, if $\mu \in M_{\rho\tau}(I, J)$, then obviously μ belongs to $(\Phi, \Psi)_{IJ}^{-1}(c) \cap \mathcal{M}_+^f$.

Conversely, if $\mu \in (\Phi, \Psi)_{IJ}^{-1}(c) \cap \mathcal{M}_+^f$, then μ verifies, for $k \in I$ and $l \in J$,

$$\sum_{(i,j)\in I\times J} \mu_{ij}\phi_k(x_i) = r_k \quad \text{and} \quad \sum_{(i,j)\in I\times J} \mu_{ij}\psi_l(y_j) = t_l.$$

Set for the marginal values of μ respectively on A and on B,

$$b_i = \sum_{j\in J} \mu_{ij} \quad \text{and} \quad d_j = \sum_{i\in I} \mu_{ij}.$$

Then $\{(b_i)_{i\in I}, (d_j)_{j\in J}\}$ is a solution of the following system with $n + m + 2$ unknowns,

$$\begin{cases} \sum_i \phi_k(x_i) b_i = r_k, & k \in I \\ \sum_j \psi_l(y_j)\, d_j = t_l, & l \in J. \end{cases}$$

By Relations (7), the family $\{(\rho_i)_{i\in I}, (\tau_j)_{j\in J}\}$ is the unique solution of this Cramer system.

Thus $b_i = \rho_i$ for $i \in I$, and $d_j = \tau_j$ for $j \in J$, which means that μ belongs to $M_{\rho\tau}(I, J)$. ■

4. Marginal constraints problem.

4.1. *Discrete extreme measures.* In order to simplify notation, we shall study the problem for $d = 2$. Hence, two sequences of reals are given,

$$\rho \equiv (\rho_{x_i})_{x_i\in A} \equiv (\rho_i)_{i\in I}, \quad \text{and} \quad \tau \equiv (\tau_{y_j})_{y_j\in B} \equiv (\tau_j)_{j\in J},$$

for I and J subsets of $\mathbb{N}$, such that

$$0 \le \rho_i, \text{ for all } i \in I, \text{ and } 0 \le \tau_j, \text{ for all } j \in J,$$
$$\text{and } \sum_I \rho_i < +\infty, \ \sum_J \tau_j < +\infty.$$

The measures can be identified with the tables of their values $(\mu_{ij})_{(i,j)\in I\times J}$ which satisfy the following condition

CONDITION 4. $\sum_{j\in J} \mu_{ij} = \rho_i$ for all $i \in I$ and $\sum_{i\in I} \mu_{ij} = \tau_j$ for all $j \in J$.

NOTATION. Let $M_{\rho\tau}(I,J)$ denote the set of measures satisfying Condition 4.

Let $m_{ij} = \min(\rho_i, \tau_j)$, for all $(i,j) \in I \times J$.

We call a table $[k,l]$, either the values of μ_{ij} for k index i and l index j, or the associated index set $I_k \times J_l$.

Let $I_n = [1, \ldots, n]$ and $I_n^n = [1, \ldots, n] \times [1, \ldots, n]$ for any $n \in \mathbb{N}$.

Set $\rho_{ij}^n = \begin{cases} \rho_i & \text{if } i \in I_n \\ 0 & \text{if not,} \end{cases}$ and $\tau_j^n = \begin{cases} \tau_j & \text{if } j \in I_n \\ 0 & \text{if not.} \end{cases}$

For any $\mu \in M_{\rho\tau}(I,J)$, set $\mu_{ij}^n = \begin{cases} \mu_{ij} & \text{if } (i,j) \in I_n^n \\ 0 & \text{if not,} \end{cases}$ and let μ^n be the associated measure defined on I_n^n.

THEOREM 4.1. *Let $I \subseteq \mathbb{N}$, $J \subseteq \mathbb{N}$. Let $\mu \in M_{\rho\tau}(I,J)$. μ is an extreme point of $M_{\rho\tau}(I,J)$ if and only if for every table $[k,l]$ in $I \times J$, $|\{(i,j) \in [k,l] / \mu_{ij} \neq 0\}| < k + l$.*

EXAMPLE.

$$\begin{array}{cccc} & \mathbf{1} & & \mathbf{2} \\ & + \circ \circ + & & + \circ + + \\ & \circ + + \circ & & \circ + \circ \circ \\ & \circ \circ + + & & \circ \circ + + \end{array}$$

$+$ represents $\mu_{ij} \neq 0$, and $\circ$ represents $\mu_{ij} = 0$.

The measure is extreme in the first case only.

Note that when the sets are finite and equal, Lindenstrauss [12] deduced this theorem from results on doubly stochastic measures, proving it by induction on the cardinal of the set of indexes.

The proof of Theorem 4.1 ensues from Theorem 4.2 and Lemmas 4.1 and 4.2.

THEOREM 4.2. *Let $I \subseteq \mathbb{N}$, $J \subseteq \mathbb{N}$. Let $\mu \in M_{\rho\tau}(I,J)$. Then μ is an extreme point of $M_{\rho\tau}(I,J)$ if and only if μ^n is an extreme point of $M_{\rho^n\tau^n}(I_n^n)$ for all $I_n^n \subset I \times J$.*

PROOF. Let us take $I = J = \mathbb{N}$. Let $\mu \in M_{\rho\tau}(\mathbb{N}, \mathbb{N})$.

If n exists such that μ^n is not an extreme point of $M_{\rho^n\tau^n}(I_n^n)$, then we have $\mu^n = (\nu^n + \eta^n)/2$, where $\nu^n \in M_{\rho^n\tau^n}(I_n^n)$ and $\eta^n \in M_{\rho^n\tau^n}(I_n^n)$. Set

$$\nu_{ij} = \begin{cases} \nu_{ij}^n & \text{if } (i,j) \in I_n^n \\ \mu_{ij} & \text{if not,} \end{cases} \quad \text{and} \quad \eta_{ij} = \begin{cases} \eta_{ij}^n & \text{if } (i,j) \in I_n^n \\ \mu_{ij} & \text{if not.} \end{cases}$$

ν and η belong to $M_{\rho\tau}(\mathbb{N}, \mathbb{N})$ so μ is not an extreme point of $M_{\rho\tau}(\mathbb{N}, \mathbb{N})$.

Conversely, suppose that μ^n is an extreme point of $M_{\rho^n\tau^n}(I_n^n)$ for all $I_n^n \subset I \times J$.

Let $\varepsilon > 0$. Let $(i_0, j_0) \in \mathbb{N} \times \mathbb{N}$ with $\mu_{i_0 j_0} \neq 0$. It exists $n \geq \max(i_0, j_0)$ such that

$$\sum_{(i,j) \notin I_n^n} \mu_{ij} \leq \frac{\varepsilon}{2}.$$

Set $e_{ij} = \mathbf{1}_{\{(i,j)\}}$ and let $\mathcal{F}_e$ be the space spanned by $(e_{ij})_{(i,j) \in I_n^n}$.

Applying Theorem 2.2 for $\phi_K = (\mathbf{1}_{\{i\} \times I_n})_{i \in I_n} \cup (\mathbf{1}_{I_n \times \{j\}})_{j \in I_n}$ and $I = I_n^n$ yields $\mathcal{F}_e = L^2(\mu^n)$. Hence

$$e_{i_0 j_0} = \sum_{i=1}^{n} a_i \mathbf{1}_{\{i\} \times I_n} + \sum_{j=1}^{n} b_j \mathbf{1}_{I_n \times \{j\}}, \quad \mu^n \text{ a.e.}$$

We have

$$\begin{cases} a_{i_0} + b_{j_0} = 1 \\ a_i + b_j = 0 \qquad \text{if } (i,j) \in I_n^n \text{ and } \mu_{ij} \neq 0, \end{cases}$$

so we can choose $(a_i)_{i \in I_n}$ and $(b_j)_{j \in I_n}$ belonging to $\{-1, 0, 1\}$. Set then

$$f = \sum_{i=1}^{n} a_i \mathbf{1}_{\{i\} \times I_n} + \sum_{j=1}^{n} b_j \mathbf{1}_{I_n \times \{j\}}.$$

We have

$$\|f\|_\infty \leq 2 \max_{i \in I_n, j \in I_n} (|a_i|, |b_j|)$$

and

$$\begin{aligned} \|e_{i_0 j_0} - f\|_{L^1(\mu)} &= \int_{I_n^n} |e_{i_0 j_0} - f| \, d\mu^n + \int_{\mathbb{N}^2 \setminus I_n^n} |e_{i_0 j_0} - f| \, d\mu \\ &= \int_{\mathbb{N}^2 \setminus I_n^n} |e_{i_0 j_0} - f| \, d\mu \leq \|f\|_\infty \sum_{(i,j) \notin I_n^n} \mu_{ij} \leq \varepsilon. \end{aligned}$$

Hence the space $\mathcal{F}_\phi$ spanned by ϕ_K is dense in $L^1(\mu)$.

Suppose $\mu = (\nu + \eta)/2$, with $\nu \in M_{\rho\tau}(\mathbb{N}, \mathbb{N})$ and $\eta \in M_{\rho\tau}(\mathbb{N}, \mathbb{N})$.

By the Radon-Nikodym theorem, $d\nu = g d\mu$ and $d\eta = h d\mu$, with $g \in L^1(\mu)$ and $h \in L^1(\mu)$. $\mathcal{F}_\phi$ is dense in $L^1(\mu)$ so $h = g$ and μ is not an extreme point of $M_{\rho\tau}(\mathbb{N}, \mathbb{N})$. ■

LEMMA 4.1. *Let $\mu \in M_{\rho\tau}(I, J)$.*
If there exists a table $[k, l]$ in $I \times J$ such that $|\{(i,j) \in [k, l] / \mu_{ij} \neq 0\}| \geq k + l$ then μ is not an extreme point of $M_{\rho\tau}(I, J)$.

PROOF. Let $\mu \in M_{\rho\tau}(I, J)$.

Let $I_k \times J_l$ be the $[k, l]$ table in which μ is non zero for at least $k + l$ points.

If (i_0, j_0) is such that $\mu_{i_0 j_0} = m_{i_0 j_0}$, we have either $\rho_{i_0} \leq \tau_{j_0}$ or $\tau_{i_0} < \rho_{j_0}$. Let us suppose the former is true. Then $\mu_{i_0 j_0} = \rho_{i_0}$ and $\mu_{i_0 j} = 0$ for $j \neq j_0$. Hence μ is not zero in $(I_k \setminus \{i_0\}) \times J_l$ for at least $k + l - 2$ points (i,j). Continuing this way, we may suppose $\mu_{ij} \neq m_{ij}$ for all $(i,j) \in I_k \times J_l$.

Set the following system with $(\nu_{ij})_{(i,j) \in I_k \times J_l}$ as unknowns, where $\nu_{ij} = 0$ if $\mu_{ij} = 0$,

$$\begin{cases} \sum_{j \in J_l} \nu_{ij} = \mu_{i.}, & i \in I_k \\ \sum_{i \in I_k} \nu_{ij} = \mu_{.j}, & j \in J_l \end{cases}$$

where $\mu_{.j} = \sum_{i \in I_k} \mu_{ij}$ and $\mu_{i.} = \sum_{j \in J_l} \mu_{ij}$, and $\sum_{i \in I_l} \mu_{i.} = \sum_{j \in J_k} \mu_{.j}$.

This system has $k + l$ equations and $k + l$ or more unknowns with a constraint.

So the dimension of the set of its solutions is 1 or more and the dimension of the set of solutions of the homogeneous associated system is 1 or more, too. Let $(\nu_{ij}^*)_{(i,j) \in I_k \times J_l}$ be a solution of the homogeneous associated system and set

$$\nu_{ij} = \begin{cases} \mu_{ij}, & \text{if } (i,j) \in I \times J \setminus I_k \times J_l, \\ 0, & \text{if } \mu_{ij} = 0, \\ M^* \nu_{ij}^*, & \text{if not,} \end{cases}$$

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where

$$M^* = \Big[\min_{I_k \times J_l}(m_{ij} - \mu_{ij}, \mu_{ij})\Big]\Big[\max_{I_k \times J_l}(|\nu^*_{ij}|)\Big]^{-1}.$$

If ν is the associated measure on $I \times J$, then both $(\mu - \nu)$ and $(\mu + \nu)$ belong to $M_{\rho\tau}(I, J)$. Thus μ is not an extreme point of $M_{\rho\tau}(I, J)$. ∎

LEMMA 4.2. *Let $|I| < +\infty$ and $|J| < +\infty$. Let $\mu \in M_{\rho\tau}(I, J)$.*

If for every table $[k, l]$ in $I \times J$, $|\{(i,j) \in [k, l] / \mu_{ij} \neq 0\}| < k + l$, then μ is an extreme point of $M_{\rho\tau}(I, J)$.

PROOF. Let $\eta \in M_{\rho\tau}(I, J)$ be supported within the support of μ. We shall prove that $\eta = \mu$.

Let us take $I = \{1, \ldots, n\}$ and J=$\{1, \ldots, m\}$ with $n \geq m$.

There exists $i_o \in I$ such that $\mu_{i_oj} = 0$ for $(m - 1)$ index j of J (or $j_o \in J$ as such), since otherwise $\mu_{ij} \neq 0$ for $2n \geq n + m$ points, which is contrary to the lemma hypothesis.

Hence η is different from zero for less than $(n + m)$ points (i,j) of $I \times J$.

Let $j_o \in J$ be such that $\mu_{i_oj_o} \neq 0$. Then $\eta_{i_oj_o} = \mu_{i_oj_o} = \rho_{i_o}$ is determined uniquely, and necessarily $\rho_{i_o} < \tau_{j_o}$.

There remains $(n + m - 2)$ points for which $\mu_{ij} \neq 0$, comprised in a $[n - 1, m]$ table, $(I \setminus \{i_o\}) \times J$.

For this table, set

$$\begin{cases} \tau^*_{j_o} = \tau_{j_o} - \rho_{i_o}, \\ \tau^*_j = \tau_j, \\ \rho^*_i = \rho_i. \end{cases}$$

Using the same argument again, we get a second point (i_1, j_1) such that $\eta_{i_1j_1} = \mu_{i_1j_1}$ $(= m^*_{i_1j_1})$.

In every table $[k, l]$, for $k \leq n$, $l \leq m$, there are less than $(k + l)$ points (i,j) such that $\mu_{ij} \neq 0$. Using the above argument again $(n + m - 1)$ times yields $\eta_{ij} = \mu_{ij}$ for all (i,j) in $I \times J$.

Thus μ is an extreme point of $M_{\rho\tau}(I, J)$. ∎

4.2. *Spanning by extreme measures.* A result similar to Theorem 2.3 is obviously true here.

LEMMA 4.3. *Let $\mu \in M_{\rho\tau}(I, J)$, with $|I| < +\infty$ and $|J| < +\infty$.*

Then μ is a convex combination of less than $|S_\mu| + 2$ extreme measures of $M_{\rho\tau}(I, J)$, supported within S_μ.

On the other hand, discrete and non-discrete measures cannot have the same marginal measures, so results such as Theorem 2.4 cannot be proved here.

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