Proceedings of the Edinburgh Mathematical Society (2008) **51**, 273–284 ©
DOI:10.1017/S0013091505001203 Printed in the United Kingdom

ON THE REGULARITY CONJECTURE FOR THE COHOMOLOGY OF FINITE GROUPS

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(Received 29 August 2005)

Abstract Let K be a field of characteristic p and let G be a finite group of order divisible by p. The regularity conjecture states that the Castelnuovo–Mumford regularity of the cohomology ring $H^*(G,K)$ is always equal to 0. We prove that if the regularity conjecture holds for a finite group H, then it holds for the wreath product $H \wr \mathbb{Z}/p$. As a corollary, we prove the regularity conjecture for the symmetric groups Σ_n. The significance of this is that it is the first set of examples for which the regularity conjecture has been checked, where the difference between the Krull dimension and the depth of the cohomology ring is large. If this difference is at most 2, the regularity conjecture is already known to hold by previous work.

For more general wreath products, we have not managed to prove the regularity conjecture. Instead we prove a weaker statement: namely, that the dimensions of the cohomology groups are polynomial on residue classes (PORC) in the sense of Higman.

Keywords: cohomology of groups; local cohomology; Castelnuovo–Mumford regularity; wreath product

2000 *Mathematics subject classification:* Primary 20J06
Secondary 13D45

1. Introduction

Let K be a field of characteristic p and let G be a finite group. Then the regularity conjecture [**2**,**3**] states that the Castelnuovo–Mumford regularity of the cohomology ring $\operatorname{Reg} H^*(G,K)$ is always equal to 0. Briefly, the definitions are as follows. We write $\mathfrak{m}$ for the maximal ideal of positive-degree elements in $H^*(G,K)$. If M is a graded $H^*(G,K)$-module, we define $\Gamma_{\mathfrak{m}} M$ to be the $\mathfrak{m}$-torsion in M: namely, $\{x \in M \mid \exists n > 0,\ \mathfrak{m}^n x = 0\}$. Then $\Gamma_{\mathfrak{m}}$ is left exact but not right exact, and its right-derived functors give the *local cohomology* of M and are written $H^i_{\mathfrak{m}} M$. Since there is also an internal grading, we write $H^{i,j}_{\mathfrak{m}} M$ for the jth graded piece. We define $a^i_{\mathfrak{m}}(M) = \max\{j \in \mathbb{Z} \mid H^{i,j}_{\mathfrak{m}} M \neq 0\}$ ($\pm\infty$ is allowed) and the *Castelnuovo–Mumford regularity* is $\operatorname{Reg} M = \max\{a^i_{\mathfrak{m}} M + i\}$. This is a measure of how far you have to go along a resolution before 'regular behaviour' sets in.

There are various motivations for studying regularity of cohomology rings. One is that it gives *a priori* bounds for how much of a projective resolution you have to compute before you can be sure that you have all the generators and relations in the cohomology ring (see Theorem 10.1 of [**3**] for further details). Another motivation is that it gives more precise information about the duality developed by Benson and Carlson [**4**]. In terms of the definitions of that paper, a regularity of zero ensures that the 'last survivor' really *is* last. A further consequence of the regularity conjecture is that regular behaviour for the dimensions of the cohomology groups $\dim_K H^n(G, K)$ begins straight away, as we prove in Theorem 5.5.

In Theorem 1.5 of [**3**] it was proved that, as long as the Krull dimension and the depth of $H^*(G, K)$ differ by at most two, the regularity conjecture holds. Unfortunately, the vast majority of the examples for which the regularity conjecture has been checked satisfy this bound; for example, this is the case for all 2-groups of order at most 64 [**5**] as well as various other classes of finite groups studied in [**11**, **13**, **17**, **20**]. So to gain more confidence in the conjecture, it is desirable to check families of examples where the difference is greater than 2.

One good way of producing examples where the difference between the Krull dimension and the depth is large is to look at wreath products. The goal of this article is to provide further evidence for the regularity conjecture by examining wreath products. In particular, we shall prove that the conjecture holds for the cohomology of the finite symmetric groups, where the difference between the Krull dimension and the depth is arbitrarily large. This follows from Quillen's stratification theorem [**18**, **19**]: Σ_{p^n} has p-rank p^{n-1}, so this is the Krull dimension; but there is also a conjugacy class of elementary abelian p-subgroups of rank n, so that the depth is at most n.

Let H be a finite group and consider the wreath product

$$H \wr \mathbb{Z}/p = H^p \rtimes \mathbb{Z}/p.$$

Our main theorem is the following, the proof of which can be found in § 3.

Theorem 1.1. *Suppose that* $\operatorname{Reg} H^*(H, K) = 0$. *Then* $\operatorname{Reg} H^*(H \wr \mathbb{Z}/p, K) = 0$.

Our proof involves using the structure of the cohomology of wreath products, as described in Nakaoka [**15**]. Using the fact that the Sylow p-subgroups of Σ_n are direct products of wreath products, we obtain the following corollary.

Corollary 1.2. *If* $\Sigma_n \geqslant G \geqslant \operatorname{Syl}_p(\Sigma_n)$, *then* $\operatorname{Reg} H^*(G, K) = 0$.

The corollary is obtained in § 3, where we also treat the alternating groups $\mathcal{A}_{2^n}$ in characteristic 2.

For more general wreath products, similar techniques probably work. But the technical details become much harder. Instead, we show that a numerical consequence of regularity does hold for more general wreath products. Namely, in Theorem 5.5 we show that one consequence of the regularity conjecture is that the dimension function $i \mapsto \dim_K H^i(G, K)$ is a polynomial on residue classes (PORC) function in the sense of Higman [**12**].

Theorem 1.3. *If* $|G|$ *is divisible by the characteristic of* K *and if* $\operatorname{Reg} H^*(G, K) = 0$, *then there exist an integer* d *and polynomials* $f_0, \dots, f_{d-1}$ *such that, for all* $i \geqslant 0$, $\dim_K H^i(G, K) = f_j(i)$, *where* j *is the remainder on dividing* i *by* d.

The fact that this regular behaviour occurs for large enough i is a simple consequence of finite generation. The interesting consequence of the regularity conjecture is that the eventual behaviour begins straight away. We prove the following theorem in § 5.

Theorem 1.4. *Let K be a field of characteristic p. Let H be a finite group and let G be a permutation group on a finite set Ω. Let $H \wr G$ be the wreath product, where G permutes a product of copies of H indexed by Ω. Suppose that p divides the order of H, the function $i \mapsto H^i(H, K)$ is PORC, and that for every subgroup J of G of order divisible by p the function $i \mapsto H^i(J, K)$ is PORC. Then the function $i \mapsto H^i(H \wr G, K)$ is PORC.*

2. Castelnuovo–Mumford regularity

Let $R = \bigoplus_{j \geqslant 0} R_j$ be a Noetherian graded commutative K-algebra. Here *graded commutative* means that for homogeneous elements x and y we have $yx = (-1)^{|x|\,|y|}xy$, where $|x|$ denotes the degree of x. We shall assume that R is *connected*, meaning that $R_0 = K$. For example, these conditions hold in the case where R is the cohomology ring of a finite group: $R = H^*(G, K)$. Let $\mathfrak{m}$ be the maximal ideal spanned by the homogeneous elements of positive degree, $\mathfrak{m} = \bigoplus_{j>0} R_j$. If M is a graded R-module (we allow positive and negative grading), then the local cohomology is doubly graded: $H^{i,j}_{\mathfrak{m}}M$. The first grading is the local cohomological degree and the second is the internal degree coming from the grading on R and M. We define the *a-invariants* of M to be

$$a^i_{\mathfrak{m}}(M) = \max\{j \in \mathbb{Z} \mid H^{i,j}_{\mathfrak{m}}M \neq 0\},$$

$a^i_{\mathfrak{m}}(M) = -\infty$ if $H^{i,j}_{\mathfrak{m}}M = 0$ for all j and $a^i_{\mathfrak{m}}(M) = \infty$ if $H^{i,j}_{\mathfrak{m}}M \neq 0$ for arbitrarily large values of j. Note that $a^i_{\mathfrak{m}}(M) = -\infty$ unless i lies between the depth and the Krull dimension of M, by a theorem of Grothendieck.

The *Castelnuovo–Mumford regularity* is defined to be

$$\operatorname{Reg} M = \max_{i \geqslant 0}\{a^i_{\mathfrak{m}}(M) + i\}.$$

See § 4 of [**3**] for further details, and § 20.5 of [**9**] for the history and geometric significance of this definition for strictly commutative K-algebras whose generators have degree equal to 1. See also [**14**] for a more general context for this definition, and [**7**] for a closely related definition. The significance of regularity in group cohomology is the following conjecture from [**2**, **3**].

Conjecture 2.1. *Let G be a finite group. Then* $\operatorname{Reg} H^*(G, K) = 0$.

We mention that it is proved in Theorem 4.2 of [**3**] that the inequality

$$\operatorname{Reg} H^*(G, K) \geqslant 0 \tag{2.1}$$

holds.

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Regularity can be reformulated in terms of free resolutions as follows. Let $\zeta_1, \dots, \zeta_r$ be a filter regular homogeneous system of parameters in R, and let

$$0 \to F_r \to \cdots \to F_1 \to F_0 \to M \to 0$$

be a minimal free resolution of M over the polynomial ring $K[\zeta_1, \dots, \zeta_r]$. Define $\beta_i^R(M)$ to be the largest degree of a generator of F_i as an R-module (or $\beta_i^R(M) = -\infty$ if $F_i = 0$). Then it is proved in Corollary 5.7 of [**3**] that

$$\operatorname{Reg} M = \max_{i \geqslant 0} \{\beta_i^R(M) - i\} - \sum_{j=1}^{r} (|\zeta_j| - 1). \tag{2.2}$$

In particular, if M is finitely generated, then $\operatorname{Reg} M$ is finite, and hence the $a_{\mathfrak{m}}^i(M)$ are either finite or equal to $-\infty$.

In order to prove the main theorem, we begin with some general properties of regularity.

Proposition 2.2. *Suppose that*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of R-modules. Then we have

$$\operatorname{Reg} M_2 \leqslant \max\{\operatorname{Reg} M_1, \operatorname{Reg} M_3\}.$$

If $\operatorname{Reg} M_1 \leqslant \operatorname{Reg} M_3$, *then* $\operatorname{Reg} M_2 = \operatorname{Reg} M_3$.

Proof. This follows from the long exact sequence in local cohomology:

$$\cdots \to H_{\mathfrak{m}}^{i,j}(M_1) \to H_{\mathfrak{m}}^{i,j}(M_2) \to H_{\mathfrak{m}}^{i,j}(M_3) \to H_{\mathfrak{m}}^{i+1,j}(M_1) \to \cdots.$$

If $\operatorname{Reg} M_1 \leqslant \operatorname{Reg} M_3$, choose i and j so that $i + j = \operatorname{Reg} M_3$ and $H_{\mathfrak{m}}^{i,j}(M_3) \neq 0$. Then $H_{\mathfrak{m}}^{i+1,j}(M_1) = 0$ and so $H_{\mathfrak{m}}^{i,j}(M_2) \neq 0$. □

Corollary 2.3. *Suppose that*

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = \{0\}$$

is a filtration of an R-module M. Then

$$\operatorname{Reg} M \leqslant \max_{0 \leqslant i < n} \{\operatorname{Reg} M_i/M_{i+1}\}.$$

If for $0 < i < n$ *we have* $\operatorname{Reg} M_i/M_{i+1} \leqslant \operatorname{Reg} M_0/M_1$, *then* $\operatorname{Reg} M = \operatorname{Reg} M_0/M_1$.

Proof. This follows from Proposition 2.2 and induction on n. □

Proposition 2.4. *If R_1 and R_2 are graded K-algebras satisfying the hypotheses of this section, with graded modules M_1 and M_2, respectively, then we may regard $M_1 \otimes_K M_2$ as an $R_1 \otimes_K R_2$-module, and its regularity is*

$$\operatorname{Reg}(M_1 \otimes_K M_2) = \operatorname{Reg} M_1 + \operatorname{Reg} M_2.$$

Proof. If $\zeta_1,\dots,\zeta_r$ is a homogeneous system of parameters for R_1 and $\eta_1,\dots,\eta_s$ is one for R_2, then

$$\zeta_1\otimes 1,\dots,\zeta_r\otimes 1, 1\otimes\eta_1,\dots,1\otimes\eta_s \tag{2.3}$$

is a homogeneous system of parameters for $R_1\otimes_K R_2$. If F_* and F'_* are minimal free resolutions of M_1 and M_2 over the respective polynomial subrings of R_1 and R_2, then $F_*\otimes_K F'_*$ is a minimal free resolution of $M_1\otimes_K M_2$ over the polynomial subring generated by the parameters (2.3). It follows that

$$\beta_i^{R_1\otimes R_2}(M_1\otimes M_2)=\max_{j+k=i}\{\beta_j^{R_1}(M_1)+\beta_k^{R_2}(M_2)\}$$

and so

$$\beta_i^{R_1\otimes R_2}(M_1\otimes M_2)-i=\max_{j+k=i}\{(\beta_j^{R_1}(M_1)-j)+(\beta_k^{R_2}(M_2)-k)\}.$$

Now use the formula (2.2) for regularity. □

Next, we write $R^{[p]}$ for the graded ring whose homogeneous elements are symbols $x^{[p]}$ with x a homogeneous element of R, and with $|x^{[p]}|=p|x|$. Since $(-1)^{(p|x|)(p|y|)}$ is equal to $(-1)^{|x|\,|y|}$ in K (check separately for $p=2$ and p odd), $R^{[p]}$ is again a Noetherian graded commutative K-algebra. Similarly, if M is a graded R-module, we write $M^{[p]}$ for the corresponding graded $R^{[p]}$-module with homogeneous elements $m^{[p]}$.

Proposition 2.5. *We have*

$$p\operatorname{Reg}M-(p-1)\operatorname{Dim}(M)\leqslant\operatorname{Reg}M^{[p]}\leqslant p\operatorname{Reg}M-(p-1)\operatorname{Depth}(M).$$

Proof. Let d be the depth of M and let s be its Krull dimension. We have

$$a^i_{\mathfrak{m}}(M^{[p]})=pa^i_{\mathfrak{m}}(M),$$

and so

$$\operatorname{Reg}M^{[p]}=\max_{d\leqslant i\leqslant s}\{p\cdot a^i_{\mathfrak{m}}(M)+i\}=\max_{d\leqslant i\leqslant s}\{p(a^i_{\mathfrak{m}}(M)+i)-(p-1)i\}.$$

□

3. Proof of the main theorem

It was proved by Nakaoka (see Theorem 3.3 of [**15**]; see also the end of § 4.1 of [**1**]) that there is a ring isomorphism

$$H^*(H\wr\mathbb{Z}/p,K)\cong H^*(\mathbb{Z}/p,H^*(H^p,K)).$$

We write t for the generator of $\mathbb{Z}/p$, and we use multiplicative notation, so that $\mathbb{Z}/p=\langle t\mid t^p=1\rangle$.

As a representation of $\mathbb{Z}/p$, $H^*(H^p,K)\cong H^*(H,K)^{\otimes p}$ decomposes as a direct sum of two pieces. One piece is spanned by the elements $x\otimes\cdots\otimes x$, as x runs over a vector

https://doi.org/10.1017/S0013091505001203 Published online by Cambridge University Press

space basis of $H^*(H,K)$, with trivial $\mathbb{Z}/p$-action; the other piece is spanned by tensors involving more than one basis element, and the $\mathbb{Z}/p$-action on this summand is free. Notice that this decomposition depends on a choice of homogeneous basis for $H^*(H,K)$, and is therefore not canonical.

Let $T \subseteq H^*(H \wr \mathbb{Z}/p, K)$ be the image of the transfer from $H^*(H^p, K)$. Then T lies inside $H^0(\mathbb{Z}/p, H^*(H^p,K)) = H^*(H^p,K)^{\mathbb{Z}/p}$ and consists of the invariants in the free summand described in the previous paragraph. Furthermore,

$$H^*(H \wr \mathbb{Z}/p, K)/T \cong H^*(\mathbb{Z}/p, K) \otimes H^*(H,K)^{[p]}, \tag{3.1}$$

where $[p]$ indicates, as in § 2, that the degrees have been multiplied by a factor of p. The elements of $H^*(H,K)^{[p]}$ in this isomorphism are spanned by the images $x^{[p]}$ of the elements $x \otimes \cdots \otimes x$.

We regard everything in sight as a module over $H^*(H \wr \mathbb{Z}/p, K)$ for the purpose of computing regularity. We may compute regularity by means of the following proposition.

Proposition 3.1. *Let H be a subgroup of a finite group G. Then the regularity of an $H^*(H,K)$-module is the same whether regarded as an $H^*(H,K)$-module or as an $H^*(G,K)$-module via restriction.*

Proof. By a theorem of Evens [**10**], $H^*(H,K)$ is finitely generated as a module over $H^*(G,K)$ via the restriction map. So if M is an $H^*(H,K)$-module, then the local cohomology of M is the same whether computed as an $H^*(H,K)$-module or as an $H^*(G,K)$-module. □

Lemma 3.2. *We have*

$$\operatorname{Reg} H^*(H \wr \mathbb{Z}/p, K)/T = \operatorname{Reg} H^*(H,K)^{[p]} \leqslant p \cdot \operatorname{Reg} H^*(H,K) - p + 1.$$

Proof. The equality follows from the isomorphism (3.1) and Proposition 2.4. The depth of $H^*(H,K)$ is at least 1 by a theorem of Duflot [**8**], so the inequality follows from Proposition 2.5. □

Lemma 3.3. *As an $H^*(H \wr \mathbb{Z}/p, K)$-module, $H^*(H,K)^{\otimes p}$ has a filtration with $p+1$ filtered quotients, which consist of $p-1$ copies of T followed by one copy of $H^*(H,K)^{[p]}$, and finally another copy of T.*

Proof. Since the image of the restriction map $H^*(H \wr \mathbb{Z}/p, K) \to H^*(H,K)^{\otimes p}$ consists of $\mathbb{Z}/p$-invariants, we have commuting actions of $H^*(H \wr \mathbb{Z}/p, K)$ and $K\mathbb{Z}/p$ on $H^*(H,K)^{\otimes p}$. Consider the action of $(1-t) \in K\mathbb{Z}/p$. As mentioned above, $H^*(H,K)^{\otimes p}$ has only trivial and free summands as a $K\mathbb{Z}/p$-module. So we have a filtration of $H^*(H \wr \mathbb{Z}/p, K)$-modules

$$\begin{aligned} H^*(H,K)^{\otimes p} \supseteq \operatorname{Ker}(1-t)^{p-1} \supseteq \operatorname{Ker}(1-t)^{p-2} \supseteq \cdots \\ \cdots \supseteq \operatorname{Ker}(1-t)^2 \supseteq \operatorname{Ker}(1-t) \supseteq \operatorname{Im}(1-t)^{p-1} \supseteq 0. \end{aligned}$$

All except one of these filtered quotients come from the free summand and are isomorphic to T. The remaining filtered quotient,

$$\operatorname{Ker}(1-t)/\operatorname{Im}(1-t)^{p-1},$$

is isomorphic to the trivial $K\mathbb{Z}/p$-summand, namely to $H^*(H,K)^{[p]}$.

We remark that by switching from kernels to images earlier in the filtration, the copy of $H^*(H,K)^{[p]}$ as a filtered quotient may be placed anywhere except at the beginning or the end. $\square$

Proposition 3.4. *If* $\operatorname{Reg} H^*(H,K)=0$*, then as an* $H^*(H\wr\mathbb{Z}/p,K)$*-module we have* $\operatorname{Reg} T=0$*.*

Proof. By Lemma 3.2, we have $\operatorname{Reg} H^*(H,K)^{[p]}<0$, and by Proposition 2.4 we have $\operatorname{Reg} H^*(H,K)^{\otimes p}=0$. Applying Corollary 2.3 to the filtration of $H^*(H,K)^{\otimes p}$ described in Lemma 3.3, we deduce that $\operatorname{Reg} T=0$. $\square$

Proof of Theorem 1.1. If $\operatorname{Reg} H^*(H,K)=0$, then by Lemma 3.2 we have

$$\operatorname{Reg} H^*(H\wr\mathbb{Z}/p,K)/T<0,$$

and by Proposition 3.4 we have $\operatorname{Reg} T=0$. It follows from Proposition 2.2 that $\operatorname{Reg} H^*(H\wr\mathbb{Z}/p,K)\leqslant 0$. Combining this with the inequality (2.1), this completes the proof of Theorem 1.1. $\square$

4. Examples

The most obvious example where we can apply our main theorem is the Sylow p-subgroups of the symmetric groups. These are direct products of iterated wreath products $\mathbb{Z}/p\wr\mathbb{Z}/p\wr\cdots\wr\mathbb{Z}/p$, and so by Theorem 1.1 and Proposition 2.4 the cohomology in characteristic p of these Sylow p-subgroups satisfies the regularity conjecture. To complete the proof of Corollary 1.2 we apply the following proposition.

Proposition 4.1. *Let K be a field of characteristic p and let S be a Sylow p-subgroup of a finite group G. Then* $\operatorname{Reg} H^*(G,K)\leqslant\operatorname{Reg} H^*(S,K)$*. In particular, if* $\operatorname{Reg} H^*(S,K)=0$*, then* $\operatorname{Reg} H^*(G,K)=0$*.*

Proof. A standard argument using the transfer map (see, for example, § XII.10 of [**6**]) shows that $H^*(G,K)$ is a direct summand of $H^*(S,K)$ as an $H^*(G,K)$-module via the restriction map. Combining this with Proposition 3.1 proves the first statement. For the second statement, we use the inequality (2.1). $\square$

There are other interesting finite groups whose Sylow p-subgroups are direct products of iterated wreath products in a similar manner. For example, if $G=\mathrm{GL}_n(\mathbb{F}_q)$ with q coprime to p, then the Sylow p-subgroups of G have this form (except in certain cases when $p=2$). However, for these groups the computations of Quillen [**20**] show that $H^*(G,K)$ is Cohen–Macaulay, so that the regularity conjecture holds (for example by Theorem 1.5 of [**3**]).

A more interesting class of examples is the alternating groups in characteristic 2.

Proposition 4.2. *The Sylow 2-subgroups of $\mathcal{A}_{2^n}$ are of the form*

$$\mathbb{Z}/2 \wr \mathbb{Z}/2 \wr \cdots \wr \mathbb{Z}/2 \wr (\mathbb{Z}/2 \times \mathbb{Z}/2).$$

Proof. This interesting and subtle fact was communicated to me by Mark Feshbach. The proof can be found in a paper of Wong [**22**]. The essential observation is that if S is a Sylow 2-subgroup of $\mathcal{A}_{2m}$, then $V^m \rtimes S$ is a Sylow 2-subgroup of $\mathcal{A}_{4m}$. Here, V is a Klein four group acting regularly on four points, and each pair of points permuted by S is replaced by a copy of V with an outer automorphism of order 2. As an abstract group, $V^m \rtimes S$ is isomorphic to $\mathbb{Z}/2 \wr S$. This is an example of Neumann's twisted wreath products [**16**]. □

So an analysis of the case of $\mathcal{A}_{2^n}$ involves understanding wreath products with $\mathbb{Z}/2 \times \mathbb{Z}/2$.

Theorem 4.3. *Let K be a field of characteristic 2. If $\operatorname{Reg} H^*(H, K) = 0$, then $\operatorname{Reg} H^*(H \wr (\mathbb{Z}/2 \times \mathbb{Z}/2), K) = 0$.*

Proof. We can mimic the proof for the wreath product of H with $\mathbb{Z}/p$ as follows. Let $V = \mathbb{Z}/2 \times \mathbb{Z}/2 = \{1, a, b, c\}$. Let T_1 be the image of transfer from $H^*(H^4, K)$ to $H^*(H \wr V, K)$. Then T_1 lies inside $H^0(V, H^*(H^4, K))$, namely inside the invariants of V on $H^*(H, K)^{\otimes 4}$. We write T_2 for the image of transfer from $H^*(H^2, K)$ to $H^*(H \wr \mathbb{Z}/2, K)$, so that there is a direct sum of three copies of $H^*(\mathbb{Z}/2, K) \otimes T_2^{[2]}$ inside $H^*(H \wr V, K)/T_1$, one for each subgroup of order 2 in V. These three copies of $T_2^{[2]}$ can be seen as the subspace of $H^*(H, K)^{\otimes 4}$ spanned by elements like $x \otimes x \otimes y \otimes y$, where the tensor factors occur in two pairs. Finally, the quotient of $H^*(H \wr V, K)/T$ by these summands is isomorphic to $H^*(V, K) \otimes H^*(H, K)^{[4]}$. We shall show that each of these pieces has regularity less than or equal to 0.

We filter $H^*(H, K)^{\otimes 4}$ in such a way that the filtered quotients consist of four copies of T_1, two copies of $T_2^{[2]}$ for each of the three subgroups of V of order 2, and one copy of $H^*(H, K)^{[4]}$. The top and bottom filtered quotients are copies of T_1. By Proposition 3.4, we have $\operatorname{Reg} T_2 = 0$, and so by Proposition 2.5 we have $\operatorname{Reg} T_2^{[2]} < 0$. Also by Proposition 2.5 we have $\operatorname{Reg} H^*(H, k)^{[4]} < 0$. Since $\operatorname{Reg} H^*(H, K)^{\otimes 4} = 0$ by Proposition 2.4, it follows from Corollary 2.3 that $\operatorname{Reg} T_1 = 0$.

Again applying Proposition 2.4, we have

$$\operatorname{Reg} H^*(\mathbb{Z}/2, K) \otimes T_2^{[2]} < 0 \quad \text{and} \quad \operatorname{Reg} H^*(V, K) \otimes H^*(H, K)^{[4]} < 0.$$

Finally, we apply Corollary 2.3 to the filtration described in the first part of the proof to deduce that $H^*(H \wr V, K)$ has regularity less than or equal to 0. Combining this with the inequality (2.1) completes the proof of the theorem. □

The same method of proof presumably works for a wreath product with any p-group, but it seems hard to set up the details of the filtrations.

Corollary 4.4. *Let K be a field of characteristic 2. If $\mathcal{A}_{2^n} \geqslant G \geqslant \operatorname{Syl}_2(\mathcal{A}_{2^n})$, then $\operatorname{Reg} H^*(G, K) = 0$.*

5. PORC functions

A function $i \mapsto c_i$ from non-negative integers to integers is said to be *PORC* if there exists a positive integer d (the *modulus*) and polynomials $f_0, \dots, f_{d-1}$ such that for all $i \geqslant 0$ we have $c_i = f_j(i)$, where j is the unique integer satisfying $0 \leqslant j < d$ and $j \equiv i \pmod{d}$. The function $i \mapsto c_i$ is said to be *almost PORC* if this condition holds for all large enough i.

It is well known that a function is almost PORC if and only if the Poincaré series $f(t) = \sum_{i=0}^{\infty} c_i t^i$ is the power series expansion of a rational function of the form

$$f(t) = \frac{p(t)}{\prod_{i=1}^{r}(1 - t^{n_i})} \tag{5.1}$$

where $p(t)$ is a polynomial with integer coefficients and where $n_1, \dots, n_r$ are positive integers. The modulus d can be taken to be any positive number divisible by all the n_i. So $f(t)$ could be rewritten in the form $p(t)/(1 - t^d)^r$ by multiplying the numerator and denominator of the rational function (5.1) by suitable polynomials.

Example 5.1. The Poincaré series of the cohomology of an elementary abelian p-group E of rank r is given by

$$\sum_{i \geqslant 0} \dim_K H^i(E, K) = \frac{1}{(1-t)^r}.$$

The coefficients are PORC if and only if $r > 0$.

The following lemma summarizes some obvious properties of PORC functions.

Lemma 5.2. *The set of Poincaré series of PORC functions is closed under the following operations:*

(i) *addition and subtraction,*

(ii) *multiplication,*

(iii) *multiplication by integers (non-zero constant functions are not PORC, so this is not a special case of (ii)),*

(iv) *division by t if $c_0 = 0$ (but not multiplication by t),*

(v) *division by a non-zero integer, if each of the c_i is divisible by that integer,*

(vi) *replacing t by t^n for some positive integer n.*

Theorem 5.3. *Let*

$$f(t) = \sum_{i=0}^{\infty} c_i t^i = \frac{p(t)}{\prod_{i=1}^{r}(1 - t^{n_i})}$$

be the Poincaré series of an almost PORC function $i \mapsto c_i$. Then the function $i \mapsto c_i$ is PORC if and only if the degree of $p(t)$ is strictly less than $\sum_{i=1}^{r} n_i$.

Proof. The hypothesis is unchanged if we multiply the top and bottom of the rational function by a suitable polynomial so that all the n_i are equal, say $n_i = d$.

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First we suppose that $\deg p(t) < \sum_{i=1}^{r} n_i$, and we will prove that $i \mapsto c_i$ is PORC. Since $\mathbb{Z}$-linear combinations of PORC functions are PORC by (i) and (iii) of Lemma 5.2, it suffices to treat the case where $p(t) = t^j$ for some $j < dr$. So $f(t)$ takes the form

$$f(t) = \frac{t^j}{(1-t^d)^r} = t^j \sum_{i=0}^{\infty} \binom{i+r-1}{i} t^{id} = \sum_{i=0}^{\infty} \binom{i+r-1}{r-1} t^{j+id}.$$

Since $j < dr$, for values of i with $j + id \geqslant 0$, we have $i + r > 0$. So the binomial coefficient $\binom{i+r-1}{r-1}$ makes sense and is a polynomial function of i that takes the value 0 for $-r < i < 0$.

Conversely, suppose that $f(t)$ is the Poincaré series of a PORC function $i \mapsto c_i$. If $\deg p(t) \geqslant dr$, then using the fact that $t^j/(1-t^d)^r$ is PORC for $j < dr$, we may repeatedly add PORC functions and divide by t (using parts (i) and (iv) of Lemma 5.2) until only the term of highest degree remains, and $p(t)$ is a non-zero multiple of

$$t^{dr}/(1-t^d)^r = t^{dr} \sum_{i=0}^{\infty} \binom{i+r-1}{r-1} t^{id} = \sum_{k=r}^{\infty} \binom{k-1}{r-1} t^{kd}.$$

The polynomial $\binom{k-1}{r-1}$ takes value $(-1)^{r-1}$ at $k = 0$, which does not agree with the value of c_0, contradicting the statement that $f(t)$ is PORC. So we conclude that $\deg p(t) < dr$. □

Proposition 5.4. *Let K be a field, and suppose that $R = \bigoplus_{i \geqslant 0} R_i$ is a connected graded commutative Noetherian K-algebra (see the comments at the beginning of § 2). Let $M = \bigoplus_{i \geqslant 0} M_i$ be a finitely generated non-negatively graded R-module. Then $i \mapsto c_i = \dim_K M_i$ is a PORC function provided that the a-invariants satisfy $a^i_{\mathfrak{m}}(M) < 0$ for all $i \geqslant 0$.*

Proof. Let $\zeta_1, \dots, \zeta_r$ be a homogeneous system of parameters in R with $\deg \zeta_i = n_i$ and set $S = K[\zeta_1, \dots, \zeta_r]$. Using Theorem 5.5 of [**3**], if $a^i_{\mathfrak{m}}(M) < 0$ for all $i \geqslant 0$, then $\beta^S_i(M) < \sum_{j=1}^{r} n_j$ for all $i \geqslant 0$. Let

$$0 \to F_r \to \cdots \to F_0 \to M \to 0$$

be the minimal free resolution of M. The Poincaré series of each F_j has the form

$$\frac{\text{polynomial of degree less than } \sum_{i=1}^{r} n_i}{\prod_{i=1}^{r}(1-x^{n_i})}.$$

The Poincaré series for M is the alternating sum of those for the F_i, so it has the same form. Now apply Theorem 5.3. □

Theorem 5.5. *Let G be a finite group and let K be a field of characteristic p dividing $|G|$. Among the following statements, we have the implications* (a) $\Rightarrow$ (b) $\Rightarrow$ (c)*:*

(a) $\operatorname{Reg} H^*(G, K) = 0$*;*

(b) *for all $i \geqslant 0$ we have $a^i_{\mathfrak{m}} H^*(G, K) < 0$;*

(c) *the function $i \mapsto \dim_K H^i(G, K)$ is a PORC function.*

Proof. By a theorem of Duflot [**8**], since p divides $|G|$, the depth of $H^*(G,K)$ is at least 1. So $a^0_{\mathfrak{m}}H^*(G,K) = -\infty$. Condition (a) states that $a^i_{\mathfrak{m}}H^*(G,K) \leqslant -i$ for all $i \geqslant 0$, so it follows that (a) implies (b). The statement that (b) implies (c) is contained in Proposition 5.4. □

Remark 5.6. It is shown in Corollary 4.7 of [**3**] that condition (b) in Theorem 5.5 is equivalent to the existence of a quasi-regular sequence in $H^*(G,K)$ in the sense of Benson and Carlson [**4**].

Proof of Theorem 1.4. Using Nakaoka's formula [**15**] for the cohomology of the wreath product, and some counting arguments of Burnside, Webb [**21**, Theorem 3.1] gives the following formula for the Poincaré series of the cohomology of the wreath product:

$$\sum_{i\geqslant 0} t^i \dim_K H^i(H \wr G, K) = \frac{1}{|G|} \sum_{J' \leqslant J} \mu(J',J)|J'|g_{J'}(t)f_J(t). \tag{5.2}$$

Here, the sum ranges over all pairs of subgroups $J' \subseteq J$ of G. For each subgroup J of G, $f_J(t)$ denotes $f(t^{|\Omega_1|})\cdots f(t^{|\Omega_n|})$, where $f(t) = \sum_{i\geqslant 0} t^i \dim_K H^i(H,K)$ and where $\Omega = \Omega_1 \cup \cdots \cup \Omega_n$ is the decomposition of Ω into orbits of J. The function $g_J(t)$ is the Poincaré series $\sum_{i\geqslant 0} t^i \dim_K H^i(J,K)$.

It follows from Theorem 5.5 and parts (ii) and (vi) of Lemma 5.2 that the functions $f_J(t)$ have PORC coefficients. We divide the terms in the sum (5.2) into two types. The terms where $|J|$ is not divisible by p have PORC coefficients by part (iii) of Lemma 5.2, whereas for the terms where $|J|$ is divisible by p we need to use parts (ii) and (iii) of the lemma. Finally, the quotient by $|G|$ has PORC coefficients by part (v) of the lemma. □

Acknowledgements. The research in this paper was partly supported by NSF grant DMS-0242909.

The author acknowledges with thanks the hospitality and support of the Centre Bernoulli EPFL, Lausanne, where this work was carried out. He also thanks Srikanth Iyengar and Peter Webb for conversations about the material in this paper.

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