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ON REFLEXIVITY OF ALGEBRAS

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For each natural number n we define $\mathscr{R}_n$ to be the class of all weakly closed algebras $\mathscr{A}$ of (bounded linear) operators on a separable Hilbert space H such that the lattice of invariant subspaces of $\mathscr{A}^{(n)}$ and (alg lat $\mathscr{A})^{(n)}$ are the same. (If A is an operator, $A^{(n)}$ denotes the direct sum of n copies of A; if $\mathscr{A}$ is a collection of operators, $\mathscr{A}^{(n)} = \{A^{(n)}: A \in \mathscr{A}\}$. Also, alg lat $\mathscr{A}$ denotes the algebra of all operators leaving all invariant subspaces of $\mathscr{A}$ invariant.) In the first section we show that $\mathscr{R}_1 \backslash \mathscr{R}_2 \neq \emptyset$. In Section 2 we prove that every weakly closed algebra containing a maximal abelian self adjoint algebra (m.a.s.a.) is in $\mathscr{R}_2$, and that $\mathscr{R}_2 \backslash \mathscr{R}_7 \neq \emptyset$. It is also shown that certain algebras containing a m.a.s.a. are necessarily reflexive. (Reflexive means $\mathscr{A}$ = alg lat $\mathscr{A}$.) In Section 3 we study the invariant operator ranges of certain algebras. For instance, we show that if a weakly closed algebra $\mathscr{A}$ contains a m.a.s.a. and if every invariant operator range of $\mathscr{A}$ is either closed or the range of a compact operator, then $\mathscr{A}$ is reflexive. A similar result is proved for reductive algebras. Also, it is shown that if $\mathscr{A}$ is a weakly closed algebra containing a m.a.s.a., then $T \in$ alg lat $\mathscr{A}$ if and only if T leaves every invariant operator range of $\mathscr{A}$ invariant.

1. A classification of algebras. Throughout the paper by an algebra we mean an algebra of (bounded linear) operators defined on a separable Hilbert space H. All algebras contain the identity on H; the algebra of all operators on H is denoted by $B(H)$.

The lattice of all invariant subspaces of a collection $\mathscr{A}$ of operators is denoted by lat $\mathscr{A}$, and the same notation is used for the lattice of orthogonal projections whose ranges are elements of lat $\mathscr{A}$. If $\mathscr{L}$ is any collection of subspaces (or projections), the algebra of all operators leaving all elements of $\mathscr{L}$ invariant is denoted by alg $\mathscr{L}$. Obviously alg $\mathscr{L}$ is weakly closed.

Definition 1. An algebra $\mathscr{A}$ is called *reflexive* if $\mathscr{A}$ = alg lat $\mathscr{A}$.

If n is a natural number and A is an operator on H, then $A^{(n)}$ and $H^{(n)}$ denote the direct sum of n copies of A and H, respectively. If $\mathscr{A}$ is a set of operators, $\mathscr{A}^{(n)}$ denotes the set $\{A^{(n)}: A \in \mathscr{A}\}$.

LEMMA 1. (**[20]**) *An operator A belongs to the weak closure of an algebra $\mathscr{A}$ if and only if* lat $A^{(n)} \supset$ lat $\mathscr{A}^{(n)}$ *for all natural numbers n. Consequently,*

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two weakly closed algebras $\mathscr{A}$ *and* $\mathscr{B}$ *are equal if and only if* lat $\mathscr{A}^{(n)} =$ lat $\mathscr{B}^{(n)}$ *for all* n.

Let $\mathscr{A}$ be a weakly closed algebra. In view of Lemma 1, $\mathscr{A}$ is non-reflexive if there exists a natural number n such that lat $\mathscr{A}^{(n)} \neq$ lat $\mathscr{B}^{(n)}$, where $\mathscr{B} =$ alg lat $\mathscr{A}$.

Notation 1. For each positive integer n, let $\mathscr{R}_n$ denote the class of all weakly closed algebras $\mathscr{A}$ such that lat $\mathscr{A}^{(n)} =$ lat $\mathscr{B}^{(n)}$, where $\mathscr{B} =$ alg lat $\mathscr{A}$.

Note that $\{\mathscr{R}_n\}$ is a decreasing chain, and an algebra $\mathscr{A}$ is reflexive if and only if $\mathscr{A} \in \bigcap_n \mathscr{R}_n$.

Arveson [**1**] has asked whether lat $\mathscr{A}^{(2)} =$ lat $(B(H))^{(2)}$ implies $\mathscr{A} = B(H)$, where $\mathscr{A}$ is assumed to be weakly closed. In our notation, this means that whether an operator algebra $\mathscr{A} \in \mathscr{R}_2$ with lat $\mathscr{A} = \{\{0\}, H\}$ is reflexive. The problem seems to be very difficult, and a negative answer to this problem would imply a negative answer to the transitive algebra problem. (We refer the reader to [**1**] or [**18**, page 196] for more detail.) However, with less restriction on lat $\mathscr{A}$, we are able to show that the answer is negative. In fact, we prove that every weakly closed algebra containing a maximal abelian self-adjoint algebra (m.a.s.a.) is of class $\mathscr{R}_2$; thus in view of [**2**, pages 504–509], $\mathscr{R}_2$ contains a non-reflexive algebra.

In the remainder of this section we show that $\mathscr{R}_1 \backslash \mathscr{R}_2 \neq \emptyset$, and in the next section we prove that $\mathscr{R}_2 \backslash \mathscr{R}_7 \neq \emptyset$. Note that $\mathscr{R}_1$ is the class of all weakly closed algebras.

Example 1. Let H be the direct sum of k copies of a Hilbert space K for some $k \geqq 2$. Let $\mathscr{B}$ be the algebra of all operators $((A_{ij}))$ such that $A_{ij} = 0$ for $i > j$ and $A_{ij} \in B(K)$ for all $i, j = 1, 2, \ldots, k$. Let $\mathscr{A}$ be the algebra consisting of all operators $((A_{ij})) \in \mathscr{B}$ such that $A_{11} = A_{22} = \ldots = A_{kk}$. Obviously $\mathscr{B} =$ alg lat $\mathscr{A} \neq \mathscr{A}$. We show that $\mathscr{A} \notin \mathscr{R}_2$. Let $\mathscr{M}$ be the set of all vectors of the form

$$\begin{pmatrix} x \\ 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \oplus \begin{pmatrix} y \\ x \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \in H \oplus H \qquad (x, y \in K).$$

It is easy to see that $\mathscr{M}$ is an invariant subspace of $\mathscr{A}^{(2)}$ but not of $\mathscr{B}^{(2)}$.

A similar argument shows that the nonreflexive algebras of [**18**, Examples 9.27 and 9.28] are not in $\mathscr{R}_2$.

Example 2. Let A be any operator on a finite-dimensional Hilbert space such that the algebra generated by A and I is not reflexive. Such algebras exist by a criterion due to [**5**], and we show that they are not in $\mathscr{R}_2$. Assume the algebra $\mathscr{A}$ generated by A and I is not reflexive and, if possible, $\mathscr{A} \in \mathscr{R}_2$. Let $B \in (\text{alg lat}\,\mathscr{A})\backslash\mathscr{A}$. Then lat $(B \oplus B) \supset$ lat $(A \oplus A)$. By the Deddens–Fillmore criterion the algebra generated by $A \oplus A$ and $I \oplus I$ is reflexive and, therefore, contains $B \oplus B$. Hence $\mathscr{A}$ contains B, a contradiction.

2. Algebras containing m.a.s.a. In this section we will show that every weakly closed algebra containing a m.a.s.a. is necessarily in $\mathscr{R}_2$. Using this fact and an example of Arveson [**2**, page 504] we show that $\mathscr{R}_2\backslash\mathscr{R}_7 \neq \emptyset$. We will also show that if a weakly closed algebra containing a m.a.s.a. is nonreflexive, then there exists a projection $P \in \text{lat}\,\mathscr{A}$ such that $(I - P)$ lat $\mathscr{A}$ contains a nontrivial Boolean algebra.

Notation 2. Let $x \in H^{(n)}$ and $\mathscr{M} \subset H^{(n)}$. The vector x has a unique representation of the form $x_1 \oplus x_2 \oplus \ldots \oplus x_n$ with $x_i \in H$, $i = 1, 2, \ldots, n$. The vectors $x_1, x_2, \ldots, x_n$ are called the first, the second, ..., the nth component of x, respectively. Similarly, the set of all ith components of vectors in $\mathscr{M}$ is denoted by $\mathscr{M}_i$ and is called the ith component of $\mathscr{M}$.

LEMMA 2. *Let A be a self-adjoint operator of multiplicity* 1. *Let $\mathscr{Q}$ be an invariant subspace of $A^{(n)}$ for some fixed integer $n \geqq 2$. Let $i \leqq n$ be a fixed positive integer. Assume the ith component of no nonzero vector of $\mathscr{Q}$ is zero. Then $A^{(n)}|\mathscr{Q}$ and $A|\overline{\mathscr{Q}}_i$ are unitarily equivalent. In particular, if $\mathscr{Q}'$ and $\mathscr{Q}''$ are complementary invariant subspaces of $A^{(n)}|\mathscr{Q}$, then the closures of $\mathscr{Q}_i'$ and $\mathscr{Q}_i''$ are complementary invariant subspaces of $A|\mathscr{Q}_i$. Conversely, if L and M are complementary invariant subspaces of $A|\mathscr{Q}_i$, then there exist complementary invariant subspaces $\mathscr{Q}'$ and $\mathscr{Q}''$ of $A^{(n)}|\mathscr{Q}$ such that L and M are the closures of $\mathscr{Q}_i'$ and $\mathscr{Q}_i''$, respectively.*

Proof. Define C_i: $\mathscr{Q} \rightarrow \overline{\mathscr{Q}}_i$ by $C_ix = x_i$. Obviously

$$C_i(A^{(n)}|\mathscr{Q}) = (A|\overline{\mathscr{Q}}_i)C_i.$$

Since C_i is injective and has dense range, it follows that $C_i = K_iU_i$, where U_i: $\mathscr{Q} \rightarrow \overline{\mathscr{Q}}_i$ is unitary and K_i: $\overline{\mathscr{Q}}_i \rightarrow \overline{\mathscr{Q}}_i$ is a positive injective operator. Thus

$$K_i[U_i(A^{(n)}|\mathscr{Q})U_i^*] = (A|\overline{\mathscr{Q}}_i)K_i$$

and hence

$$A|\overline{\mathscr{Q}}_i = U_i(A^{(n)}|\mathscr{Q})U_i^* \quad [\mathbf{11}, \text{page } 306].$$

In particular, $C_iF(\delta) = E(\delta)C_i$ for all Borel sets δ, where F and E are the resolutions of the identity for $A^{(n)}|\mathscr{Q}$ and $A|\overline{\mathscr{Q}}_i$, respectively. There-

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fore, $A^{(n)}|\mathscr{Q}$ is a self-adjoint operator of multiplicity 1. Now the rest of the lemma follows from the fact that C_i maps each $F(\delta)\mathscr{Q}$ densely into $E(\delta)\bar{\mathscr{Q}}_i$, and that every invariant subspace of a self-adjoint operator of multiplicity 1 is the range of some spectral projection.

LEMMA 3. *Let* $A \in B(H)$ *be a self-adjoint operator of multiplicity* 1. *For a fixed integer* $n \geqq 2$, *let* $\mathscr{P}$ *be an invariant subspace of* $A^{(n)}$ *such that no nonzero vector of* $\mathscr{P}$ *has some zero component. Then* $\bar{\mathscr{P}}_1 = \ldots = \bar{\mathscr{P}}_n$, *and there exist closable operators* G_i *from* $\mathscr{P}_1$ *onto* $\mathscr{P}_{i+1}$ $(i = 1, 2, \ldots, n-1)$ *such that*

$$\mathscr{P} = \{x \oplus G_1x \oplus \ldots \oplus G_{n-1}x : x \in \mathscr{P}_1\},$$

the closures of $G_1, \ldots G_{n-1}$ *are normal, and*

$$\mathscr{P}_1 = \cap \{\text{Domain } (\bar{G}_i) : i = 1, \ldots, n-1\}.$$

Proof. Since $A^{(n)}|\mathscr{P}$ and $A|\bar{\mathscr{P}}_i$ are unitarily equivalent (Lemma 2), it follows that $\bar{\mathscr{P}}_1 = \ldots = \bar{\mathscr{P}}_n$, and there exists a unitary operator V: $\bar{\mathscr{P}}_1 \rightarrow \mathscr{P}$ such that

$$(A^{(n)}|\mathscr{P})V = V(A|\bar{\mathscr{P}}_1).$$

Define C_i: $\mathscr{P} \rightarrow \bar{\mathscr{P}}_1$ by $C_ix = x_i (i = 1, \ldots, n)$. Observe that

$$C_iV(A|\bar{\mathscr{P}}_1) = C_i(A^{(n)}|\mathscr{P})V = (A|\bar{\mathscr{P}}_1)C_iV.$$

This implies that C_iV belongs to the commutant of $A|\bar{\mathscr{P}}_1$, and thus

$$C_iV = f_i(A|\bar{\mathscr{P}}_1) \quad \text{and} \quad (C_iV)^{-1} = g_i(A|\bar{\mathscr{P}}_1),$$

where f_i and g_i are Baire functions for $i = 1, 2, \ldots, n-1$. Thus

$$C_i(C_1)^{-1} = C_iV(C_1V)^{-1} = f_i(A|\bar{\mathscr{P}}_1)g_1(A|\bar{\mathscr{P}}_1) \subset (f_ig_1)(A|\bar{\mathscr{P}}_1)$$

and hence $C_i(C_1)^{-1}$ has a normal closure $(f_ig_1)(A|\bar{\mathscr{P}}_1)$, $i = 1, \ldots, n-1$. (See [**7**, pages 1196–1200 and Problem 3 (page 1257)].) Let

$$G_i = C_{i+1}C_1^{-1}, i = 1, 2, \ldots, n-1.$$

It is easy to see that

$$\begin{aligned} \mathscr{P} &= \{C_1x \oplus \ldots \oplus C_nx : x \in \mathscr{P}\} \\ &= \{y \oplus G_1y \oplus \ldots \oplus G_{n-1}y : y \in \mathscr{P}_1\}. \end{aligned}$$

It remains to show that $\mathscr{P}_1 = \cap_i \text{Domain } (\bar{G}_i)$. Let

$$\mathscr{M} = \{x \oplus \bar{G}_1x \oplus \ldots \oplus \bar{G}_{n-1}x : x \in \cap_i \text{Domain } (\bar{G}_i)\}.$$

Obviously $\mathscr{M}$ is closed, and $\mathscr{P}_1$ and $\mathscr{M}_1$ have the same closures. Let $\mathscr{Q} = \mathscr{M} \ominus \mathscr{P}$. In view of Lemma 2, the closures of $\mathscr{Q}_1$ and $\mathscr{P}_1$ are complementary orthogonal subspaces of $\bar{\mathscr{M}}_1$, from which it follows that $\mathscr{Q}_1 = \{0\}$. Thus $\mathscr{Q} = \{0\}$ and $\mathscr{M} = \bar{\mathscr{P}}$.

The following is the key theorem.

THEOREM 1. *Let* $\mathscr{A} \subset B(H)$ *be a weakly closed algebra containing a* m.a.s.a. *Let* $\mathscr{M}$ *be an invariant subspace of* $\mathscr{A}^{(n)}$ *for some fixed integer* $n \geqq 2$. *Let* $\mathscr{N}$ *be the span of all vectors in* $\mathscr{M}$ *having at least one zero component, and assume* $\mathscr{M}$ *is the smallest invariant subspace of* $\mathscr{A}^{(n)}$ *containing* $\mathscr{P} = \mathscr{M} \ominus \mathscr{N}$. *Let* $T \in B(H)$ *be such that* lat $T^{(n-1)} \supset$ lat $\mathscr{A}^{(n-1)}$. *Then* $\mathscr{N}$ *is an invariant subspace of* $\mathscr{A}^{(n)}$ *and* $T^{(n)}$, *and*

(a) $\bar{\mathscr{M}}_i = \bar{\mathscr{P}}_i \oplus \bar{\mathscr{N}}_i, i = 1, 2, \ldots, n,$

(b) $\bar{\mathscr{P}}_1 = \ldots = \bar{\mathscr{P}}_n$ *and* $\bar{\mathscr{N}}_1 = \ldots = \bar{\mathscr{N}}_n$.

Moreover, for every vector $x \in \mathscr{P}$, *the vector* $T^{(n)}x$ *is the direct sum of a vector* $y \in \mathscr{P}$ *and a vector* z *of the form*

$$z = z_1 \oplus z_2 \oplus \ldots \oplus z_n \in \bar{\mathscr{N}}_1 \oplus \bar{\mathscr{N}}_2 \oplus \ldots \oplus \bar{\mathscr{N}}_n.$$

Proof. Let $\mathscr{N}'$ be the set of all vectors $x \in \mathscr{M}$ whose first components are zero and let

$$\mathscr{N}'' = \{x \in H^{(n-1)} : 0 \oplus x \in \bar{\mathscr{N}'}\}.$$

Obviously $\mathscr{N}' \in \text{lat}\,\mathscr{A}^{(n)}$ and hence

$$\mathscr{N}'' \in \text{lat}\,\mathscr{A}^{(n-1)} \subset \text{lat}\ T^{(n-1)}.$$

Thus

$$\mathscr{N}' \in \text{lat}\ T^{(n)} \cap \text{lat}\,\mathscr{A}^{(n)}.$$

Similar arguments for other components show that

$$\mathscr{N} \in \text{lat}\ T^{(n)} \cap \text{lat}\,\mathscr{A}^{(n)}.$$

In particular, $\mathscr{P}$ is an invariant subspace of $A^{(n)}$, where A is a self-adjoint operator of multiplicity 1 which generates a m.a.s.a. in $\mathscr{A}$. Note that no nonzero vector of $\mathscr{P}$ has some zero component. Thus, by Lemma 3, $\bar{\mathscr{P}}_1 = \ldots = \bar{\mathscr{P}}_n$ and, in view of the minimality of $\mathscr{M}$, $\bar{\mathscr{M}}_1 = \ldots = \bar{\mathscr{M}}_n$.

Let $\mathscr{Q} = \mathscr{M} \ominus \mathscr{N}'$ and $\mathscr{Q}' = \mathscr{N} \ominus \mathscr{N}'$. The sets $\mathscr{Q}$ and $\mathscr{Q}'$ are invariant subspaces of $A^{(n)}$. Moreover, $\mathscr{Q}_1 = \mathscr{M}_1$ and $\mathscr{Q}_1' = \mathscr{N}_1$. Considering the operators $A^{(n)}|\mathscr{Q}$ and $A|\bar{\mathscr{Q}}_1$, and the orthogonal subspaces $\mathscr{Q}'$ and $\mathscr{P}$, one can apply Lemma 2 to see that $\bar{\mathscr{N}}_1$ and $\bar{\mathscr{P}}_1$ are orthogonal and span $\bar{\mathscr{M}}_1$. Similar results hold for other components of $\mathscr{M}$, $\mathscr{N}$, and $\mathscr{P}$.

Let $x = x_1 \oplus \ldots \oplus x_n \in \mathscr{P}$. Since $\bar{\mathscr{M}}_i$ is an invariant subspace of T, it follows that $Tx_i = y_i \oplus z_i$, where $y_i \in \bar{\mathscr{P}}_i$ and $z_i \in \bar{\mathscr{N}}_i$ $(i = 1, 2, \ldots, n)$. It remains to show that $y_1 \oplus \ldots \oplus y_n \in \mathscr{P}$.

For each $B \in \text{alg lat}\,\mathscr{A}$, define $B^\# : \bar{\mathscr{P}}_1 \to \bar{\mathscr{P}}_1$ by $B^\# u = (I - P)Bu$, where P is the orthogonal projection from H onto $\bar{\mathscr{N}}_1$. Let $\mathscr{A}^\#$ be the weakly closed algebra generated by $\{B^\# : B \in \mathscr{A}\}$. The algebra $\mathscr{A}^\#$ con-

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tains the m.a.s.a. generated by the self-adjoint operator $A^\# = A|\mathscr{P}_1$, and $\mathscr{A}^{\#(n)}$ leaves $\mathscr{P}$ invariant. In view of Lemma 3, $\mathscr{P}$ is of the form

$$\{u \oplus G_1 u \oplus \ldots \oplus G_{n-1}u : u \in \mathscr{P}_1 = \cap_i \text{Domain } (\bar{G}_i)\}.$$

Fix $0 < i \leqq n - 1$ and consider the closed subspace

$$\mathscr{Q} = \{u \oplus \bar{G}_i u : u \in \text{Domain } (\bar{G}_i)\}$$

of $H^{(2)}$. Obviously $\mathscr{Q}$ is an invariant subspace of $\mathscr{A}^{\#(2)}$. Hence $B^\# \bar{G}_i u = \bar{G}_i B^\# u$ for all $u \in$ Domain $(\bar{G}_i)$, and $B^\#$ leaves Domain $(\bar{G}_i)$ invariant. In particular, every spectral subspace of the normal operator $\bar{G}_i$ is an invariant subspace of $\mathscr{A}^\#$.

We claim $T^\#$ commutes with $\bar{G}_i$. Let D be an arbitrary invariant subspace of $\mathscr{A}^\#$. Obviously $D \oplus \bar{\mathscr{N}}_i$ is an invariant subspace of $\mathscr{A}$ and hence that of T. Thus D is an invariant subspace of $T^\#$ and, therefore, lat $T^\# \supset$ lat $\mathscr{A}^\#$. Thus every spectral subspace of $\bar{G}_i$ is left invariant by $T^\#$, and hence $T^\#$ leaves Domain $(\bar{G}_i)$ invariant and $T^\# \bar{G}_i u = \bar{G}_i T^\# u$ for all $u \in$ Domain $(\bar{G}_i)$. (See [**7**, pages 1258–1259].) Now since i is arbitrary, it follows that $T^\#$ leaves $\mathscr{P}_1$ invariant and $T^\# G_i = G_i T^\#$, $i = 1, 2, \ldots, n - 1$. We conclude that $T^{\#(n)}$ leaves $\mathscr{P}$ invariant and hence $y_1 \oplus \ldots \oplus y_n = (T^\# x_1) \oplus \ldots \oplus (T^\# x_n) \in \mathscr{P}$.

THEOREM 2. *Every weakly closed algebra containing a* m.a.s.a. *is of class* $\mathscr{R}_2$.

Proof. Let $\mathscr{A}$ be an algebra containing a m.a.s.a., and let $T \in$ alg lat $\mathscr{A}$. Let $\mathscr{M}$ be an arbitrary invariant subspace of $\mathscr{A}^{(n)}$ and let $\mathscr{N}$ be the span of all vectors in $\mathscr{M}$ having some zero component. Let $\mathscr{P} = \mathscr{M} \ominus \mathscr{N}$. To show that $\mathscr{M}$ is an invariant subspace of $T^{(2)}$ it is enough, in view of Theorem 1 and its proof, to show that $T^{(2)}x \in \mathscr{M}$ for all $x \in \mathscr{P}$. Therefore, we can assume without loss of generality that $\mathscr{M}$ is the smallest invariant subspace of $\mathscr{A}^{(2)}$ containing $\mathscr{P}$.

Since $\mathscr{N}$ is the span of vectors of the form $u \oplus 0$ and $0 \oplus v$, $\mathscr{N}_1$ and $\mathscr{N}_2$ are closed and $\mathscr{N} = \mathscr{N}_1 \oplus \mathscr{N}_2$. Now, if x is an arbitrary vector in $\mathscr{P}$, it follows from Theorem 1 that $T^{(2)}x$ is the direct sum of vectors in $\mathscr{P}$ and $\mathscr{N}_1 \oplus \mathscr{N}_2$. Thus $T^{(2)}x \in \mathscr{M}$ and the proof is complete.

The following example shows that $\mathscr{R}_2 \backslash \mathscr{R}_7 \neq \emptyset$.

Example 3. We show that the nonreflexive algebra containing a m.a.s.a. given by Arveson [**2**, pages 504–509] is in $\mathscr{R}_2 \backslash \mathscr{R}_7$. We first review the example.

Fix a function $u \in C_0^\infty(\mathbf{R}^3)$ such that

$$\int_{\mathbf{R}^3} u(t)\overline{u(t-x)}dt > 0$$

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for all $x \in S^2$. For $x = (x_1, x_2, x_3) \in \mathbf{R}^3$, define

$$\begin{aligned} a_1(x) &= u(x), & b_1(x) &= (x_1^2 + x_2^2 + x_3^2 - 1)u(x), \\ a_2(x) &= x_1 u(x), & b_2(x) &= -2x_1 u(x), \\ a_3(x) &= x_2 u(x), & b_3(x) &= -2x_2 u(x), \\ a_4(x) &= x_3 u(x), & b_4(x) &= -2x_3 u(x), \\ a_5(x) &= x_1^2 u(x), & b_5(x) &= u(x), \\ a_6(x) &= x_2^2 u(x), & b_6(x) &= u(x), \\ a_7(x) &= x_3^2 u(x), & b_7(x) &= u(x). \end{aligned}$$

Note that $a_1, \ldots, a_7, b_1, \ldots, b_7$ are elements of $L^2(\mathbf{R}^3)$. In [**2**, Proposition 2.5.5] it is shown that there exists a linear space of operators on $L^2(\mathbf{R}^3)$ denoted by $\mathscr{A}_{\min}(\Sigma)$, and an operator T such that if the elements of $L(\mathbf{R}^3)$ are viewed as multiplications, then

(i) $L^\infty(\mathbf{R}^3)\mathscr{A}_{\min}(\Sigma)L^\infty(\mathbf{R}^3) \subset \mathscr{A}_{\min}(\Sigma)$ [**2**, page 488],

(ii) $b_1 \oplus \ldots \oplus b_7$ is perpendicular to $Sa_1 \oplus \ldots \oplus Sa_7$ for all $S \in \mathscr{A}_{\min}(\Sigma)$,

(iii) lat $T \supset$ lat $\mathscr{A}_{\min}(\Sigma)$,

(iv) $b_1 \oplus \ldots \oplus b_7$ is not perpendicular to $Ta_1 \oplus \ldots \oplus Ta_7$.

Let $\mathscr{A}$ be the algebra of all operators on $L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3)$ which admit a 2 by 2 matrix representation

$$\begin{pmatrix} A & S \\ 0 & B \end{pmatrix},$$

where A, B belong to $L^\infty(\mathbf{R}^3)$ and S is in the weak closure of $\mathscr{A}_{\min}(\Sigma)$. In view of (i), $\mathscr{A}$ is a weakly closed algebra containing the m.a.s.a. $L^\infty(\mathbf{R}^3) \oplus L^\infty(\mathbf{R}^3)$. Therefore, by Theorem 2, $\mathscr{A} \in \mathscr{R}_2$. Let

$$\tilde{T} = \begin{pmatrix} 0 & T \\ 0 & 0 \end{pmatrix}.$$

By (iii), $\tilde{T} \in$ alg lat $\mathscr{A}$; by (ii) and (iv) the smallest invariant subspace $\mathscr{M}$ of $\mathscr{A}^{(7)}$ containing the vector

$$\begin{pmatrix} 0 \\ a_1 \end{pmatrix} \oplus \ldots \oplus \begin{pmatrix} 0 \\ a_7 \end{pmatrix} \in [L^2(\mathbf{R}^3) \oplus L^2(\mathbf{R}^3)]^{(7)}$$

is not left invariant by $T^{(7)}$. Hence $\mathscr{A} \notin \mathscr{R}_7$.

Question 1. Does the algebra $\mathscr{A}$ in Example 3 belong to $\mathscr{R}_3$?

Question 2. Is $\mathscr{R}_2 \neq \mathscr{R}_3$? What about $\mathscr{R}_n$ and $\mathscr{R}_{n+1}$ in general? Note that we have so far shown that $\mathscr{R}_1 \neq \mathscr{R}_2$ and $\mathscr{R}_2 \neq \mathscr{R}_7$.

In [**18**, page 197] it is asked whether the algebra generated by $A \oplus A$

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is reflexive for every $A \in B(H)$. The following proposition shows that this is not true for a general algebra.

PROPOSITION 1. *Let $\mathscr{A}$ be a nonreflexive algebra in $\mathscr{R}_n$. Then $\mathscr{A}^{(n)}$ is not reflexive. In particular, there exists an algebra $\mathscr{A} \in \mathscr{R}_2$ such that $\mathscr{A}^{(2)}$ is not reflexive.*

Proof. Assume $\mathscr{A} \in \mathscr{R}_n$ is not reflexive. Let $A \in (\text{alg lat } \mathscr{A}) \backslash \mathscr{A}$. Obviously $A^{(n)} \notin \mathscr{A}^{(n)}$. Since $\mathscr{A} \in \mathscr{R}_n$, $A^{(n)} \in \text{alg lat} \mathscr{A}^{(n)}$, which implies that $\mathscr{A}^{(n)}$ is nonreflexive. Now the algebra $\mathscr{A}$ of Example 3 is an element of $\mathscr{R}_2$ and $\mathscr{A}^{(2)}$ is nonreflexive.

The following theorem is a generalization of a result of Radjavi–Rosenthal [**16**], [**18**, Theorem 9.24].

THEOREM 3. *Let $\mathscr{A} \subset B(H)$ be a weakly closed algebra containing a* m.a.s.a. *Assume for no projection $P \in \text{lat} \mathscr{A}$ the lattice*

$$(I - P) \text{ lat} \mathscr{A} = \{(I - P)Q \colon Q \in \text{lat} \mathscr{A}\}$$

contains a nontrivial Boolean algebra. Then every invariant subspace $\mathscr{M}$ of $\mathscr{A}^{(n)}$ is spanned by invariant subspaces of the form

$$(*) \quad \{x_1 \oplus x_2 \oplus \ldots \oplus x_n : x_j \in M_j \text{ for } j \in J \text{ and } x_i = \textstyle\sum_{j \in J} a_{ij} x_j \quad \text{for } i \notin J\},$$

where $J \subset \{1, 2, \ldots, n\}$, $\{M_j : j \in J\} \subset \text{lat} \mathscr{A}$ and the complex numbers a_{ij} are independent of $x_1, \ldots, x_n$. In particular, $\mathscr{A}$ is reflexive.

Proof. We prove the theorem by induction on n. The case $n = 1$ is trivial. Assume every invariant subspace of $\mathscr{A}^{(k)}$ is spanned by invariant subspaces of the form (*) for all $k \leqq n - 1$. Let $\mathscr{M}$ be an invariant subspace of $\mathscr{A}^{(n)}$ and let $\mathscr{Q} \subset \mathscr{M}$ be the orthogonal complement of all invariant subspaces of the form (*) included in $\mathscr{M}$. Assume without loss of generality that $\mathscr{M}$ is the smallest invariant subspace of $\mathscr{A}^{(n)}$ containing $\mathscr{Q}$. We have to show that $\mathscr{M} = \{0\}$. Let $\mathscr{N}$ be the span of all vectors in $\mathscr{M}$ having some zero component. By induction assumption, $\mathscr{N}$ is spanned by invariant subspaces of the form (*) and hence $\mathscr{Q} \subset \mathscr{P} = \mathscr{M} \ominus \mathscr{N}$. In particular, $\mathscr{M}$ is the smallest invariant subspace of $\mathscr{A}^{(n)}$ containing $\mathscr{P}$. Let P be the projection from H onto $\overline{\mathscr{N}_1}$ and let $\mathscr{A}^{\#}$ be as in the proof of Theorem 1. Let

$$\mathscr{P} = \{x \oplus G_1 x \oplus \ldots \oplus G_{n-1} x : x \in \mathscr{P}_1\}$$

as in Lemma 3. We observed in the proof of Theorem 1 that $\mathscr{A}^{\#}$ leaves the spectral subspaces of each $\bar{G}_i$ invariant, and that $D \in \text{lat } \mathscr{A}^{\#}$ if and only if $D \oplus P \in \text{lat} \mathscr{A}$ and $D \subset \overline{\mathscr{P}}_1$. (Note that the same notation is used for a projection and its range.) Therefore, if $\mathscr{B}_i$ is the Boolean algebra of all spectral projections of $\bar{G}_i$, then

$$\mathscr{B}_i \subset (I - P) \text{ lat} \mathscr{A}.$$

Thus $\mathscr{B}_i$ is trivial, which implies that $G_i = \bar{G}_i$ is a multiple b_i of the identity on $\mathscr{P}_1 = \bar{\mathscr{P}}_1$. Hence

$$\mathscr{P} = \{x \oplus b_1 x \oplus \ldots \oplus b_{n-1}x : x \in \mathscr{P}_1\}.$$

Since $\mathscr{Q} \subset \mathscr{P}$, it follows from the definition of $\mathscr{Q}$ that $\mathscr{Q} = \{0\}$ and thus $\mathscr{M} = \{0\}$.

To show that $\mathscr{A}$ is reflexive, let $T \in$ alg lat $\mathscr{A}$. Since every invariant subspace of the form (*) is invariant under $T^{(n)}$, it follows that lat $T^{(n)} \supset$ lat $\mathscr{A}^{(n)}$ for all n. Thus $\mathscr{A}$ is reflexive and the proof is complete.

COROLLARY 1. ([**16**], [**18**]) *Let $\mathscr{A}$ be a weakly closed algebra containing a* m.a.s.a. *Assume* lat $\mathscr{A}$ *is a chain. Then $\mathscr{A}$ is reflexive.*

Proof. If $P \in$ lat $\mathscr{A}$, then $(I - P)$ lat $\mathscr{A}$ is a chain and cannot contain any nontrivial Boolean algebra.

An algebra $\mathscr{A}$ is called *pre-reflexive* if $\mathscr{A} \cap \mathscr{A}^* = (\text{lat } \mathscr{A})'$. In [**2**, Theorem 2.1.8] it is shown that every ultraweakly closed algebra containing a m.a.s.a. is pre-reflexive. Here we include an operator-theoretic proof of this fact for weakly closed algebras.

COROLLARY 2. *Every weakly closed algebra containing a* m.a.s.a. *is pre-reflexive.*

Proof. Let $\mathscr{A}$ be a weakly closed algebra which contains a m.a.s.a. Obviously $\mathscr{A} \cap \mathscr{A}^* \subset (\text{lat } \mathscr{A})'$. For the converse inclusion assume $T \in (\text{lat } \mathscr{A})'$. Every invariant subspace of $\mathscr{A}$ is reduced by T. We show by induction on n that lat $T^{(n)} \supset$ lat $\mathscr{A}^{(n)}$. The statement is trivially true for $n = 1$. Assume the statement is true for all $k \leqq n - 1$. Let $\mathscr{M}$ be an invariant subspace of $\mathscr{A}^{(n)}$. Let $\mathscr{P}$ and $\mathscr{N}$ be as in Theorem 1, and assume without loss of generality that $\mathscr{M}$ is the smallest invariant subspace of $\mathscr{A}^{(n)}$ containing $\mathscr{P}$. (Note that $\mathscr{N} \in$ lat $T^{(n)}$ by the induction assumption.) Let $x \in \mathscr{P}$ be arbitrary. In view of Theorem 1, $Tx_i = y_i \oplus z_i$, $y_i \in \mathscr{P}_i$, $z_i \in \mathscr{N}_i$ $(i = 1, 2, \ldots, n)$ and $y = y_1 \oplus y_2 \oplus \ldots \oplus y_n \in \mathscr{P}$. Since $\mathscr{N}_i$ is a reducing invariant subspace of T and $\mathscr{N}_i \perp \mathscr{P}_i$, it follows that $z_i = 0$ (for all i). Thus

$$T^{(n)}x = y \in \mathscr{P} \subset \mathscr{M}$$

and hence $\mathscr{M}$ is an invariant subspace of $T^{(n)}$.

Therefore, $T \in \mathscr{A}$ and by a similar argument $T^* \in \mathscr{A}$. The proof is complete.

COROLLARY 3. ([**1**]) *Let $\mathscr{A} \subset B(H)$ be a weakly closed transitive algebra containing a* m.a.s.a. *Then $\mathscr{A} = B(H)$. (This is also a special case of Corollary* 1.)

The proof follows from the following stronger corollary.

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COROLLARY 4. ([**17**], [**21**]) *Let $\mathscr{A}$ be a weakly closed reductive algebra containing a* m.a.s.a. *Then $\mathscr{A}$ is self-adjoint.* (*Note that $\mathscr{A}$ being reductive means that every invariant subspace of $\mathscr{A}$ is reducing.*)

Proof. Observe that

$$\mathscr{A}^* \subset (\text{lat}\,\mathscr{A})' = \mathscr{A} \cap \mathscr{A}^* \subset \mathscr{A}$$

which implies that $\mathscr{A}$ is self-adjoint.

3. Invariant operator ranges of algebras.

Definition 2. By an *operator range* we mean a linear manifold which is the range of a Hilbert-space operator. An *invariant* operator range of a collection $\mathscr{A}$ of operators in an operator range which is an invariant linear manifold of $\mathscr{A}$.

THEOREM 4. *Let $\mathscr{A} \subset B(H)$ be a weakly closed algebra of operators containing a* m.a.s.a., *and let $T \in B(H)$. Then $T \in$* alg lat $\mathscr{A}$ *if and only if T leaves every invariant operator range of $\mathscr{A}$ invariant.*

Proof. Let KH be an invariant operator range of $\mathscr{A}$, where K is an operator. Using polar decomposition, assume without loss of generality $0 \leqq K \leqq I$. By a result of Foias [**10**, page 892] there exists a positive number $\lambda < 1$ such that $\mathscr{A}E[t, 1]H \subset E[\lambda t, 1]\,H$ for all $t \in [0, 1]$, where E is the resolution of the identity for K. Let $T \in$ alg lat $\mathscr{A}$. Since the closure of $\mathscr{A}E[t, 1]H$ is an invariant subspace of $\mathscr{A}$ and $E[t, 1]H \subset \mathscr{A}E[t, 1]H$, it follows that

$$TE[t, 1]H \subset E[\lambda t, 1]H \text{ for all } t \in [0, 1].$$

Let $H_i = E(\lambda^i, \lambda^{i-1}]H, i = 1, 2, 3, \ldots$. Then $\overline{KH} = H_1 \oplus H_2 \oplus \ldots$, and the operators $T^\# = T|\overline{KH}$ and $K^\# = K|\overline{KH}$ are respectively of the forms $((T_{ij}))$, $((K_{ij}))$, where T_{ij} and K_{ij} have H_j as their domains and H_i as their ranges. Moreover, $T_{ij} = 0$ for $i \geqq j + 3$ and $K_{ij} = 0$ for $i \neq j$. (Note that $\overline{KH} \in$ lat $\mathscr{A}$ and that some H_i may be trivial.) Let $J = \{j : H_j \neq \{0\}\}$; then $\lambda^j \leqq K_{jj} \leqq \lambda^{j-1}$ for $j \in J$. Therefore,

$$\|K_{ii}^{-1}T_{ij}K_{jj}\| \leqq \lambda^{-i+j-1}\|T\| \text{ for } i, j \in J, i < j + 3,$$

and hence $(K^\#)^{-1}T^\#K^\#$ has a matrix representation $((K_{ii}^{-1}T_{ij}K_{jj}))$ whose entries are majorized by the entries of the numerical matrix $((c_{ij}))$, where

$$c_{ij} = \begin{cases} \lambda^{j-i-1}\|T\| & \text{if } i < j + 3, \\ 0 & \text{if } i \geqq j + 3. \end{cases}$$

Since $((c_{ij}))$ defines a bounded operator, it follows from [**13**, Lemma 1] that $(K^\#)^{-1}T^\#K^\#$ is bounded and hence $T^\#$ leaves the range of $K^\#$ invariant. This completes the proof of the theorem.

THEOREM 5. *Let* $\mathscr{A} \in \mathscr{R}_{n-1} \backslash \mathscr{R}_n$ *for some integer* $n \geqq 2$. *Then there exists an invariant subspace* $\mathscr{M}$ *of* $\mathscr{A}^{(n)}$ *such that* $\mathscr{M} = \mathscr{P} \oplus \mathscr{N}$, *where* $\mathscr{N}$ *is the span of all vectors in* $\mathscr{M}$ *having some zero components, and* $\mathscr{M}$ *is the smallest invariant subspace of* $\mathscr{A}^{(n)}$ *containing* $\mathscr{P}$. *Moreover,* $\mathscr{M} \notin \text{lat } T^{(n)}$ *for some* $T \in \text{alg lat} \mathscr{A}$. *Also, the following statements are true.*

(a) *The linear manifolds* $\mathscr{M}_i$ *and* $\mathscr{N}_i$ *are invariant operator ranges of* $\mathscr{A}$, $i = 1, 2, \ldots, n$.

(b) *If* $\mathscr{I}$ *is the maximal invariant subspace of* $(\text{alg lat} \mathscr{A})^{(n)}$ *contained in* $\mathscr{M}$ *and if* $\mathscr{Q} = \mathscr{M} \ominus \mathscr{I}$, *then* $\mathscr{Q} \neq \{0\}$ *and for all nonzero vectors* $x \in \mathscr{Q}$ *the components* $x_1, \ldots, x_n$ *are linearly independent.*

(c) *If* $\mathscr{A}$ *contains a* m.a.s.a., *then no* $\mathscr{M}_i$ *is the range of a compact operator.*

Proof. The existence of $\mathscr{M}$, $\mathscr{N}$ and $\mathscr{P}$ with the required properties is easy and follows from an argument similar to the one used in the proof of Theorems 1, 2 and 3.

For (a) observe that each $\mathscr{M}_i$ (respectively $\mathscr{N}_i$) is the range of the operator $x \mapsto x_i$ from $\mathscr{M}$ (resp. $\mathscr{N}$) onto $\mathscr{M}_i$ (resp. $\mathscr{N}_i$).

Let $\mathscr{I}$ and $\mathscr{Q}$ be as in (b). Since $\mathscr{M} \neq \mathscr{I}$, $\mathscr{Q} \neq \{0\}$. Let $\bar{x} \in \mathscr{M}$ be such that $\sum a_i \bar{x}_i = 0$, where $a_1, \ldots, a_n$ are complex numbers and $a_i = 1$ for some i which can be assumed without loss of generality to be 1. Let

$$\mathscr{S} = \{x \in \mathscr{M} : \sum a_i x_i = 0\} \quad \text{and}$$
$$\mathscr{S}' = \{x_2 \oplus \ldots \oplus x_n : x_1 \oplus x_2 \oplus \ldots \oplus x_n \in \mathscr{S}\}.$$

It is easy to see that $\mathscr{S}'$ is a (closed) invariant subspace of $\mathscr{A}^{(n-1)}$ and, consequently, that of $(\text{alg lat} \mathscr{A})^{(n-1)}$. So $\mathscr{S}$ is an invariant subspace of $(\text{alg lat} \mathscr{A})^{(n)}$ which implies that $\bar{x} \in \mathscr{S} \subset \mathscr{I}$.

Finally we prove (c). Let $\mathscr{I}$ and $\mathscr{Q}$ be as in (b), and let $A \in \mathscr{A}$ be a self-adjoint operator of multiplicity 1. It is easy to see that $\mathscr{N} \subset \mathscr{I}$, $\mathscr{Q} \subset \mathscr{P}$ and $\mathscr{Q}$ is a reducing invariant subspace of $A^{(n)}$. Thus, in view of Lemma 2 and its proof, $\bar{\mathscr{Q}}_1, \ldots, \bar{\mathscr{Q}}_n$ are equal spectral subspaces of A, and reduce the normal operators $C_i V : \bar{\mathscr{P}}_1 \rightarrow \bar{\mathscr{P}}_1$ of the proof of Lemma 3. Assume, if possible, that some $\mathscr{M}_i$ is the range of a compact operator. By a reordering of the copies of $H^{(n)}$, one can assume without loss of generality that $i = 1$. The operator $x \mapsto x_1$ from $\mathscr{M}$ onto $\mathscr{M}_1$ is compact. In particular, $C_1 V | \bar{\mathscr{Q}}_1$ is a compact normal operator. Hence the bounded normal operators $C_1 V | \bar{\mathscr{Q}}_1$ and $C_2 V | \bar{\mathscr{Q}}_1$ have a common reducing finite-dimensional invariant subspace and thus the linear transformation $G_1 = (C_2 V)(C_1 V)^{-1}$ has an eigenvector in $\mathscr{Q}_1$. It follows that $\mathscr{Q}$ contains a nonzero vector x such that $x_1, x_2, \ldots, x_n$ are not linearly independent, a contradiction.

COROLLARY 5. *Let* $\mathscr{A}$ *be a weakly closed algebra containing a* m.a.s.a. *Assume every invariant operator range of* $\mathscr{A}$ *is either closed or the range of a compact operator. Then* $\mathscr{A}$ *is reflexive.*

Proof. Assume, if possible, that $\mathscr{A} \in \mathscr{R}_{n-1} \backslash \mathscr{R}_n$ for some $n \geqq 2$. Let $\mathscr{M}, \mathscr{N}$ and $\mathscr{P}$ be as in Theorem 5. Since no $\mathscr{M}_i$ is the range of a compact operator, each $\mathscr{M}_i$ is closed and hence, in view of Lemma 2, each $\mathscr{P}_i$ is closed. Thus the linear transformations $G_1, G_2, \ldots, G_{n-1}$ of Lemma 3 are bounded normal operators and $G_i B^\# = B^\# G_i$ for all i and all $B \in \mathscr{A}$, where $B^\# = (I - P)B|\overline{\mathscr{P}}_1$ and P is the orthogonal projection with range $\overline{\mathscr{N}}_1$. Let $\lambda \in \sigma(G_1)$. Since $B^\#$ commutes with $G_1 - \lambda$ for all $B \in \mathscr{A}$, it follows from Lemma 2 that the operator range

$$R = \{u \oplus v : u \in \text{Range } (G_1 - \lambda) \text{ and } v \in \overline{\mathscr{N}}_1\}$$

is an invariant operator range of $\mathscr{A}$ and, hence, either Range $(G_1 - \lambda)$ is closed or $G_1 - \lambda$ is compact. Let $\mathscr{Q}$ be as in Theorem 5. We saw in the proof of Theorem 5(c) that $\mathscr{Q}_1$ $(= \overline{\mathscr{Q}}_1)$ is a reducing invariant subspace of G_1 and hence $(G_1 - \lambda)|\mathscr{Q}_1$ is either compact or has a closed range for all $\lambda \in \sigma(G_1|\mathscr{Q}_1)$. In any case, G_1 has an eigenvector in $\mathscr{Q}_1$ which implies that $\mathscr{Q}$ contains a nonzero vector x such that $x_1, x_2, \ldots, x_n$ are not linearly independent, a contradiction.

Remark 1. Corollary 5 is not true for a general algebra $\mathscr{A}$. In Examples 1 and 2 we saw that nonreflexive algebras exist on finite-dimensional Hilbert spaces; for such algebras all invariant operator ranges are closed ranges of compact operators.

Remark 2. In view of Corollary 5, on finite-dimensional Hilbert spaces every algebra containing a m.a.s.a. is reflexive [**2**, page 484].

Definition 3. A weakly closed algebra $\mathscr{A} \subset B(H)$ is called *k-reductive* if

$$\operatorname{lat} \mathscr{A}^{(k)} = \operatorname{lat} \mathscr{A}^{*(k)};$$

and is called *k-transitive* if

$$\operatorname{lat} \mathscr{A}^{(k)} = \operatorname{lat} [B(H)]^{(k)}.$$

The definition of a k-transitive algebra first appeared in [**6**].

THEOREM 6. *A reductive (transitive) algebra $\mathscr{A}$ is k-reductive (k-transitive) if and only if $\mathscr{A} \in \mathscr{R}_k$. Moreover, if $\mathscr{A} \in \mathscr{R}_{n-1} \backslash \mathscr{R}_n$ is reductive and if $\mathscr{M}$ is an invariant subspace of $\mathscr{A}^{(n)}$ not invariant under* (alg lat $\mathscr{A})^{(n)}$, *then $\mathscr{M}$ contains an invariant subspace $\mathscr{P}$ of $\mathscr{A}^{(n)}$ with the following properties.*

(a) *$\mathscr{P}$ contains no nontrivial reducing invariant subspace of $\mathscr{A}^{(n)}$ and the components $x_1, \ldots, x_n$ of any nonzero vector $x \in \mathscr{P}$ are linearly independent.*

(b) *If $n \geqq 3$, no $\mathscr{P}_i$ is closed.*

(c) *If $\mathscr{A}$ is transitive and if $\{i(1), \ldots, i(k)\}$ is a set of integers such that $1 \leqq i(1) < i(2) < \ldots < i(k) \leqq n$ for some positive integer $k < n$, then*

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the linear manifold

$$\mathscr{Q} = \{x_{i(1)} \oplus \ldots \oplus x_{i(k)} : x_1 \oplus \ldots \oplus x_n \in \mathscr{P}\}$$

is dense in $H^{(k)}$. *In particular, if* $n \geqq 3$, *then* $\mathscr{Q}$ *is not closed.*

(d) *If* $\mathscr{A}$ *is transitive, no* $\mathscr{P}_i$ *is the range of a compact operator.*

Proof. Assume $\mathscr{A}$ is reductive (transitive). Since every von-Neumann algebra is reflexive (Double Commutant Theorem), alg lat $\mathscr{A}$ is the von-Neumann algebra generated by $\mathscr{A} \cup \mathscr{A}^*$. (If $\mathscr{A}$ is transitive, then alg lat $\mathscr{A} = B(H)$). This shows that $\mathscr{A}$ is k-reductive (k-transitive) if and only if $\mathscr{A} \in \mathscr{R}_k$. Now assume $\mathscr{A} \in \mathscr{R}_{n-1} \backslash \mathscr{R}_n$ is reductive. Note that $\mathscr{M}$ is a non-reducing invariant subspace of $\mathscr{A}^{(n)}$ if and only if $\mathscr{M}$ is not left invariant by (alg lat $\mathscr{A})^{(n)}$.

To prove (a), let $\mathscr{M}$ be an arbitrary non-reducing invariant subspace of $\mathscr{A}^{(n)}$. Let $\mathscr{P}$ be the orthogonal complement of the maximal reducing invariant subspace of $\mathscr{A}^{(n)}$ contained in $\mathscr{M}$. In view of Theorem 5(b), $\mathscr{P}$ is the required subspace.

For part (b) assume, if possible, that $\mathscr{P}_i$ is closed for some i, which can be assumed to be 1. It follows that the operator $C_1 : \mathscr{P} \rightarrow \mathscr{P}_1$ is invertible and

$$\mathscr{P} = \{u \oplus G_1 u \oplus \ldots \oplus G_{n-1} u : u \in \mathscr{P}_1\},$$

where $G_i = C_{i+1} C_1^{-1}$ $(i = 1, \ldots, n-1)$ are bounded linear transformations. Since each $\{u \oplus G_i u : u \in \mathscr{P}_1\}$ is a closed invariant subspace of $\mathscr{A}^{*(2)}$, $G_i T^* = T^* G_i$ (on $\mathscr{P}_1$) for all $T \in \mathscr{A}$ and hence $\mathscr{P}$ is an invariant subspace of $\mathscr{A}^{*(n)}$, a contradiction.

Next let $\mathscr{A}$ and $i(1), \ldots, i(k)$ be as in (c), and assume without loss of generality that $i(j) = j$, $j = 1, 2, \ldots, k$. Let

$$\mathscr{Q} = \{x_1 \oplus x_2 \oplus \ldots \oplus x_k : x_1 \oplus x_2 \oplus \ldots \oplus x_k \oplus \ldots \oplus x_n \in \mathscr{P}\}.$$

The set $\mathscr{Q}$ is an invariant linear manifold of $\mathscr{A}^{(k)}$ and hence $\overline{\mathscr{Q}}$ is an invariant subspace of $[B(H)]^{(k)}$. Let $y_1 \oplus \ldots \oplus y_k \in H^{(k)}$ be arbitrary. Take $0 \neq x \in \mathscr{P}$. Since $x_1, \ldots, x_n$ are linearly independent, we can define an operator B such that $Bx_i = y_i$, $i = 1, \ldots, k$. It follows that

$$y_1 \oplus \ldots \oplus y_k = B^{(k)}(x_1 \oplus \ldots \oplus x_k) \in \overline{\mathscr{Q}}.$$

This shows that $\mathscr{Q}$ is dense in $H^{(k)}$.

Let $n \geqq 3$. If $k = 1$, it follows from (b) that $\mathscr{Q} = \mathscr{P}_1$ is not closed. If $k \geqq 2$, it follows from (a) that $\mathscr{Q} \neq H^{(k)}$.

Finally assume $\mathscr{A}$ is as in (d) and, if possible, $\mathscr{P}_i$ is the range of a compact operator. Assume without loss of generality that $i = n$. If $n = 2$ and $\mathscr{P}_1$ is closed, then

$$\mathscr{P} = \{x \oplus Kx : x \in H\},$$

where K is a compact operator commuting with $\mathscr{A}$, a contradiction [**12**]. Otherwise, in view of (c), the manifold

$$\mathscr{Q} = \{C_1 x \oplus \ldots \oplus C_{n-1}x : x \in \mathscr{P}\}$$

is not closed, where $C_i : \mathscr{P} \to \mathscr{P}_i$ is defined by $C_i x = x_i$ $(i = 1, \ldots, n)$. Let $y_1 \oplus \ldots \oplus y_{n-1} \notin \mathscr{Q}$. Let $\{x(k)\}$ be a sequence in $\mathscr{P}$ such that $y_i = \lim C_i x(k), i = 1, 2, \ldots, n - 1$. We claim $\|C_n x(k)\|$ diverges to ∞. If not, then $\{x(k)\}$ has a subsequence converging weakly to a vector of the form $y_1 \oplus \ldots \oplus y_{n-1} \oplus y_n \in \mathscr{P}$, a contradiction.

Consider the bounded sequence

$$z(k) = x(k)/\|C_n x(k)\|, \; k = 1, 2, \ldots .$$

Obviously, $\lim C_i z(k) = 0$ for $i = 1, 2, \ldots, n - 1$, and there exists a subsequence $\{z(k_m)\}$ such that the sequence $\{C_n z(k_m)\}$ is (strongly) convergent (note that C_n is a compact linear transformation). But $\|C_n z(k_m)\| = 1$, which implies that a nonzero vector of the form $0 \oplus \ldots \oplus 0 \oplus u$ belongs to $\mathscr{P}$, again a contradiction.

COROLLARY 6. ([**14**]) *Let $\mathscr{A}$ be a weakly closed transitive algebra. If every invariant operator range of $\mathscr{A}$ is either closed or the range of a compact operator, then $\mathscr{A} = B(H)$.*

The proof follows easily from Theorem 6. However, in Corollary 9 below, we prove a similar result for reductive algebras.

LEMMA 4. *Let $\mathscr{A} \in \mathscr{R}_1 \backslash \mathscr{R}_2$ be a reductive algebra. Let $\mathscr{P}$ be an invariant subspace of $\mathscr{A}^{(2)}$ which contains no nontrivial reducing invariant subspace of $\mathscr{A}^{(2)}$. Assume $\mathscr{P}_1$ is closed and let P be the projection from H onto $\mathscr{P}_1$. Then the set $\{x_1 \oplus Px_2 : x_1 \oplus x_2 \in \mathscr{P}\}$ is an invariant subspace of $\mathscr{A}^{(2)}$ which contains no nontrivial reducing invariant subspace of $\mathscr{A}^{(2)}$.*

Proof. Since x_1 and x_2 are linearly independent for all nonzero $x_1 \oplus x_2 \in \mathscr{P}$, it follows that $\mathscr{P} = \{x \oplus Kx : x \in \mathscr{P}_1\}$, where $K : \mathscr{P}_1 \to \mathscr{P}_2$ is a bounded operator commuting with $\mathscr{A}$ (on $\mathscr{P}_1$). Thus $B(I - P)K = (I - P)KB$ and $B(PK) = (PK)B$ for all $B \in \mathscr{A}$ (on $\mathscr{P}_1$). Hence the set

$$\{x + (I - P)Kx : x \in \mathscr{P}_1\} \subset H$$

is an invariant subspace of $\mathscr{A}$ and consequently of $\mathscr{A}^*$. So

$$B^*(I - P)K = (I - P)KB^*$$

(on $\mathscr{P}_1$) for all $B \in \mathscr{A}$. Also, the set

$$\mathscr{Q} = \{x \oplus PKx : x \in \mathscr{P}_1\}$$

is an invariant subspace of $\mathscr{A}^{(2)}$.

It remains to show that $\mathscr{Q}$ contains no nontrivial reducing invariant subspace of $\mathscr{A}^{(2)}$. Let $\mathscr{S} \subset \mathscr{Q}$ be a reducing invariant subspace of $\mathscr{A}^{(2)}$.

Then

$$\mathscr{S} = \{x \oplus PKx : x \in \mathscr{S}_1\},$$

$\mathscr{S}_1 \subset \mathscr{P}_1$ is closed, and

$$B^*(PK|\mathscr{S}_1) = (PK|\mathscr{S}_1)B^*$$

(on $\mathscr{S}_1$) for all $B \in \mathscr{A}$. Hence the set $\{x \oplus Kx : x \in \mathscr{S}_1\}$ is a reducing invariant subspace of $\mathscr{A}^{(2)}$, which implies that $\mathscr{S}_1$ is zero. (Note that $K = PK + (I - P)K$.) Thus $\mathscr{S} = \{0\}$ and the proof is complete.

COROLLARY 7. *Let $\mathscr{A} \in \mathscr{R}_{n-1}\backslash\mathscr{R}_n$ be a reductive algebra, and let $\mathscr{P}$ be an invariant subspace of $\mathscr{A}^{(n)}$ which contains no nontrivial reducing invariant subspace of $\mathscr{A}^{(n)}$. Then not all $\mathscr{P}_i$ are the ranges of compact operators.* (*In particular, every reductive algebra in a finite-dimentional Hilbert space is self-adjoint* [**4**].)

Proof. Assume, if possible, that all $\mathscr{P}_i$ are the ranges of compact operators which implies that $\mathscr{P}$ itself is the range of a compact operator. Hence $\mathscr{P}$ is finite-dimensional and all $\mathscr{P}_i$ are closed. Thus $n = 2$ and

$$\mathscr{P} = \{x \oplus Kx : x \in \mathscr{P}_1\}.$$

In view of Lemma 4, we can assume without loss of generality that $\mathscr{P}_1 = \mathscr{P}_2$. So K has an eigenvector (in $\mathscr{P}_1$) and, therefore, $\mathscr{P}$ has a nonzero vector x such that x_1 and x_2 are not linearly independent, a contradiction.

The following corollary is known for transitive algebras [**18**, page 146]. In the following by a *graph transformation* of an algebra $\mathscr{A}$ we mean any linear transformation T for which there exist an integer n and an invariant subspace $\mathscr{M}$ of $\mathscr{A}^{(n)}$ such that $C_1, \ldots, C_n$ are injective and $T = C_iC_j^{-1}$ for some distinct pair i and j, where $C_i : \mathscr{M} \rightarrow \mathscr{M}_i$ is defined by $C_ix = x_i$ $(i = 1, \ldots, n)$. The range of a graph transformation of $\mathscr{A}$ is called a *graph operator range* of $\mathscr{A}$. Note that any graph operator range of $\mathscr{A}$ is an invariant operator range of $\mathscr{A}$.

COROLLARY 8. *Let $\mathscr{A}$ be a weakly closed reductive algebra. Assume every graph transformation of $\mathscr{A}$ has an eigenvalue. Then $\mathscr{A}$ is self-adjoint.*

Proof. If $\mathscr{A} \neq \mathscr{A}^*$, then $\mathscr{A} \in \mathscr{R}_{n-1}\backslash\mathscr{R}_n$ for some integer $n \geqq 2$. Let $\mathscr{P}$ be an invariant subspace of $\mathscr{A}^{(n)}$ which contains no nontrivial reducing invariant subspace of $\mathscr{A}^{(n)}$. Define $C_i : \mathscr{P} \rightarrow \mathscr{P}_i$ by $C_ix = x_i$. Since $C_1x, \ldots, C_nx$ are linearly independent for all nonzero $x \in \mathscr{P}$, it follows that no $C_iC_j^{-1}$ has an eigenvalue $(i \neq j)$, a contradiction.

COROLLARY 9. *Let $\mathscr{A}$ be a weakly closed reductive algebra such that every graph operator range of $\mathscr{A}$ is of the form $\{u \oplus v : u \in M, v \in R\}$, where*

M is an invariant subspace of $\mathscr{A}$, R is an invariant compact-operater range of $\mathscr{A}$, and M, $\bar{R}$ are perpendicular. Then $\mathscr{A}$ is self-adjoint. In particular, if every invariant operator range of a weakly closed reductive algebra is either closed or the range of a compact operator, then it is self-adjoint.

Proof. Assume that $\mathscr{A} \neq \mathscr{A}^*$. Then $\mathscr{A} \in \mathscr{R}_{n-1} \backslash \mathscr{R}_n$ for some $n \geqq 2$. Let $\mathscr{P}$ be a nontrivial invariant subspace of $\mathscr{A}^{(n)}$ which contains no nontrivial reducing invariant subspace of $\mathscr{A}^{(n)}$. Each $\mathscr{P}_i$ is of the form $\{u \oplus v: u \in M_i, v \in R_i\}$ as in the statement of the theorem. Since not all $\mathscr{P}_i$ are the ranges of compact operators, $M_i \neq \{0\}$ for some i which can be assumed without loss of generality that $i = 1$ and $\mathscr{P}_1 = M_1$. Therefore, in view of Theorem 6(b), $n = 2$. Using Lemma 4, if necessary, we can modify $\mathscr{P}$ such that

$$\mathscr{P}_2 \subset \mathscr{P}_1 = \overline{\mathscr{P}}_1 \quad \text{and} \quad \mathscr{P} = \{x \oplus Kx: x \in \mathscr{P}_1\}.$$

The operator $K: \mathscr{P}_1 \rightarrow \mathscr{P}_1$ has no eigenvalue and hence Range $(K - \lambda)$ is nonclosed for some $\lambda \in \sigma(K)$. Since $\{x \oplus (K - \lambda)x : x \in \mathscr{P}_1\}$ is an invariant subspace of $\mathscr{A}^{(2)}$, it follows that

$$\text{Range } (K - \lambda) = M \oplus R,$$

where M is an invariant subspace of $\mathscr{A}$ and $R \neq \{0\}$ is an invariant compact-operator range of $\mathscr{A}$. Let

$$\mathscr{Q} = \{x \oplus (K - \lambda)x: x \in \mathscr{P}_1 \quad \text{and} \quad (K - \lambda)x \in R\}.$$

It is easy to see that $\mathscr{Q} \in \text{lat}\,\mathscr{A}^{(2)}$, $\mathscr{Q}_1$ is closed, and $\mathscr{Q}_2 = R$. Define $S : H \rightarrow H$ by $Sx = (K - \lambda)x$ for $x \in \mathscr{Q}_1$ and $Sx = 0$ for $x \perp \mathscr{Q}_1$. The operator S is compact and $SB = BS$ for all $B \in \mathscr{A}$. Hence $SB^* = B^*S$ for all $B \in \mathscr{A}$, [**19**], which implies that

$$(K - \lambda)B^*x = B^*(K - \lambda)x \text{ for all } x \in \mathscr{Q}_1.$$

Thus the set $\{x \oplus Kx: x \in \mathscr{Q}_1\} \subset \mathscr{P}$ is a reducing invariant subspace of $\mathscr{A}^{(2)}$, a contradiction.

COROLLARY 10. ([**9**]) *If $\mathscr{A}$ is a weakly closed reductive algebra such that every operator range invariant under $\mathscr{A}$ is closed, then $\mathscr{A}$ is self-adjoint.*

COROLLARY 11. ([**9**]) *If $\mathscr{A}$ is a weakly closed reductive algebra and if every graph transformation of $\mathscr{A}$ is bounded, then $\mathscr{A}$ is self-adjoint.*

Proof. Let R be an arbitrary graph operator range of $\mathscr{A}$ and let $\mathscr{M} \in \text{lat}\,\mathscr{A}^{(n)}$ be such that $R = \mathscr{M}_1$ and the mappings $C_i: \mathscr{M} \rightarrow \mathscr{M}_i$ $(i = 1, 2, \ldots, n)$ are injective. We show that R is closed. Let $\{x(k)\}$ be a sequence in $\mathscr{M}$ such that $\{C_1 x(k)\}$ converges to y_1. Since each $C_i C_1^{-1}$ is bounded, it follows that $\{C_i x(k)\}$ converges to a vector y_i, $i = 1, 2, \ldots, n$. Hence the sequence $\{x(k)\}$ converges to $y_1 \oplus \ldots \oplus y_n$ which implies that $y_1 \in \mathscr{M}_1 = R$. This shows that R is closed and hence $\mathscr{A}$ is self-adjoint.

Added in proof. We thank Professor Peter Rosenthal who informed us of some known results which led to the following remarks.

(a) Let k be a natural number. An algebra $\mathscr{A}$ is called *k-reflexive* if $\mathscr{A}^{(k)}$ is reflexive. Let $n \geqq 3$ be a natural number. Let $\mathscr{A}$ be an arbitrary algebra on an n-dimensional Hilbert space H_n. Azoff [**3**, Theorem 3.1] has shown that $\mathscr{A}$ is $(n - 1)$-reflexive. In view of Proposition 1, if $\mathscr{A}$ is nonreflexive, then $\mathscr{A} \notin \mathscr{R}_{n-1}$. Azoff [**3**, Example 3.2] also gives an example of an algebra $\mathscr{A}$ on H_n which is not $(n - 2)$-reflexive. By an argument similar to the one given in our Example 1, one can show that the algebra $\mathscr{A}$ of [**3**, Example 3.2] belongs to $\mathscr{R}_1 \backslash \mathscr{R}_2$. Therefore, Proposition 1 is the most that can be said about the relation between the class $\mathscr{R}_k$ and the class of k-reflexive operators. For $n \geqq 4$, in view of [**3**, Theorem 4.1], every nonreflexive commutative algebra on H_n does not belong to $\mathscr{R}_{n/2}$, where $n/2$ is to be interpreted as the greatest integer in $n/2$.

(b) The existence of a nonreflexive $\mathscr{A}^{(2)}$ in Proposition 1 is due to Feintuch [**8**].

(c) An operator-theoretic proof of Corollary 2 is also given by Nordgren–Radjavi–Rosenthal [**15**]. There are similarities between our techniques and those of [**8**] and [**15**].

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