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# ON THE DIFFERENCE PROPERTY OF THE CLASS OF POINTWISE DISCONTINUOUS FUNCTIONS AND OF SOME RELATED CLASSES

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**1. Introduction.** Let $\mathbf{R}$ denote the set of real numbers. For $f:\mathbf{R} \rightarrow \mathbf{R}$ and $h \in \mathbf{R}$, the difference function $\Delta_h f$ is defined by

$$\Delta_h f(x) = f(x + h) - f(x) \quad (x \in \mathbf{R}).$$

The function $H:R \rightarrow R$ is called additive if it satisfies Cauchy's equation

$$H(x + y) = H(x) + H(y) \quad \text{for every } x, y \in \mathbf{R}.$$

Let $\mathcal{F}$ be a class of real valued functions defined on $\mathbf{R}$. $\mathcal{F}$ is said to have the difference property if, for every function $f:\mathbf{R} \rightarrow \mathbf{R}$ satisfying $\Delta_h f \in \mathcal{F}$ for every $h \in \mathbf{R}$, there exists an additive function $H$ such that $f - H \in \mathcal{F}$.

It was conjectured by P. Erdős that the class of continuous functions has the difference property. This conjecture was proved by N. G. de Bruijn in [**1**], where the difference property of several other classes was verified as well. (For other references, see [**6**].)

In this paper we are going to investigate the classes of real functions having bounded oscillation on a set everywhere dense in $\mathbf{R}$. Let $\omega(f, x)$ denote the oscillation of $f$ at $x$. For every $K \geqq 0$, we shall denote by $C_K$ the class of those functions $f:\mathbf{R} \rightarrow \mathbf{R}$ for which $\{x \in R;\ \omega(f, x) \leqq K\}$ is everywhere dense in $R$.

Then $f \in C_0$ if and only if $f$ is continuous at the points of an everywhere dense set i.e., if $f$ is pointwise discontinuous ([**5**], p. 105). If $f \in C_K$ then $\{x \in R;\ \omega(f, x) < K + \epsilon\}$ is everywhere dense and open for $\epsilon > 0$ and hence $\{x \in R;\ \omega(f, x) \leqq K\}$ is an everywhere dense $G_\delta$ set. Since the intersection of countable many everywhere dense $G_\delta$ sets is likewise everywhere dense, it follows that

(i) $\quad C_0 = \bigcap_{K>0} C_K$ and

(ii) $\quad f \in C_{K_1}, g \in C_{K_2}$ implies $f + g \in C_{K_1+K_2} \quad (K_1, K_2 \geqq 0)$.

Our main purpose is to prove the following theorems.

THEOREM 1. *The class $C_0$ has the difference property.*

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THEOREM 2. *If $K > 0$ and if $f: R \to R$ satisfies $\Delta_h f \in C_K$ for every $h \in \mathbf{R}$, then there is an additive function $H$ such that $f - H \in C_{3K}$.*

Let $\alpha$ denote the greatest lower bound of those positive numbers $\lambda$ for which $\Delta_h f \in C_K (h \in \mathbf{R})$ implies $f - H \in C_{\lambda K}$ for some additive $H$. By Theorem 2, $\alpha \leqq 3$. On the other hand, we shall prove in the next section that $\alpha \geqq 1.5$. This implies, in particular, that for $K > 0 C_K$ does not have the difference property. We do not know the exact value of $\alpha$. It is very likely that either $\alpha = 1.5$ or $\alpha = 3$ holds.

Theorem 2 might suggest that the class $\cup_{K>0} C_K$ has the difference property. However, this is not the case. Let $\mathscr{M}$ denote the class of those functions $f: \mathbf{R} \to \mathbf{R}$ which are bounded on some interval.

An example given by de Bruijn shows that no class between the class of bounded functions and $\mathscr{M}$ can have the difference property. We shall discuss this example, together with some other classes not having the difference property in Section 4.

The proof of Theorems 1 and 2, making use of the results of Section 2, will be given in Section 3.

**2. An example and some preliminary results.** Suppose $|\Delta_h f(x)| \leqq 1$ holds for every $h$ and $x$. Then diam $R(f) \leqq 1$ and hence $\omega(f, x) \leqq 1$ for every $x \in \mathbf{R}$. Now suppose that, for every $h \in R$, $|\Delta_h f(x)| \leqq 1$ holds on $R$ except for at most finitely many $x \in \mathbf{R}$. Our next theorem shows that in this case $\omega(f, x) > 1$ can hold for every $x \in R$.

THEOREM 3. *There exists a function $f: \mathbf{R} \to \mathbf{R}$ such that*

(i) $\{x \in R; |\Delta_h f(x)| > 1\}$ *is finite for every* $h \in \mathbf{R}$;

(ii) $\omega(f, x) \geqq 3$ *for every* $x \in \mathbf{R}$.

*Proof.* Let $\mathbf{Q}$ denote the set of rationals and let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of $\mathbf{Q}$. Since $\mathbf{R}$ is a linear space over the field

$$\mathbf{Q}(\sqrt{2}) = \{r + \sqrt{2}s; r, s \in \mathbf{Q}\}$$

we can select a basis $U \subset \mathbf{R}$ such that every $x \in \mathbf{R}$ has a unique representation of form

$$x = \sum_{u \in U} \alpha_x(u) \cdot u \quad \text{where } \alpha_x(u) \in \mathbf{Q}(\sqrt{2})$$

for every $u \in U$ and only a finite number of $\alpha_x(u)$'s is different from zero. We can suppose $1 \in U$. Now we put

$$\sqrt{2}\mathbf{Q} = \{\sqrt{2}r; r \in \mathbf{Q}\},$$
$$A = \{x \in R \backslash (\mathbf{Q} \cup \sqrt{2}\mathbf{Q}); \alpha_x(1) = r_k + r_m\sqrt{2}, k > m\},$$
$$B = \{x \in R \backslash (\mathbf{Q} \cup \sqrt{2}\mathbf{Q}); \alpha_x(1) = r_k + r_m\sqrt{2}, k \leqq m\}$$

and define

$$f(x) = \begin{cases} 0 \text{ if } x \in \mathbf{Q} \\ 1 \text{ if } x \in A \\ 2 \text{ if } x \in B \\ 3 \text{ if } x \in \sqrt{2}\mathbf{Q}\setminus\{0\}. \end{cases}$$

Then $f$ is well-defined and (ii) is fulfilled. In order to prove (i) we show first that $(\mathbf{Q} + h) \cap B$ is finite for every $h \in \mathbf{R}$. ($X + h$ stands for $\{x + h; x \in X\}$.) In fact, if $\alpha_h(1) = r_k + r_m\sqrt{2}$ then $r \in \mathbf{Q}$, $r + h \in B$ implies

$$r + r_k \in \{r_1, r_2, \dots, r_m\}$$

and hence

$$|(\mathbf{Q} + h) \cap B| = m.$$

This implies that $\mathbf{Q} \cap (B + h) = [(\mathbf{Q} - h) \cap B] + h$ is finite for every $h$ as well. We can see in the same way that $(\sqrt{2}\mathbf{Q} + h) \cap A$ and $\sqrt{2}\mathbf{Q} \cap (A + h)$ are finite for every $h \in \mathbf{R}$. Since

$$|(\mathbf{Q} + h) \cap \sqrt{2}\mathbf{Q}| \leqq 1,$$

we obtain that the set

$$\begin{aligned} V_h = {} & [(\mathbf{Q} + h) \cap B] \cup [\mathbf{Q} \cap (B + h)] \cup [\sqrt{2}\mathbf{Q} \cap (A + h)] \\ & \cup [(\sqrt{2}\mathbf{Q} + h) \cap A] \cup [(\mathbf{Q} + h) \cap \sqrt{2}\mathbf{Q}] \\ & \cup [\mathbf{Q} \cap (\sqrt{2}\mathbf{Q} + h)] \end{aligned}$$

is finite. Now $x + h \notin V_h$ implies $\Delta_h f(x) = -1$, 0 or 1 which proves (i).

COROLLARY 4. *If $K > 0$ then the class $C_K$ does not have the difference property. Moreover, there exists a function $f:\mathbf{R} \rightarrow \mathbf{R}$ such that $\Delta_h f \in C_K$ for every $h \in \mathbf{R}$ and there is no additive function $H$ with*

$$f - H \in \bigcup_{L<1.5K} C_L.$$

*Proof.* We can suppose $K = 2$. If $f$ denotes the function constructed in Theorem 3 then $\Delta_h f \in C_2$ for every $h$. If $H$ is additive and $g = f - H \in C_L$ then

$$H = f - g \in C_{L+3} \subset \mathcal{M}.$$

By a well-known theorem, this implies that $H$ is linear. Therefore $f = g + H \in C_L$ from which, according to (ii) of Theorem 3, it follows that $L \geqq 3$.

Our next theorem shows that the constant 3 in (ii) of Theorem 3 is the best possible. In the sequel, $I$ and $J$ will denote non-degenerate intervals.

If $f$ is defined on $I$, then we denote

$$I(f > c) = \{x \in I; f(x) > c\} \quad (c \in \mathbf{R}).$$

The definition of $I(f < c)$, $I(f \geqq c)$, $I(f \leqq c)$ is similar.

THEOREM 5. *Let $f$ be defined on the interval $I$ and let $K \geqq 0$ be given. Suppose that the set*

$$\{x \in I \cap (I - h); |\Delta_h f(x)| > K\}$$

*is nowhere dense for every $h \in \mathbf{R}$. Then there exists a $c_0 \in R$ such that $I(f < c_0 - \epsilon)$ and $I(f > c_0 + 3K + \epsilon)$ are nowhere dense for every $\epsilon > 0$. In particular, $\omega(f, x) \leqq 3K$ holds on an everywhere dense subset of $I$.*

For the proof we shall need the following simple lemma.

LEMMA 6. *Let $f:I \rightarrow \mathbf{R}$, $d \geqq 0$ be given and suppose that for every $c \in \mathbf{R}$, at least one of the sets $I(f < c)$, $I(f > c + d)$ is nowhere dense. Then there exists a $c_0 \in \mathbf{R}$ such that $I(f < c_0 - \epsilon)$ and $I(f > c_0 + d + \epsilon)$ are both nowhere dense for every $\epsilon > 0$.*

*Proof.* We put

$$A = \{c \in R; I(f < c) \text{ is nowhere dense}\}.$$

Obviously, $c_1 \in A$, $c_2 < c_1$ implies $c_2 \in A$. Since

$$I = \bigcup_{n=1}^{\infty} I(f < n)$$

hence, by Baire's category theorem, there is an $n \notin A$. Therefore $A$ is bounded from above. On the other hand,

$$I = \bigcup_{n=1}^{\infty} I(f > -n)$$

and thus there is an $n$ such that $I(f > -n)$ is not nowhere dense. By assumption, this implies $-n - d \in A$ which proves $A \neq \emptyset$. Now it is easy to check that $c_0 = \sup A$ satisfies the requirement of the lemma.

*Proof of Theorem* 5. Suppose that $I$, $f$, $K$ satisfy the conditions of Theorem 5. By Lemma 6, it is enough to show that for every $c \in \mathbf{R}$, at least one of the sets

$$C = I(f < c) \quad \text{and} \quad D = I(f > c + 3K)$$

is nowhere dense. Suppose this is not true. Then there are bounded closed intervals $J_1, J_2 \subset \operatorname{int} I$ such that $C$ and $D$ are everywhere dense in $J_1$ and $J_2$, respectively. We select countable and dense subsets

$$\{c_n\}_{n=1}^{\infty} \subset C \cap J_1 \quad \text{and} \quad \{d_m\}_{m=1}^{\infty} \subset D \cap J_2$$

and define

$$A_{n,m} = \{x \in I \cap (I + (d_m - c_n));$$
$$|f(x + c_n - d_m) - f(x)| > K\}(n, m = 1, 2, \dots).$$

By our assumption, $A_{n,m}$ is nowhere dense for every $n$, $m$ and hence

$$A = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} (A_{nm} - d_m)$$

is of first category. Let $\delta > 0$ be such that

$$J_1 + \delta \subset I \quad \text{and} \quad J_2 + \delta \subset I$$

and let a point $y \in (0, \delta)\backslash A$ be selected. Then

$$y + c_n \in I, y + d_m \in I \quad \text{and} \quad y + d_m \notin A_{n,m}$$

for every $n$, $m = 1, 2, \dots$. Hence, by the definition of $A_{n,m}$, we have

$$|f(y + c_n) - f(y + d_m)| \leqq K \quad (n, m = 1, 2, \dots).$$

By assumption, the set

$$B = \{x \in I \cap (I - y); |f(x + y) - f(x)| > K\}$$

is nowhere dense. On the other hand, $\{c_n\}_{n=1}^{\infty}$ is dense in $J_1$, therefore there is a $c_n \notin B$. Since $c_n \in I \cap (I - y)$, this implies

$$|f(c_n + y) - f(c_n)| \leqq K.$$

Similarly

$$|f(d_m + y) - f(d_m)| \leqq K$$

holds true for at least one $m$. For these $c_n$, $d_m$ we have

$$\begin{aligned} |f(d_m) - f(c_n)| \leqq\ & |f(d_m) - f(d_m + y)| \\ & + |f(d_m + y) - f(c_n + y)| \\ & + |f(c_n + y) - f(c_n)| \leqq 3K. \end{aligned}$$

However, $d_m \in I(f > c + 3K)$ and $c_n \in I(f < c)$, from which

$$f(d_m) - f(c_n) > 3K,$$

a contradiction. This completes the proof of Theorem 5.

*Remark* 7. The argument we used above is a refinement of that used in [**2**] and [**3**]. Accordingly, our theorem is a generalization of the following result mentioned in [**2**]: if $A \subset \mathbf{R}$ is such that $(A + h)\backslash A$ is nowhere dense for every $h$ then either $A$ or $\mathbf{R}\backslash A$ is nowhere dense. In fact, let $f$ denote the characteristic function of $A$. If $(A + h)\backslash A$ is nowhere dense for every $h$ then it is easy to check that

$$\{x \in \mathbf{R}: \Delta_h f(x) \neq 0\}$$

is also nowhere dense for every $h$. Hence, applying Theorem 5 with $K = \epsilon = 1/5$, we get a $c \in \mathbf{R}$ with $\mathbf{R}(f < c)$ and $\mathbf{R}(f > c + 4/5)$ nowhere dense. Since $f$ only takes the values 0 and 1, this implies that either $A$ or $\mathbf{R}\backslash A$ is nowhere dense.

Our next theorem establishes a special case of Theorem 2. We recall that $\mathscr{M}$ stands for the class of functions which are defined on $\mathbf{R}$ and are bounded on some interval.

THEOREM 8. *If $K > 0$, $f \in \mathscr{M}$ and $\Delta_h f \in C_K$ for every $h \in \mathbf{R}$, then $f \in C_{3K}$.*

*Proof.* Suppose that $f$ is bounded on $J$. Since

$$f(x + h) = f(x) + \Delta_h f(x) \quad \text{and} \quad \Delta_h f \in C_K,$$

this implies that $\omega(f, x) < \infty$ holds on an everywhere dense subset of $J + h$. This is true for every $h$, hence $\{x \in \mathbf{R};\ \omega(f, x) < \infty\}$ is everywhere dense in $\mathbf{R}$.

We have to show that $\{x \in \mathbf{R};\ \omega(f, x) \leqq 3K\}$ is everywhere dense. Suppose this is not true and let $I$ be an interval with $\omega(f, x) > 3K$ $(x \in I)$. By our preceding remark, there is a point $x \in I$ with $\omega(f, x) < \infty$ and hence there is an interval $I_1 \subset I$ such that $f$ is bounded on $I_1$.

The (finite) function $\omega(f, x)$ is upper semi-continuous on $I_1$ so it has a continuity point $x_0 \in \operatorname{int} I_1$. Let $\omega(f, x_0) = a > 3K$ and put

$$\epsilon = \frac{a - 3K}{13}.$$

Let $I_2$ be an open interval such that

$$x_0 \in I_2 \subset I_1,\ \operatorname{diam} f(I_2) < a + \epsilon \quad \text{and}$$

$$\omega(f, x) > a - \epsilon\ (x \in I_2).$$

If $u = \inf_{x \in I_2} f(x)$then it follows that

$$u \leqq f(x) < u + a + \epsilon\ (x \in I_2)$$

and the sets

$$A \overset{\text{def}}{=} I_2(f < u + 3\epsilon) \quad \text{and} \quad B \overset{\text{def}}{=} I_2(f > u + a - 2\epsilon)$$

are everywhere dense in $I_2$.

Let $h$ be arbitrary. Then $x \in A$, $x + h \in I_2$ implies

$$f(x + h) - f(x) > u - (u + 3\epsilon) = -3\epsilon$$

and hence $\{x;\ \Delta_h f(x) > -3\epsilon\}$ is everywhere dense in $I_2 \cap (I_2 - h)$. Since

$\Delta_h f \in C_K$ this implies that

$$\{x \in I_2 \cap (I_2 - h); \Delta_h f(x) < -K - 4\epsilon\}$$

is nowhere dense. Similarly, if $x \in B$, $x + h \in I_2$ then

$$f(x + h) - f(x) < (u + a + \epsilon) - (u + a - 2\epsilon) = 3\epsilon$$

implying that

$$\{x \in I_2 \cap (I_2 - h); \Delta_h f(x) > K + 4\epsilon\}$$

is nowhere dense. Therefore, by Theorem 5, there is a point $x \in I_2$ with

$$\omega(f, x) \leqq 3K + 12\epsilon = a - \epsilon$$

which is impossible, since

$$\omega(f, x) > a - \epsilon \quad \text{for every } x \in I_2.$$

This contradiction proves Theorem 8.

**3. Proof of theorems 1 and 2.** First we deduce Theorem 1 from Theorem 2. Suppose Theorem 2 and let $f:\mathbf{R} \to \mathbf{R}$ be such that $\Delta_h f \in C_0$ for every $h \in R$. Then for every $K > 0$ we have an additive function $H_K$ such that

$$g_K = f - H_K \in C_{3K}.$$

Then $g_1 + H_1 = g_K + H_K$ and hence

$$H_1 - H_K = g_K - g_1 \in C_{3K+3} \subset \mathscr{M} \quad \text{for every } K > 0.$$

This implies that $H_1 - H_K$ is linear and thus

$$f - H_1 = g_1 = g_K + (H_K - H_1) \in C_{3K}$$

for every $K > 0$. Therefore

$$f - H_1 \in \bigcap_{K>0} C_{3K} = C_0$$

which proves Theorem 1.

LEMMA 9. *Let $F:I \to \mathbf{R}$, $G:J \to \mathbf{R}$ and $K > 0$ be given and suppose that the set*

$$A = \{ (x, y) \in I \times J; |F(x) - G(y)| > K\}$$

*is nowhere dense in $I \times J$. Then for every $\epsilon > 0$ there is a $c \in \mathbf{R}$ such that $I( |F - c| > K + \epsilon)$ is nowhere dense.*

*Proof.* By Lemma 6 it is enough to show that for every $a \in \mathbf{R}$, at least one of the sets $I(F < a)$, $I(F > a + 2K)$ is nowhere dense. Since

$$B \overset{\text{def}}{=} [I(F < a) \times J(G \geqq a + K)] \\ \cup [I(F > a + 2K)] \times J(G < a + K)] \subset A,$$

$B$ is nowhere dense. On the other hand,

$$J(G \geqq a + K) \cup J(G < a + K) = J$$

and hence either $J(G \geqq a + K)$ or $J(G < a + K)$ is not nowhere dense. Thus $B$ can be nowhere dense only if either $I(F < a)$ or $I(F > a + 2K)$ is nowhere dense.

Our next lemma states the global and local stability of Cauchy's functional equation. The global stability was proved by Hyers [**4**] and rediscovered by de Bruijn [**1**]. For the sake of completeness (and, since the proofs given by Hyers and de Bruijn are not the simplest), we provide a simple proof.

LEMMA 10. (i) *If* $f:\mathbf{R} \rightarrow \mathbf{R}$ *satisfies the inequality*

$$|f(x + y) - f(x) - f(y)| \leqq K \quad \textit{for every } x, y \in \mathbf{R},$$

*then there exists an additive function* $H$ *such that*

$$|f(x) - H(x)| \leqq K \quad \textit{for every } x \in \mathbf{R}.$$

(ii) *If* $\delta > 0$, $f$ *is defined on* $[0, 2\delta]$ *and*

$$|f(x + y) - f(x) - f(y)| \leqq K$$

*holds for every* $x, y \in [0, \delta]$, *then there exists an additive function* $H$ *such that*

$$|f(x) - H(x)| \leqq 2K \quad \textit{for every } x \in [0, \delta].$$

*Proof.* (i) The condition implies

$$|f(nx) - nf(x)| \leqq nK \quad \text{for every } x \in \mathbf{R} \text{ and } n = 1, 2, \ldots.$$

Let $\epsilon > 0$ be arbitrary. If $n, m > 2K/\epsilon$ then

$$\left|\frac{f(nx)}{n} - \frac{f(mx)}{m}\right| = \left|\frac{mf(nx) - nf(mx)}{nm}\right|$$

$$\leqq \left|\frac{mf(nx) - f(mnx)}{nm}\right| + \left|\frac{f(mnx) - nf(mx)}{nm}\right|$$

$$\leqq \frac{mK}{nm} + \frac{nK}{nm} = K\left(\frac{1}{n} + \frac{1}{m}\right) < \epsilon$$

and hence the sequence $\left\{\dfrac{f(nx)}{n}\right\}_{n=1}^{\infty}$ converges. We put

$$H(x) \overset{\text{def}}{=} \lim_{n\rightarrow\infty} \frac{f(nx)}{n} \quad (x \in \mathbf{R}).$$

For every $x, y \in \mathbf{R}$ we have

$$\begin{aligned}|H(x+y) - H(x) - H(y)| &= \lim_{n\to\infty} \frac{|f(nx+ny)-f(nx)-f(ny)|}{n} \\ &= 0\end{aligned}$$

proving that $H$ is additive. Finally,

$$|f(x) - H(x)| = \lim_{n\to\infty} \frac{|nf(x)-f(nx)|}{n} \leqq K \quad \text{for every } x \in \mathbf{R}.$$

(ii) We can suppose $\delta = 1$. Subtracting from $f$ a suitable linear function we also can assume $f(1) = 0$. We put

$$g(x) = f(\{x\}) \quad (x \in \mathbf{R}),$$

where $\{x\}$ denotes the fractional part of $x$. Then

$$|g(x+y) - g(x) - g(y)| \leqq 2K \quad \text{for every } x, y \in \mathbf{R}.$$

Indeed, this is obvious if $\{x + y\} = \{x\} + \{y\}$. On the other hand, if $\{x + y\} = \{x\} + \{y\} - 1$ then

$$\begin{aligned}|g(x+y) - g(x) - g(y)| &= |f(\{x\} + \{y\} - 1) - f(\{x\}) \\ &\quad - f(\{y\})| \\ &\leqq |f(\{x\} + \{y\} - 1) + f(1) - f(\{x\} + \{y\})| \\ &\quad + |f(\{x\} + \{y\}) - f(\{x\}) - f(\{y\})| \leqq 2K.\end{aligned}$$

Thus, applying (i) to the function $g$, the result follows.

Now we turn to the proof of Theorem 2. We can suppose $K = 1$. Let $f:\mathbf{R} \to \mathbf{R}$ be given and assume $\Delta_h f \in C_1$ for every $h \in \mathbf{R}$. We define

$$G_h = \{x \in \mathbf{R};\ \omega(\Delta_h f, x) < 2\},$$

$$A_h = \{(x, y) \in \mathbf{R}^2;\ |\Delta_h f(x) - \Delta_h f(y)| > 6\}$$

and

$$B_h = \{(x, y) \in \mathbf{R}^2;\ |\Delta_h f(x) - \Delta_h f(y)| > 2\} \quad (h \in \mathbf{R}).$$

Since $\Delta_h f \in C_1$, $G_h$ is an everywhere dense open subset of $\mathbf{R}$ for every $h$. First we prove that, if $A_h$ is dense in a rectangle $I \times J$, then

(1) $\quad B_h \supset (I \times J) \cap (G_h \times G_h).$

Indeed, if $x_0 \in I \cap G_h$, $y_0 \in J \cap G_h$ then there is $\eta > 0$ such that $|x - x_0| < \eta$, $|y - y_0| < \eta$ imply

$$|\Delta_h f(x) - \Delta_h f(x_0)| < 2 \quad \text{and} \quad |\Delta_h f(y) - \Delta_h(x_0)| < 2.$$

Since $A_h$ is dense in $I \times J$, we can choose a point $(x, y) \in A_h$ with $|x - x_0| < \eta$, $|y - y_0| < \eta$ and hence we get

$$\begin{aligned} |\Delta_h f(x_0) - \Delta_h f(y_0)| \geqq\ & |\Delta_h f(x) - \Delta_h f(y)| \\ & - |\Delta_h f(x) - \Delta_h f(x_0)| \\ & - |\Delta_h f(y_0) - \Delta_h f(y)| > 6 - 2 - 2 = 2. \end{aligned}$$

Thus (1) is proved.

Our next aim is to show that

(2) there are intervals, $I$, $J$ and $\delta > 0$ such that $A_h$ is nowhere dense in $I \times J$ for every $|h| \leqq \delta$.

Let $\{(I_n, J_n)\}_{n=1}^{\infty}$ be an enumeration of all pairs of intervals with rational end-points. Suppose that (2) is not true. Then for every $n = 1, 2, \ldots$ there is $|h_n| < 1/n$ such that $A_{h_n}$ is not nowhere dense in $I_n \times J_n$. Let $A_{h_n}$ be dense in the rectangle $T_n \subset I_n \times J_n$. Then, by our preceding argument,

$$B_{h_n} \supset T_n \cap (G_{h_n} \times G_{h_n}) \quad \text{and} \quad \operatorname{int} (B_{h_n} \cap (I_n \times J_n)) \neq \emptyset.$$

This implies that $\operatorname{int} \bigcup_{n=N}^{\infty} B_{h_n}$ is everywhere dense in $\mathbf{R}^2$ for every $N$ and

$$B \overset{\text{def}}{=} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} B_{h_n}$$

is residual. By a well-known theorem, we can find a straight line

$$e_k = \{(x, y); y = x + k\}$$

such that $e_k \cap [R^2 \setminus B]$ is of first category relative to $e_k$ ( [**5**], p. 247). Then the set $\{x \in \mathbf{R}; (x, x + k) \in B\}$ is everywhere dense in $\mathbf{R}$ and hence we can select a point $x_0 \in G_k$ with

$$(x_0, x_0 + k) \in B.$$

Now we arrive at a contradiction as follows. Since $x_0 \in G_k$, there is an $N$ such that

$$|\Delta_k f(x) - \Delta_k f(x_0)| < 2 \quad \text{if } |x - x_0| \leqq \frac{1}{N}.$$

On the other hand, by

$$(x_0, x_0 + k) \in B \subset \bigcup_{n=N}^{\infty} B_{h_n},$$

we have $(x_0, x_0 + k) \in B_{h_m}$ for some $m \geqq N$. Hence, by the definition of $B_h$ and, by $|h_m| \leqq 1/N$, we obtain

$$2 < |\Delta_{h_m} f(x_0 + k) - \Delta_{h_m} f(x_0)| = |\Delta_k f(x_0 + h_m) - \Delta_k f(x_0)| < 2$$

which is impossible. This contradiction proves (2).

Suppose that the intervals $I$, $J$ and $\delta > 0$ satisfy the condition formulated in (2). Applying Lemma 9 with

$$F(x) = G(x) = \Delta_h f(x), K = 6, \epsilon = 1,$$

we get a real number $c(h)$ for every $|h| \leqq \delta$ such that

$$D_h = \{x \in I; |\Delta_h f(x) - c(h)| > 7\}$$

is nowhere dense. Let $\delta_1 = \min (\delta/2, \frac{1}{2} |I|)$ and let $h, k \in [0, \delta_1]$. We prove

(3) $\quad |c(h + k) - c(h) - c(k)| \leqq 21.$

Indeed, the sets $D_h$, $D_k$, $D_{h+k}$ are nowhere dense, therefore we can select a point

$$x \in [I \cap (I - k)] \setminus [(D_h - k) \cup D_k \cup D_{h+k}].$$

For this $x$ we have

$$|\Delta_h f(x + k) - c(h)| \leqq 7, |\Delta_k f(x) - c(k)| \leqq 7 \quad \text{and}$$
$$|\Delta_{h+k} f(x) - c(h + k)| \leqq 7$$

from which we obtain

$$\begin{aligned} &|c(h + k) - c(h) - c(k)| \\ &\qquad \leqq |c(h + k) - [f(x + h + k) - f(x)]| \\ &\qquad + |f(x + h + k) - f(x + k) - c(h)| \\ &\qquad + |f(x + k) - f(x) - c(k)| \leqq 21 \end{aligned}$$

proving (3).

By Lemma 10, (ii) there is an additive function $H$ such that

$$|c(h) - H(h)| \leqq 42$$

holds for every $h \in [0, \delta_1]$.

Now we define $g = f - H$; we prove $g \in C_3$. Since

$$\Delta_h g = \Delta_h f - H(h) \in C_1 \text{ for every } h,$$

hence, by Theorem 8, it is enough to show $g \in \mathcal{M}$. If $h \in [0, \delta_1]$ and $x \in I \setminus D_h$ then

$$\begin{aligned} |\Delta_h g(x)| &= |\Delta_h f(x) - H(h)| \\ &\leqq |\Delta_h f(x) - c(h)| + |c(h) - H(h)| \leqq 7 + 42 = 49. \end{aligned}$$

Therefore the set $\{x \in I; |\Delta_h g(x)| > 49\}$ is nowhere dense for every $h \in [0, \delta_1]$. Let $I_1$ be a subinterval of $I$ with $|I_1| < \delta_1$. Then

$$E_h = \{x \in I_1 \cap (I_1 - h); |\Delta_h g(x)| > 49\}$$

is nowhere dense for every $h \in \mathbf{R}$. In fact, for $h \in [0, \delta_1]$ we have just proved it. For $h > \delta_1$ we have $E_h = \emptyset$. If $h < 0$, then the identity

$$\Delta_h g(x) = -\Delta_{-h} g(x + h)$$

proves the assertion. Hence, by Theorem 5, there is a point $x \in I_1$ with

$$\omega(g, x) \leqq 147.$$

Then $g$ is bounded in a neighbourhood of $x$ and thus $g \in \mathscr{M}$, which completes the proof of Theorem 2.

**4. A family of classes not having the difference property.** The following example, showing that the class $\mathscr{M}$ does not have the difference property, is due to de Bruijn [**1**]. Let $U$ be a Hamel-basis in $\mathbf{R}$ and let

$$x = \sum_{u \in U} \alpha_x(u)\, u \quad (x \in \mathbf{R}),$$

where $\alpha_x(u) \in \mathbf{Q}$ for every $u \in U$ and only a finite number of $\alpha_x(u)$'s is different from zero. Let $u_0 \in U$ be fixed. Then the function

$$f(x) = \log\left[ (\alpha_x(u_0))^2 + 1 \right]$$

has the following properties:

(i) $\Delta_h f$ is bounded for every $h \in \mathbf{R}$;

(ii) $f - H \notin \mathscr{M}$ for every additive $H$.

This proves that, if a class $\mathscr{F}$ contains the bounded functions and $\mathscr{F} \subset \mathscr{M}$ (for example, $\mathscr{F} = \cup_{K>0} C_K$ is such a class), then $\mathscr{F}$ does not have the difference property.

Our next theorem gives another family of classes not having the difference property. In particular, it will follow that the class of functions having at least one continuity point does not have the difference property.

THEOREM 11. *Let $\mathscr{Z}$ denote the class of functions $f$:$\mathbf{R} \to \mathbf{R}$ for which $f(x) = 0$ holds on an open set not bounded from above or below. If $\mathscr{Z} \subset \mathscr{F} \subset \mathscr{M}$, then $\mathscr{F}$ does not have the difference property.*

*Proof.* Let $U$ be a Hamel-basis and let $u_n$ be different elements of $U (n = 1, 2, \dots)$. Using the same notation as above, we define $f$ by

$$f(x) = (\alpha_x(u_n))^2 \quad \text{if } (n-1)^2 \leqq |x| < n^2 \quad (n = 1, 2, \dots).$$

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We shall prove that

(i) $\Delta_h f \in \mathscr{X}$ for every $h \in \mathbf{R}$ and

(ii) $f - H \notin \mathscr{M}$ for every additive $H$.

Let $h \in \mathbf{R}$ be given. If $N$ is large enough then

$$\alpha_h(u_n) = 0 \quad \text{for every } n > N.$$

This implies

$$f(x + h) - f(x) = 0$$

for every

$$x \in (n^2 - n, n^2 - n + 1) \quad \text{and} \quad x \in (-n^2 + n - 1, -n^2 + n)$$
$$(n > \max (N, |h| + 1)).$$

Thus (i) holds. If $H$ is additive and $f - H$ is bounded on some interval then, by $f \geqq 0$, $H$ is bounded from below on this interval. Then $H$ must be linear. On the other hand, for every fixed $u \in U$, $\alpha_x(u)$ is not bounded on any interval and hence the same is true for $f - H$ which is impossible. This proves (ii) and Theorem 11.

*Remark* 12. The class $\mathscr{X}$ in Theorem 11 cannot be substituted by the class of functions $f:\mathbf{R} \to \mathbf{R}$ for which $f(x) = 0$ if $|x|$ is large enough. It can be proved that the class of those functions $f:\mathbf{R} \to \mathbf{R}$ for which there exist $c \in \mathbf{R}$ and $K > 0$ such that

$$f(x) = c \text{ if } |x| > K,$$

possesses the difference property.

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