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Remarks on Hopf Images and Quantum Permutation Groups S_n^+

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Abstract. Motivated by a question of A. Skalski and P. M. Sołtan (2016) about inner faithfulness of S. Curran's map of extending a quantum increasing sequence to a quantum permutation, we revisit the results and techniques of T. Banica and J. Bichon (2009) and study some group-theoretic properties of the quantum permutation group on 4 points. This enables us not only to answer the aforementioned question in the positive for the case where $n = 4, k = 2$, but also to classify the automorphisms of S_4^+, describe all the embeddings $O_{-1}(2) \subset S_4^+$ and show that all the copies of $O_{-1}(2)$ inside S_4^+ are conjugate. We then use these results to show that the converse to the criterion we applied to answer the aforementioned question is not valid.

Introduction

Let $\mathbb{G}$ be a compact quantum group (in the sense of Woronowicz, but throughout the note we will not need any of the functional analytic features of the associated Hopf-C^*-algebra), let $\mathcal{O}(\mathbb{G})$ be its associated coordinate ring, and assume that $\beta: \mathcal{O}(\mathbb{G}) \to \mathcal{B}$ is a *-representation of $\mathcal{O}(\mathbb{G})$ as a *-algebra in some *-algebra. Via abstract Gelfand–Naimark duality, such a maps corresponds to a map $\widehat{\beta}: \mathbb{X} \to \mathbb{G}$, and it is natural to ask what is the smallest quantum subgroup containing $\widehat{\beta}(\mathbb{X})$, or, in other words, what the quantum subgroup generated by $\widehat{\beta}(\mathbb{X}) \subset \mathbb{G}$ is. The answer to these types of questions was studied earlier in [2, 4, 8, 22] in the case of compact quantum groups and later extended to locally compact quantum groups in [13, 14].

The concept of a subgroup is central to treating quantum groups from the group-theoretic perspective, and many efforts were made to provide accurate descriptions of various aspects of this concept, as well as providing some nontrivial examples; see *e.g.*, [3, 7, 10, 20]. This note deals with subgroups of the quantum permutation groups, introduced by Wang in [23]. It was observed in [17] that quantum permutations can be used to study distributional symmetries of infinite sequences of non-commutative random variables that are identically distributed and free modulo the tail algebra, thus extending the classical de Finetti's theorem to the quantum/free realm.

Another extension of de Finetti's theorem was given by Ryll–Nardzewski; he observed that instead of invariance of joint distributions under permutations of random variables it is enough to consider subsequences and compare these types of joint distributions to obtain the same conclusion. What this theorem really boils down to is the fact that one can canonically treat the set $I_{k,n}$ of increasing sequences (of indices)

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as subset of all permutations S_n, and this subset is big enough to generate the whole symmetric group $\langle I_{k,n}\rangle = S_n$, unless $k = 0$ or $k = n$.

This viewpoint was utilized by Curran [9] to extend a theorem of Ryll-Nardzewski to the quantum case: he introduced the quantum space of quantum increasing sequences $I^+_{k,n}$ and described how to canonically extend the quantum increasing sequence to a quantum permutation in S^+_n. The analytical properties of the C^*-algebra $C(I^+_{k,n})$ were strong enough to provide an extension of Ryll-Nardzewski to the quantum/free case. However, these results did not say anything about the subgroup of quantum permutation group that is generated by quantum increasing sequences.

If the analogy with the classical world is complete, one would expect that in fact $\overline{\langle I^+_{k,n}\rangle} = S^+_n$ for all n and $k \neq 0, n$. This was already ruled out in [22], where it was observed that $\overline{\langle I^+_{k,n}\rangle} = S_n$ whenever $k = 1, n-1$. The second best thing one could hope for is that $\overline{\langle I^+_{k,n}\rangle} = S^+_n$ for at least one $k \in \{2, \dots, n-2\}$, as this would explain the results of Curran in a more group-theoretic manner. In general, [22, Question 7.3] asks for the complete description of all $\overline{\langle I^+_{k,n}\rangle}$ and emphasizes the case $n = 4$ and $k = 2$ as the first non-trivial case to study. We give a positive answer in this case using the following lower bound criterion for the Hopf image: assume $\beta\colon C^u(\mathbb{G}) \to \mathsf{B}$ is a morphism and assume X is the set of all characters of B. Denoting by $\mathbb{H}$ the Hopf image of β, we have that $\overline{\langle X\rangle} \subset \mathbb{H}$. The C^*-algebra language is mainly used for convenience, and it is straightforward to adapt this criterion to the purely algebraic situation. We refer the readers unfamiliar with Gelfand–Naimark duality and other C^*-algebraic aspects of the theory of quantum groups to [15, Chapters 3–4]. It should be noted that the inclusion can be proper for some analytical reasons, but it can be shown that even when restricting to the setting of topological generation of quantum groups in the spirit of [8], such an inclusion can still be proper.

In the course of analyzing the impossibility of getting strict equality in the aforementioned criterion even in the analytically best-behaved case of coamenable compact quantum group of Kac type, we study some group-theoretic properties of the quantum permutation group S^+_4. Namely, we classify all Hopf automorphisms of $C(S^+_4)$ and show that there are three copies of $O_{-1}(2)$ appearing as quantum subgroups of S^+_4, and that they are conjugate.

The paper is organized as follows. Section 1 serves mainly as preliminaries needed to settle the notation for compact quantum groups (Subsection 1.1), Hopf images (Subsection 1.2) and quantum permutations groups together with quantum increasing sequences (Subsection 1.3). However, the main criterion is also contained there as Theorem 1.1, as well as the answer to [22, Question 7.3], as Theorem 1.4. In Section 2 we turn to studying group-theoretic properties of S^+_4. We introduce the objects we need in Subsection 2.1 and later revise the technique of cocycle twists in Subsection 2.2. We also introduce the concept of characteristic subgroups in Subsection 2.3 in the context of compact quantum groups. In Subsection 2.4 we recall how the technique of cocycle-deformation is applied to S^+_4 and use the results of Subsection 2.3 to classify quantum automorphisms of S^+_4. In Subsection 2.5 we classify embeddings $O_{-1}(2) \subset S^+_4$ and use them to show in Subsection 2.6 that the inclusion in our criterion, Theorem 1.1, can be proper even in the analytically best-behaved setting. We also include an elementary proof that $(L^\infty(S_4^+), h_{S_4^+})$ satisfies the Connes' embedding property.

1 Compact Quantum Groups, Hopf Image, and Criterion

Throughout the article, we will use tensor products of different structures, mainly *-algebras (the algebraic tensor product) and C^*-algebras (the minimal tensor product). They will be denoted using the same symbol $\otimes$, as this should be clear from the context which tensor product procedure is evoked at the time. The C^*-algebra formalism is used only for convenience, as all the results rely only on their algebraic features. The C^*-algebra of compact operators on a Hilbert space $\mathcal{H}$ is denoted $\mathsf{K}(\mathcal{H})$, and for a C^*-algebra A we denote by $\mathsf{M}(\mathsf{A})$ its multiplier algebra.

1.1 Compact Quantum Groups

In this section we recall the basic definitions from the theory of compact quantum groups. We stick to the formalism established in [24, 25]. A unital C^*-algebra A endowed with a *-homomorphism $\Delta\colon \mathsf{A} \to \mathsf{A}\otimes\mathsf{A}$ satisfying the coassociativity condition: $(\Delta\otimes\mathrm{id})\circ\Delta = (\mathrm{id}\otimes\Delta)\circ\Delta$ is called a *Woronowicz algebra*, if the cancellation laws hold:

$$\mathrm{span}_{\mathbb{C}}{}^{-\|\cdot\|}\big((\mathbb{1}\otimes\mathsf{A})\Delta(\mathsf{A})\big) = \mathsf{A}\otimes\mathsf{A} = \mathrm{span}_{\mathbb{C}}{}^{-\|\cdot\|}\big((\mathsf{A}\otimes\mathbb{1})\Delta(\mathsf{A})\big),$$

where $\mathrm{span}_{\mathbb{C}}{}^{-\|\cdot\|}$ denotes the norm closure of the linear span.

Such an algebra corresponds to a *compact quantum group* $\mathbb{G}$ via the identification $\mathsf{A} = C(\mathbb{G})$, the algebra of continuous functions on $\mathbb{G}$. It can be endowed with a unique state $h \in \mathsf{A}^*$, called the *Haar state*, which is left and right invariant:

$$(\mathrm{id}\otimes h)\circ\Delta = (h\otimes\mathrm{id})\circ\Delta = h(\cdot)\mathbb{1}.$$

A contains a unique dense Hopf *-subalgebra $\mathcal{O}(\mathbb{G})$ (*i.e.,* the coproduct Δ restricts to $\mathcal{O}(\mathbb{G})$); it is spanned by matrix coefficients of unitary representations of $\mathbb{G}$. The Hopf *-algebra $\mathcal{O}(\mathbb{G})$ can have, a priori, a plethora of different C^*-norms: the norm coming from the GNS-representation of the Haar state ($C^r(\mathbb{G}) = \overline{\mathcal{O}(\mathbb{G})} \subseteq \mathsf{B}(L^2(\mathbb{G}))$), the norm of A and the universal C^*-norm need not coincide. For further discussion on this topic, see *e.g.,* [18]. In any case, there are always quotient maps

$$C^u(\mathbb{G}) \longrightarrow C(\mathbb{G}) \longrightarrow C^r(\mathbb{G}),$$

where $C(\mathbb{G})$ denotes a general C^*-completion. If the quotient map $\Lambda\colon C^u(\mathbb{G}) \to C^r(\mathbb{G})$ is injective, we call $\mathbb{G}$ *coamenable* and declare that $C^u(\mathbb{G}) = C^r(\mathbb{G})$ and $\Lambda = \mathrm{id}$. In this note we mainly deal with coamenable compact quantum groups and use the symbol $C(\mathbb{G})$ to describe that C^*-algebra.

The most studied examples are the *compact matrix quantum groups*: $\mathbb{G}$ is a compact matrix quantum group if the Woronowicz algebra $C^u(\mathbb{G})$ can be endowed with a *fundamental* corepresentation $u \in M_n(C^u(\mathbb{G})) = \mathsf{M}(\mathsf{K}(\mathbb{C}^n)\otimes C^u(\mathbb{G}))$: denoting $u_{i,j} = (\langle e_i|\cdot|e_j\rangle\otimes\mathrm{id})u$ for a fixed basis $(e_i)_{1\le i\le n} \subset \mathbb{C}^n$, orthonormal with respect to an inner product $\langle\cdot|\cdot\rangle$, we ask for

$$\Delta(u_{i,j}) = \sum_{k=1}^n u_{i,k}\otimes u_{k,j} \quad \text{and} \quad \big\langle\{u_{i,j} : 1\le i, j\le n\}\big\rangle = \mathcal{O}(\mathbb{G})$$

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where $\langle X \rangle$ denotes the *-algebra generated by elements of X (note that we used the symbol $\langle S \rangle$ also to denote the subgroup generated by a subset S, but this shall cause no confusion).

Any compact quantum group $\mathbb{G}$ has its maximal classical subgroup $\mathrm{Gr}(\widehat{\mathbb{G}})$, also called the group of characters of $\mathbb{G}$ or the intrinsic subgroup of $\widehat{\mathbb{G}}$. It is given as follows: consider the universal enveloping C^*-algebra $C^u(\mathbb{G})$ and the commutator ideal of it, *i.e.,* the ideal generated by $\{xy - yx : x, y \in C^u(\mathbb{G})\}$; call this ideal I. Then the quotient map

$$q_{\mathbb{G}} : C^u(\mathbb{G}) \longrightarrow C^u(\mathbb{G}) /_I =: C(\mathrm{Gr}(\widehat{\mathbb{G}}))$$

identifies the spectrum of the (commutative) C^*-algebra $C^u(\mathbb{G}) /_I$, denoted $\mathrm{Gr}(\widehat{\mathbb{G}})$, with a closed (quantum) subgroup of $\mathbb{G}$. The commutativity of the C^*-algebra $C(\mathrm{Gr}(\widehat{\mathbb{G}}))$ ensures us that it is the only possible completion of $\mathcal{O}(\mathrm{Gr}(\widehat{\mathbb{G}}))$, so we drop the $\cdot^u$ decoration. A thorough description of the group of characters of a given (locally) compact quantum group $\mathbb{G}$ can be found in [16]

1.2 Hopf Image

The Hopf image construction, studied in detail in the case of compact quantum groups in [4, 22] and in the case of locally compact quantum groups in [14], is concerned with the following situation. Consider a (closed) subset in a (locally) compact group $X \subseteq G$. We are looking for the closed subgroup of G, say H, which is generated by the set X, *i.e.,* $\overline{\langle X \rangle} = H$. Under Gelfand–Naimark duality, this corresponds to finding the final/terminal object in the category, whose objects are defined with the aid of the following diagram:

$$\begin{tikzcd} C(G) \arrow[rr, two heads, "\beta"] \arrow[dr, "\pi"'] & & C(X) \\ & C(H). \arrow[ur, "\widetilde{\beta}"'] & \end{tikzcd}$$

In the above diagram, β is the Gelfand dual to the embedding $X \subset G$, β' is the Gelfand dual to the embedding $X \subset H$, and π is the Gelfand dual to the embedding $H \subset G$. The objects of the aforementioned category are triples consisting of commutative Woronowicz algebras $C(H)$ and maps π, $\widetilde{\beta}$ so that π intertwines the coproduct and $\widetilde{\beta} \circ \pi = \beta$. In terms of spectra of these C^*-algebras, this category consists of closed subgroups of G containing the set X, and we are looking for a minimal one.

Dropping commutativity enables us to discuss *closed quantum subgroup of* $\mathbb{G}$ *generated by a map* β: the Hopf image of a *-homomorphism $\beta : C^u(\mathbb{G}) \to \mathsf{B}$ is the final object of the category, whose objects are triples consisting of Woronowicz algebras $C^u(\mathbb{H})$, Hopf *-homomorphism $\pi : C^u(\mathbb{G}) \to C^u(\mathbb{H})$ and a *-homomorphism $\widetilde{\beta} : C^u(\mathbb{H}) \to \mathsf{B}$ such that the following diagram commutes

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$$\begin{tikzcd} C^u(\mathbb{G}) \arrow[rr, two heads, "\beta"] \arrow[dr, two heads, "\pi"'] & & \mathsf{B} \\ & C^u(\mathbb{H}). \arrow[ur, two heads, "\widetilde{\beta}"'] & \end{tikzcd}$$

It should be stressed that the universal C^*-completions have the greatest number of possible *-homomorphisms β, so looking for the Hopf images of the maps defined on the universal C^*-completions is the only reasonable from the group-theoretic perspective. For instance, the compact quantum group $\widehat{\mathbb{F}_2}$, the dual to the free group on two generators, has simple reduced completion, and thus in the reduced world the only possible map is inclusion, and if we were to speak of Hopf images treating different C^*-algebras as different quantum groups, the Hopf image of any morphism from $C_r^*(\mathbb{F}_2)$ is always the whole of $\widehat{\mathbb{F}_2}$ – even the trivial group is excluded!

Let $\mathbb{H}_1, \mathbb{H}_2 \subset \mathbb{G}$ be two closed quantum subgroups (identified via $\pi_i: C^u(\mathbb{G}) \to C^u(\mathbb{H})$). Then we say that $\mathbb{G}$ is topologically generated by $\mathbb{H}_1$ and $\mathbb{H}_2$, and write $\mathbb{G} = \overline{\langle \mathbb{H}_1, \mathbb{H}_2 \rangle}$, if the Hopf image of the either of the maps $(\pi_1 \otimes \pi_2) \circ \Delta$ or $\pi_1 \oplus \pi_2$, is the whole $C^u(\mathbb{G})$. This notion has several equivalent descriptions; see [8, Proposition 3.5] and [14, Section 3].

We now give a criterion showing "how big" the Hopf image of a *-homomorphism $\beta: C^u(\mathbb{G}) \to \mathsf{B}$ has to be. The content of the forthcoming observation is best seen in the following diagram:

$$\begin{tikzcd} C^u(\mathbb{G}) \arrow[rr, two heads, "\beta"] \arrow[dr, two heads, "\pi"] \arrow[ddd, two heads, "q_{\mathbb{G}}"] & & \mathsf{B} \arrow[ddd, two heads, "q_{\mathsf{B}}"] \\ & C^u(\mathbb{H}) \arrow[ur, two heads, "\widetilde{\beta}"] \arrow[d, two heads, "q_{\mathbb{H}}"] & \\ & C(\mathrm{Gr}(\widehat{\mathbb{H}})) \arrow[dr, two heads, "\widetilde{b}"] & \\ C(\mathrm{Gr}(\widehat{\mathbb{G}})) \arrow[ur, two heads, "p"] \arrow[rr, two heads, "b"] & & C(\sigma(\mathsf{B})). \end{tikzcd}$$

Here $q_{\mathbb{G}}: C^u(\mathbb{G}) \to C(\mathrm{Gr}(\widehat{\mathbb{G}}))$ is the canonical embedding of the group of characters $\mathrm{Gr}(\widehat{\mathbb{G}}) \subset \mathbb{G}$ (likewise for $\mathbb{H}$); $C(\sigma(\mathsf{B}))$ is the quotient of B by the commutator ideal and $\sigma(\mathsf{B})$ denotes the spectrum of this commutative C^*-algebra, q_{B} denotes this particular quotient map. Now p is obtained as follows: as $q_{\mathbb{H}} \circ \pi$ has commutative target, it factors through $C(\mathrm{Gr}(\widehat{\mathbb{G}}))$ and $p \circ q_{\mathbb{G}} = q_{\mathbb{H}} \circ \pi$. Similarly, we obtain b as the map completing the factorization of $q_{\mathsf{B}} \circ \beta$ through $q_{\mathbb{G}}$, and $\widetilde{b}$ completes the factorization of $q_{\mathsf{B}} \circ \widetilde{\beta}$ through $q_{\mathbb{H}}$.

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Theorem 1.1 *If $\mathbb{H}$ is the Hopf image of the map β, then the Hopf image of b contains* $\mathrm{Gr}(\widehat{\mathbb{H}})$. *In other words, the Gelfand dual map $\widehat{b}\colon \sigma(\mathsf{B}) \to \mathrm{Gr}(\widehat{\mathbb{G}})$ satisfies*

$$\overline{\langle \widehat{b}[\sigma(\mathsf{B})] \rangle} \subseteq \mathrm{Gr}(\widehat{\mathbb{H}}).$$

With slight abuse of notation, this should be understood as $\overline{\langle \sigma(\mathsf{B}) \rangle} \subset \mathbb{H} \subset \mathbb{G}$, where $\mathbb{H}$ is the Hopf image of β. As a general motto, this means that the farther from being simple the C^*-algebra B is, the better lower bound on $\mathbb{H}$ we obtain.

Proof That $\sigma(\mathsf{B}) \subseteq \mathrm{Gr}(\widehat{\mathbb{H}})$ follows from the commutativity of the above diagram, and hence $\overline{\langle \sigma(\mathsf{B}) \rangle} \subseteq \mathrm{Gr}(\widehat{\mathbb{H}})$, as the latter is a closed subgroup of $\mathrm{Gr}(\widehat{\mathbb{G}})$. ∎

It is clear that the inclusion of Theorem 1.1 can be proper, as the example of $\Lambda_{\widehat{\mathbb{F}_2}}\colon C^*(\mathbb{F}_2) \to C^*_r(\mathbb{F}_2)$ shows. Moreover, the result is not formulated in the optimal way, as one could replace B with the image of β (*cf.* [14, Section 2.2]), and the smaller subalgebra is more likely to have characters, as the example $\underline{a} \mapsto \mathrm{diag}(\underline{a})\colon c_0 \hookrightarrow \mathsf{K}(\ell^2)$ shows. We will later see that even restricting our attention to the best-behaved case of coamenable compact quantum groups of Kac type, with finitely many characters, and the generating set coming from two proper subgroups (in the spirit of [8]), the inclusion cannot be reversed.

1.3 The Quantum Permutation Group S_n^+ and Quantum Increasing Sequences

Quantum permutation groups S_n^+ were introduced in [23] (*cf.* [22, Section 3]). Consider the universal C^*-algebra generated by n^2-elements $u_{i,j}$, $1 \le i, j \le n$ subject to the following relations:

(a) the generators $u_{i,j}$ are all projections;
(b) $\sum_{i=1}^n u_{i,j} = \mathbb{1} = \sum_{j=1}^n u_{i,j}$.

This C^*-algebra will be denoted $C^u(S_n^+)$. The matrix $U = [u_{i,j}]_{1\le i,j\le n}$ is a fundamental corepresentation of $C^u(S_n^+)$; this gives all the quantum group-theoretic data. Moreover, $S_n^+ = S_n$ for $n \le 3$ and $S_n^+ \supsetneq S_n$ for $n \ge 4$ and S_n^+ is coamenable only if $n \le 4$ ([1]).

The algebra of continuous functions on the set of quantum increasing sequences was defined by Curran in [9, Definition 2.1]. Let $k \le n \in \mathbb{N}$ and let $C(I^+_{k,n})$ be the universal C^*-algebra generated by $p_{i,j}$, $1 \le i \le n$, $1 \le j \le k$ subject to the following relations:

(a) the generators $p_{i,j}$ are all projections;
(b) each column of the rectangular matrix $P = [p_{i,j}]$ forms a partition of unity: $\sum_{i=1}^n p_{i,j} = \mathbb{1}$ for each $1 \le j \le k$.
(c) increasing sequence condition: $p_{i,j} p_{i'j'} = 0$ whenever $j < j'$ and $i \ge i'$.

This definition is obtained by the liberation philosophy (see [5]): if one denotes by $I_{k,n}$ the set of increasing sequences of length k and values in $\{1, \dots, n\}$, then it is possible to write a matrix representation: with an increasing sequence $\underline{i} = (i_1 < \cdots < i_k)$ one associates its matrix representation $A(\underline{i}) \in M_{n\times k}(\{0,1\})$ as follows: $A(\underline{i})_{i_l,l} = 1$ and all other entries are set to be 0. One can check that the space of continuous functions on these matrices $C(\{A(\underline{i}) : \underline{i} \in I_{k,n}\})$ is generated by the coordinate functions

$x_{i,j}$ subject to the relations introduced above and the commutation relation (*cf.* the discussion after [9, Remark 2.2]).

Curran also defined a *-homomorphism $\beta_{k,n}: C(S_n^+) \to C(I_{k,n}^+)$ ([9, Proposition 2.5]) by:

- $u_{i,j} \mapsto p_{i,j}$ for $1 \le i \le n, 1 \le j \le k$,
- $u_{i,k+m} \mapsto 0$ for $1 \le m \le n-k$ and $i < m$ or $i > m+k$,
- for $1 \le m \le n-k$ and $0 \le p \le k$,

$$u_{m+p,k+m} \longmapsto \sum_{i=0}^{m+p-1} p_{i,p} - p_{i+1,p+1},$$

where we set $p_{0,0} = \mathbb{1}$, $p_{0,i} = p_{0,i} = p_{i,k+1} = 0$ for $i \ge 1$.

This *-homomorphism is well defined thanks to [9, Proposition 2.4], where some new relations were identified, and the universal property of $C(S_n^+)$, and the $\beta_{k,n}$ are defined in such a way that when applied to the commutative C^*-algebras $C(S_n) \to C(I_{k,n})$ (which satisfy the same relations plus commutativity), it is precisely the "completing an increasing sequence to a permutation" map. More precisely, one draws the diagram of an increasing sequence $\underline{i} = (i_1 < \cdots < i_k)$ in the following way: drawing k dots in one row and additional n dots in the row below, one connects l-th dot in the upper row to the i_l-th dot in the lower row. Then one draws additional $n-k$ dots in the upper row next to the previously drawn k dots and connects them as follows: $(k+j)$-th dot is connected to the j-th leftmost non-connected dot in the bottom row. Finally, one obtains the diagram of a permutation on n letters, which is then called $\beta_{k,n}(\underline{i})$ (for the version of $\beta_{k,n}$ as a map between appropriate commutative C^*-algebras), see the example below.

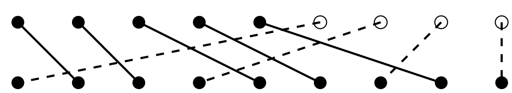

FIGURE. The sequence $(2<3<5<6<8) \in I_{5,9}$, drawn with full circles and segments, is completed, with the aid of empty circles and dashed segments, to the permutation $(1,2,3,5,8,7,4,6)$.

Fact 1.2 *$\langle I_{k,n}\rangle = S_n$ for all n and all $k \ne 0, n$, where $I_{k,n} \subseteq S_n$ is seen via the above map.*

Proposition 1.3 *Let $\mathbb{H} \subseteq S_n^+$ be the Hopf image of the map $\beta_{k,n}: C(S_4^+) \to C(I_{k,n}^+)$ for $k \ne 0, n$. Then $S_n \subseteq \mathbb{H} \subseteq S_n^+$.*

Proof The abelianization of $C(I_{k,n}^+)$ is $C(I_{k,n})$ and the map $\beta_{k,n}$ on the level of abelianizations is the canonical map, as noted above. We conclude by Theorem 1.1 together with Fact 1.2. ∎

In what follows, we restrict our attention to the case $n = 4$, $k = 2$.

Theorem 1.4 *The Hopf image of the map $\beta_{2,4}: C(S_4^+) \to C(I_{2,4}^+)$ is the whole S_4^+.*

Proof From Proposition 1.3 we see that the group of characters of $\mathbb{H}$, the Hopf image of β, is the permutation group $\mathrm{Gr}(\widehat{\mathbb{H}}) = S_4$. In particular, $\mathbb{H}$ contains the diagonal Klein subgroup, so is one of the groups listed in [3, Theorem 6.1]. It is easy to check that the group of characters of subgroups contained in [3, Theorem 6.1] are equal to S_4 only for the following two groups: S_4 and S_4^+. On the other hand, in [22, Proposition 7.4] it was shown that $C(I_{2,4}^+) \cong (\mathbb{C}^2 * \mathbb{C}^2) \oplus \mathbb{C}^2$ (the free product is amalgamated over $\mathbb{C}\mathbb{1}$) is infinite dimensional, hence $\mathbb{H} \neq S_4$. Consequently, $\mathbb{H} = S_4^+$ is the only possibility left. ■

2 Group-theoretical Properties of $SO_{-1}(3)$

2.1 The C^*-algebras $C(SO_{-1}(3))$, $C(SO(3))$, and $C(O_{-1}(2))$.

Let us now introduce the main players of this note (studied earlier in [3]).

Definition 2.1 The C^*-algebra of continuous functions on a compact quantum group $SO_{-1}(3)$ is the universal C^*-algebra generated by $a_{i,j}$, $1 \le i, j \le 3$ subject to the following relations:

(i) The matrix $A = (a_{i,j})_{1\le i,j\le 3} \in M_3(C(SO_{-1}(3))$ is orthogonal, *i.e.*, $AA^\top = A^\top A = \mathbb{1} \in M_3(C(SO_{-1}(3)))$. In particular, the generators $a_{i,j}$ are self-adjoint.
(ii) $a_{i,j}a_{i,k} = -a_{i,k}a_{i,j}$ for $k \neq j$.
(iii) $a_{i,j}a_{k,j} = -a_{k,j}a_{i,j}$ for $k \neq i$.
(iv) $a_{i,j}a_{k,l} = a_{k,l}a_{i,j}$ for $i \neq k$, $j \neq l$.
(v) $\sum_{\sigma\in S_3} a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)} = \mathbb{1}$.

Then A is the fundamental corepresentation of $C(SO_{-1}(3))$: this defines the quantum group structure.

In the same spirit we can define the C^*-algebra of continuous functions on $SO(3)$ as the universal C^*-algebra generated by $x_{i,j}$, $1 \le i, j \le 3$ subject to the following relations:

(a) The matrix $X = (x_{i,j})_{1\le i,j\le 3} \in M_3(C(SO(3))$ is orthogonal, *i.e.*, $AA^\top = A^\top A = \mathbb{1} \in M_3(C(SO(3)))$. In particular, the generators $x_{i,j}$ are self-adjoint.
(b) $x_{i,j}x_{k,l} = x_{k,l}x_{i,j}$ for all $1 \le i, j, k, l \le 3$.
(c) $\sum_{\sigma\in S_3} \mathrm{sgn}(\sigma)a_{1,\sigma(1)}a_{2,\sigma(2)}a_{3,\sigma(3)} = \mathbb{1}$

It is routine to conclude from the Stone–Weierstrass theorem that $C(SO(3))$ is indeed the C^*-algebra of continuous functions on $SO(3)$. The matrix multiplication in $SO(3)$ is encoded by X being a fundamental corepresentation.

We will also need the C^*-algebra of continuous functions on a compact quantum group $O_{-1}(2)$.

Definition 2.2 The C^*-algebra of continuous functions on a compact quantum group $O_{-1}(2)$ is the universal C^*-algebra generated by $\widetilde{a}_{i,j}$, $1 \le i, j \le 2$ subject to the relations (i)–(iv) of Definition 2.1, *mutati mutandis*. As previously, the matrix $\widetilde{A} = (\widetilde{a}_{i,j})_{1\le i,j\le 2} \in M_2(C(O_{-1}(2)))$ is a fundamental corepresentation of $C(O_{-1}(2)))$.

The following map yields a surjective *-homomorphism interpreted as $O_{-1}(2) \subset SO_{-1}(3)$:

$$(2.1)\qquad a_{i,j} \mapsto \left.\begin{cases} \widetilde{a}_{i,j} & \text{for } 1 \le i,j \le 2, \\ \widetilde{a}_{1,1}\widetilde{a}_{2,2} + \widetilde{a}_{1,2}\widetilde{a}_{2,1} & \text{for } i = j = 3, \\ 0 & \text{otherwise.} \end{cases}\right\} : C(SO_{-1}(3)) \twoheadrightarrow C(O_{-1}(2))$$

But there are more embeddings, $O_{-1}(2) \subset SO_{-1}(3)$. In order to classify them all, let us recall that these quantum groups can be described as cocycle-twists of their classical versions. Let us remark that (as will later become clear) these quantum groups are coamenable, thus (2.1) gives the proper description of the notion of subgroup.

2.2 Twistings: General Theory.

In what follows, we briefly discuss the twisting procedure and introduce the notation. We stick to the theory of Hopf *-algebras, although the procedure works well for general Hopf algebras over any field.

Let H be a Hopf *-algebra with coproduct Δ. Recall that the algebra $H \otimes H$ can be given the Hopf *-algebra structure: the coproduct is $\Delta_2 = (\mathrm{id} \otimes \Sigma \otimes \mathrm{id}) \circ (\Delta \otimes \Delta)$, where Σ denotes the flip map. We will use the Sweedler–Heyneman notation: $\Delta(x) = x_{(1)} \otimes x_{(2)}$. A linear map $\sigma: H \otimes H \to \mathbb{C}$ is called a 2-*cocycle* if:

(a) it is convolution invertible: the neutral element of convolution is $m_{\mathbb{C}} \circ (\varepsilon \otimes \varepsilon)$, the convolution of $\sigma, \sigma': H \otimes H \to \mathbb{C}$ is given by $\sigma * \sigma' = m_{\mathbb{C}} \circ (\sigma \otimes \sigma') \circ \Delta_2$,

(b) it satisfies the cocycle identity:

$$(2.2)\qquad \sigma(x_{(1)}, y_{(1)})\sigma(x_{(2)}y_{(2)}, z) = \sigma(y_{(1)}, z_{(1)})\sigma(x, y_{(2)}z_{(2)})$$

and $\sigma(x,1) = \varepsilon(x) = \sigma(1,x)$ for $x, y, z \in H$.

Here and in what follows, $m_W: W \otimes W \to W$, for a given algebra W, is the multiplication map

$$W \otimes W \ni x \otimes y \xmapsto{m_W} x \cdot y \in W.$$

Note also that we used the notation $\sigma(x,y) = \sigma(x \otimes y)$.

Following [3, 11, 21], a 2-cocycle σ provides a new Hopf *-algebra H^σ. As a coalgebra, $H^\sigma = H$, whereas the product of H^σ is defined as

$$[x][y] = \sigma(x_{(1)}, y_{(1)})\sigma^{-1}(x_{(3)}, y_{(3)})[x_{(2)}y_{(2)}],$$

where an element $x \in H$ is denoted $[x]$ when viewed as an element of H^σ. In other words, $m_{H^\sigma} = (\sigma \otimes m_H \otimes \sigma^{-1}) \circ \Delta_2^2$. The antipode of H^σ can be expressed via the following formula:

$$S^\sigma([x]) = \sigma\big(x_{(1)}, S(x_{(2)})\big)\,\sigma^{-1}\big(S(x_{(4)}), x_{(5)}\big)\,[S(x_{(3)})].$$

The Hopf algebras H and H^σ have equivalent tensor categories of comodules ([21]). In our considerations we are interested in the case when the 2-cocycle is induced from a Hopf *-algebra quotient (quantum subgroup). Let $\pi: H \to K$ be a Hopf surjection and let $\sigma: K \otimes K \to \mathbb{C}$ be a 2-cocycle on K. Then $\sigma_\pi = \sigma \circ (\pi \otimes \pi): H \otimes H \to \mathbb{C}$ is a 2-cocycle.

Proposition 2.3 ([3, Lemma 4.3]) *Let $\pi: H \to K$ be a Hopf surjection and let $\sigma: K \otimes K \to \mathbb{C}$ be a 2-cocycle. Then there is a bijection between:*

(i) *Hopf surjections $f: H \to L$ such that there exists a Hopf surjection $g: L \to K$ satisfying $g \circ f = \pi$, and*

(ii) *Hopf surjections $\widetilde{f}: H^{\sigma_\pi} \to \widetilde{L}$ such that there exists a Hopf surjection $\widetilde{g}: \widetilde{L} \to K^\sigma$ satisfying $\widetilde{g} \circ \widetilde{f} = [\pi(\cdot)]$.*

The bijection is given by $\widetilde{f}(\cdot) = [f(\cdot)]$ and sends $L \mapsto L^{\sigma_g} =: \widetilde{L}$ and $\widetilde{L} \mapsto \widetilde{L}^{\sigma_g^{-1}} =: L$.

2.3 Characteristic Subgroups

Let $\mathbb{G}$ be a compact quantum group and let $\mathbb{H}$ be a subgroup of $\mathbb{G}$: let $\pi: C^u(\mathbb{G}) \to C^u(\mathbb{H})$ be the quotient map intertwining the respective coproducts.

Definition 2.4 We will say that $\mathbb{H}$ is a *characteristic subgroup* of $\mathbb{G}$ if for any automorphism of $\mathbb{G}$, *i.e.*, a Hopf *-isomorphism $\theta: C^u(\mathbb{G}) \to C^u(\mathbb{G})$ (*cf.* [19, Section 3]), $\mathbb{H}$ is mapped onto $\mathbb{H}$, *i.e.*, $\theta(\ker(\pi)) = \ker(\pi)$, or in other words, there exists an automorphism $\chi: C^u(\mathbb{H}) \to C^u(\mathbb{H})$ such that $\pi \circ \theta = \chi \circ \pi$.

Clearly, this notion can be described equivalently in terms of the underlying Hopf *-algebra, but we will use this without further mention. An example of a characteristic subgroup is as follows.

Proposition 2.5 *The intrinsic subgroup* $\mathrm{Gr}(\widehat{\mathbb{G}})$ *of* $\mathbb{G}$ *is characteristic.*

Proof Let $\theta: C^u(\mathbb{G}) \to C^u(\mathbb{G})$ be an automorphism of $\mathbb{G}$. As the kernel of the quotient map $q: C^u(\mathbb{G}) \to C(\mathrm{Gr}(\widehat{\mathbb{G}}))$ is an ideal generated by commutators, and as $\theta([x,y]) = [\theta(x), \theta(y)]$, $\theta(\ker(q_{\mathbb{G}})) \subseteq \ker(q_{\mathbb{G}})$. The other inclusion follows by applying θ^{-1}. ■

There is another, more concrete, example of a characteristic subgroup, and we will use it in what follows. Let $K = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ denote the group algebra of the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2 = \langle t_1, t_2 | t_1^2 = t_2^2 = 1, t_1 t_2 = t_2 t_1 \rangle$. Let us also denote $t_3 = t_1 t_2 \in \mathbb{Z}_2 \times \mathbb{Z}_2$ and the neutral element $t_0 \in \mathbb{Z}_2 \times \mathbb{Z}_2$. The Klein group can be embedded into $SO_{-1}(3)$ as follows:

$$\mathcal{O}(SO_{-1}(3)) \ni a_{i,j} \overset{\pi_d^-}{\longmapsto} \delta_{i,j} t_i \in K \tag{2.3}$$

There are other occurrences of the Klein group as a subgroup of $SO_{-1}(3)$; this one will be called diagonal. Let $\pi: \mathcal{O}(SO_{-1}(3)) \to K$ be a Klein subgroup in $SO_{-1}(3)$ and consider the following factorization:

$$\begin{tikzcd} \mathcal{O}(SO_{-1}(3)) \arrow[r, two heads, "ab"] \arrow[rr, bend left, two heads, "\pi"] & \mathcal{O}(SO_{-1}(3))_{ab} \arrow[r, two heads, "\theta"] & K \end{tikzcd}$$

In the above diagram, $\mathcal{O}(SO_{-1}(3))_{ab}$ denotes the abelianization of $\mathcal{O}(SO_{-1}(3))$. The Hopf *-algebra quotient of $\mathcal{O}(SO_{-1}(3))$ by the commutator ideal, ab, denotes this quotient map.

It is clear that all quotients π onto the group algebra of the Klein group enjoy the above factorization. Let us describe it more explicitly.

Lemma 2.6 *$\mathcal{O}(SO_{-1}(3))_{ab}$ is precisely the Hopf*-algebra $C(S_4)$, and the map ab is given as follows: consider the canonical representation $\rho\colon S^4 \to O(4)$ and consider the restriction to the subspace $(1,1,1,1)^\perp$, written in the basis*

$$(1,-1,1,-1),(1,1,-1,-1),(1,-1,-1,1).$$

This gives an embedding $\rho\colon S_4 \to O(3)$, $ab\colon \mathcal{O}(SO_{-1}(3)) \to C(S_4)$ acts as $a_{i,j} \overset{ab}{\longmapsto} x_{i,j} \circ \rho$.

Proof The proof is a straightforward computation. ■

Thus, any Klein subgroup of $SO_{-1}(3)$ is a Klein subgroup in S_4; there are two types of Klein groups embedded into S_4: the diagonal one, of the form

$$\{\mathrm{id},(12)(34),(13)(24),(14)(23)\},$$

and the non-diagonal ones, of the form: $\{\mathrm{id},(1i),(kl),(1i)(kl)\}$.

Remark 2.7 The diagonal Klein subgroup, in the above map, consists of the matrices

$$\left\{\mathbb{1}_{M_3},\begin{pmatrix}-1&0&0\\0&1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}1&0&0\\0&-1&0\\0&0&-1\end{pmatrix},\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&1\end{pmatrix}\right\}.$$

Lemma 2.8 *The diagonal Klein subgroup of $SO_{-1}(3)$ is a characteristic subgroup.*

Proof Any automorphism of $SO_{-1}(3)$ restricts to an automorphism of $\mathrm{Gr}(\widehat{SO_{-1}(3)}) = S_4$ and any occurrence of a Klein subgroup in $SO_{-1}(3)$ appears as a Klein subgroup of S_4, so it suffices to check that the diagonal (in $SO_{-1}(3)$) Klein subgroup of $SO_{-1}(3)$ is precisely the diagonal (in S_4) Klein subgroup of S_4 and that the latter is characteristic in S_4. Both assertions are obvious. ■

Just to complete the picture, let us elucidate the non-diagonal Klein subgroups, providing a non-example of a characteristic subgroup.

Lemma 2.9 *All non-diagonal Klein subgroups of S_4 are conjugate; the corresponding automorphism of S_4 extends to $SO_{-1}(3)$.*

Proof Let $\{\mathrm{id},(12),(34),(12)(34)\}$ and $\{\mathrm{id},(1i),(2j),(1i)(2j)\}$ be two distinct Klein subgroups of S_4. It is easy to check that conjugation by $(2i)$ gives the first part of the lemma. In order to get the automorphism $u\colon \mathcal{O}(SO_{-1}(3)) \to \mathcal{O}(SO_{-1}(3))$ extending it, simply consider the map $A \mapsto \rho(2i)A\rho(2i)$, where ρ is the map from Lemma 2.6. ■

2.4 Twistings Applied to $SO(3)$

Recall that $K = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$. The restriction of functions on $SO(3)$ to its diagonal subgroup gives a Hopf *-algebra surjection

$$\mathcal{O}(SO(3)) \ni x_{i,j} \overset{\pi_d}{\longmapsto} \delta_{i,j} t_i \in K.$$

Let $\sigma\colon K \otimes K \cong \mathbb{C}[(\mathbb{Z}_2 \times \mathbb{Z}_2)^2] \to \mathbb{C}$ be the unique linear extension of the mapping

$$\sigma(t_i, t_j) = \begin{cases} -1 & \text{for } (i,j) \in \{(1,1),(1,2),(1,3),(2,2),(2,3),(3,2)\}, \\ 1 & \text{otherwise.} \end{cases}$$

In other words, for $1 \le i, j \le 2$ we have that $\sigma(t_i, t_j) = -1$ if and only if $i \le j$, and we extend this definition by bimultiplicativity. Then σ is a 2-cocycle in the sense of (2.2). We will work with the cocycle $\sigma_d = \sigma \circ (\pi_d \otimes \pi_d)$ on $\mathcal{O}(SO(3))$. Note that $\sigma_d^{-1} = \sigma_d$.

Similarly, define the 2-cocycle

$$\sigma_d^- \colon \mathcal{O}(SO_{-1}(3)) \otimes \mathcal{O}(SO_{-1}(3)) \longrightarrow \mathbb{C}$$

via $\pi_d^- \colon \mathcal{O}(SO_{-1}(3)) \to K$ (recall the definition of π_d^- given in (2.3)).

Theorem 2.10 ([3, Theorem 5.1]) *The map $[x_{i,j}] \mapsto a_{i,j}$ is an isomorphism between the Hopf *-algebras $\mathcal{O}(SO_{-1}(3))^{\sigma_d}$ and $\mathcal{O}(SO_{-1}(3))$.*

With this in hand, and the results of Subsections 2.2 and 2.3, we are able to classify all the automorphisms of $SO_{-1}(3)$.

Theorem 2.11 *Every automorphism of $SO_{-1}(3)$ is given by $A \mapsto \rho(x)^\top A \rho(x)$ for some $x \in S_4$. In other words, $\operatorname{Aut}(SO_{-1}(3)) \cong S_4$.*

Proof Consider an automorphism $\theta\colon C(SO_{-1}(3)) \to C(SO_{-1}(3))$ and the following diagram:

$$\begin{tikzcd} \mathcal{O}(SO_{-1}(3)) \arrow[r, two heads, "\theta"] \arrow[rr, bend left, "\chi\circ\pi_d^-"] & \mathcal{O}(SO_{-1}(3)) \arrow[r, two heads, "\pi_d^-"] & K. \end{tikzcd}$$

Here $\chi\colon K \to K$ is an automorphism satisfying $\pi_d^- \circ \theta = \chi \circ \pi_d^-$; it pops out of the fact that diagonal Klein subgroup is characteristic; see Definition 2.4 and Lemma 2.8. Thus, we can use Proposition 2.3 to "untwist" this diagram and obtain an automorphism of $SO(3)$ (which should be easier to classify). Apply the cocycle σ_d^- (recall that the Klein groups have no nontrivial twist; *cf.* [3, Lemma 6.2]), and Proposition 2.3 gives us the following diagram:

$$\begin{tikzcd} \mathcal{O}(SO(3)) \arrow[r, two heads, "\theta^{\sigma_d^-}"] \arrow[rr, bend left, "\chi\circ\pi_d"] & \mathcal{O}(SO(3)) \arrow[r, two heads, "\pi_d"] & K \end{tikzcd}$$

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and $\theta^{\sigma_d^-} = [\theta]$, where $[\,\cdot\,]$ is understood as in Subsection 2.2. As any automorphism of $SO(3)$ is inner, it is enough to check which of them preserve the diagonal Klein subgroup. It is clear that conjugation by $\rho(x)$, $x \in S_4$ (where $\rho\colon S_4 \to O(3)$ is introduced in Lemma 2.6), is such an automorphism (as the diagonal Klein subgroup is characteristic in S_4). It is clear that these automorphisms, when seen as automorphisms of S_4^+, are distinct, as their composition with $q_{S_4^+}$ are, so it suffices to compute the total number of automorphisms.

We will show, via brute-force computations, that there are precisely $24 = \#S_4$ matrices $F = (f_{ij})_{1\le i,j\le 3} \in SO(3)$ that conjugate the diagonal Klein subgroup in $SO(3)$. Indeed, using Remark 2.7, such matrix F necessarily satisfies:

$$F\begin{pmatrix}\epsilon_1 & 0 & 0\\ 0 & \epsilon_2 & 0\\ 0 & 0 & \epsilon_3\end{pmatrix}F^\top = \begin{pmatrix}\epsilon_1 f_{11}^2+\epsilon_2 f_{12}^2+\epsilon_3 f_{13}^2 & * & *\\ * & \epsilon_1 f_{21}^2+\epsilon_2 f_{22}^2+\epsilon_3 f_{23}^2 & *\\ * & * & \epsilon_1 f_{31}^2+\epsilon_2 f_{32}^2+\epsilon_3 f_{33}^2\end{pmatrix},$$

where two out of three $\epsilon_1, \epsilon_2, \epsilon_3$ are equal to -1 and the remaining one is equal to 1. The matrix on the right-hand side is again a matrix of the form described in Remark 2.7. Thus, analyzing the diagonal entries, we arrive at

$$\begin{aligned} -f_{i1}^2 - f_{i2}^2 + f_{i3}^2 &= \varepsilon_1, & -f_{i1}^2 + f_{i2}^2 - f_{i3}^2 &= \varepsilon_2,\\ +f_{i1}^2 - f_{i2}^2 - f_{i3}^2 &= \varepsilon_3, & +f_{i1}^2 + f_{i2}^2 + f_{i3}^2 &= 1,\end{aligned}$$

where, again, two out of three $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are equal to -1 and the remaining one is equal to 1. Adding a fourth equation to the two with negative right-hand side we arrive at a system of constraints forcing F to have two zero entries in each row. Repeating the same argument with the roles of F and $F^\top$ reversed, we see that, as well, F has to have two zero entries in each column. The remaining entry needs to be ± 1, because of norm-one condition. It is now easy to check that there are precisely $24 = \#S_4$ of such matrices, which finishes the proof. ∎

2.5 The Embeddings $O_{-1}(2) \subset SO_{-1}(3)$

Recall an embedding $O_{-1}(2) \subset SO_{-1}(3)$ from (2.1):

$$a_{i,j} \longmapsto \left\{\begin{matrix}\widetilde{a}_{i,j} & \text{for } 1 \le i,j \le 2,\\ \widetilde{a}_{1,1}\widetilde{a}_{2,2} + \widetilde{a}_{1,2}\widetilde{a}_{2,1} & \text{for } i = j = 3,\\ 0 & \text{otherwise.}\end{matrix}\right\} : C(SO_{-1}(3)) \twoheadrightarrow C(O_{-1}(2))$$

There are other embeddings $O_{-1}(2) \subset SO_{-1}(3)$, which are classified by the following theorem.

Theorem 2.12 *There are three embeddings of* $O_{-1}(2)$ *into* $SO_{-1}(3)$*. The three copies are conjugate via an automorphism described in* Theorem 2.11.

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Proof Recall from [3, Theorem 7.1 & Proposition 7.3] that the group of characters of $O_{-1}(2)$ is isomorphic to D_4, the dihedral group of a square (or one can simply compute using relations (ii)–(iii) of Definition 2.2 together with commutation relations: one obtains the only eight orthogonal matrices with entries $0, \pm 1$ and uses classification of groups of order 8). Let $\Phi\colon \mathcal{O}(SO_{-1}(3)) \to \mathcal{O}(O_{-1}(2))$ be a Hopf *-algebra quotient. Consider the following diagram:

$$\begin{tikzcd}
\mathcal{O}(SO_{-1}(3)) \arrow[rr, two heads, "\Phi"] \arrow[dr, "q_{SO_{-1}(3)}"'] & & \mathcal{O}(O_{-1}(2)) \arrow[dr, "q_{O_{-1}(2)}"] & \\
& C(S_4) \arrow[rr, two heads, "\varphi"] & & C(D_4).
\end{tikzcd}$$

The existence of the map φ as in the diagram above follows from the universal property of abelianization. Because all the involved morphisms are Hopf *-algebra morphisms, so is φ. Similarly, because all the involved morphisms are surjections, so is φ. Thus $\widehat{\varphi}$, the Gelfand transform of φ, is a monomorphism $\widehat{\varphi}\colon D_4 \hookrightarrow S_4$. Let us take for granted that the image of $\widehat{\varphi}$ contains the diagonal Klein subgroup of S_4 (the proof of this statement is postponed to Lemma 2.13 just below the end of the proof of Theorem 2.12).

As the diagonal Klein subgroup is characteristic in $SO_{-1}(3)$, this gives us the following diagram of morphisms:

$$\begin{tikzcd}
\mathcal{O}(SO_{-1}(3)) \arrow[r, two heads, "\Phi"] \arrow[rr, bend right, "\pi_d^-"'] & \mathcal{O}(O_{-1}(2)) \arrow[r, two heads, "\chi \circ \widetilde{\pi}"] & K,
\end{tikzcd}$$

where $\widetilde{\pi}$ is obtained by composing $q_{O_{-1}(2)}$ with Hopf *-algebra quotient map corresponding to restriction to the diagonal Klein subgroup in $\widehat{\varphi}(D_4)$ and χ is an automorphism of the Klein group (its existence is a consequence of the diagonal Klein group being characteristic). Using Proposition 2.3, we untwist this diagram and arrive at

$$\begin{tikzcd}
\mathcal{O}(SO(3)) \arrow[r, two heads, "{[\Phi]}"] \arrow[rr, bend right, "\pi_d"'] & \mathcal{O}(O(2)) \arrow[r, two heads, "\Pi"] & K.
\end{tikzcd}$$

The closed subgroups of $SO(3)$ isomorphic to $O(2)$ are all of the form

$$\left\{ F \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix} F^\top : A \in O(2) \right\}$$

for some matrix $F \in SO(3)$ (see, *e.g.*, [12, Theorem 6.1]). The occurrence of $O(2)$ in $SO(3)$ coming from the above diagram contains the diagonal Klein subgroup. Because $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq D_4$ is characteristic, we know from (the proof of) Theorem 2.11 that the matrices F are necessarily of the form $\rho(x)$ for some $x \in S_4$. In accordance with Definitions 2.1, 2.2, and 2.4, let us denote by $\widetilde{X} = (\widetilde{x}_{i,j})_{1 \le i,j \le 2}$ the fundamental matrix of $C(O(2))$. Recall that for a $k \times k$ matrix $B = (b_{i,j})_{1\le i,j\le k}$ the permanent is defined as $\mathrm{perm}(B) = \det_{-1}(B) = \sum_{\tau\in S_k}\prod_{i=1}^{k} b_{i,\tau(i)}$. To verify (2.1) it is then enough to check that

$$\begin{aligned}[\det(\widetilde{X})] &= [\widetilde{x}_{1,1}\widetilde{x}_{2,2}] - [\widetilde{x}_{1,2}\widetilde{x}_{2,1}]\\ &= \sigma(t_1,t_2)\sigma(t_1,t_2)[\widetilde{x}_{1,1}][\widetilde{x}_{2,2}] + \sigma(t_1,t_2)\sigma(t_2,t_1)[\widetilde{x}_{1,2}][\widetilde{x}_{2,1}]\\ &= \widetilde{a}_{1,1}\widetilde{a}_{2,2} + \widetilde{a}_{1,2}\widetilde{a}_{2,1} = \mathrm{perm}(\widetilde{A}) \qquad \blacksquare\end{aligned}$$

Lemma 2.13 *The image of $\widehat{\varphi}$ contains the diagonal Klein subgroup of S_4.*

Proof Up to an inner automorphism, the only way to embed the dihedral group into the symmetric group is via

$$\widehat{\varphi}(D_4) = \{\mathrm{id}, (13), (24), (1234), (1432), (13)(24), (12)(34), (14)(23)\}$$

and the diagonal Klein subgroup of S_4 is precisely $\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$, which is characteristic (hence it appears as a subgroup of any possible occurrences of D_4 in S_4). ■

In summary, Theorem 2.12 says that any embedding $O_{-1}(2) \subset SO_{-1}(3)$ is given by the following map:

$$A \xmapsto{\Phi_x} \rho(x)\begin{pmatrix}\widetilde{A} & 0\\ 0 & \mathrm{perm}(\widetilde{A})\end{pmatrix}\rho(x)^{\top},$$

where $x \in S_4$ and ρ is as in Lemma 2.6.

2.6 Conclusions

Recall from [6] that S_4^τ is a cocycle twist of S_4 by the cocycle τ induced from the cocycle σ defined on a non-diagonal Klein subgroup. From [3, Theorem 6.1] we know that $S_4^\tau \subseteq S_4^+$ and that $\mathrm{Gr}(\widehat{S_4^\tau}) = D_4$ by [3, Lemma 6.7]. Then let $O_{-1}(2) \subseteq S_4^+$ be embedded in such a way that $\mathrm{Gr}(\widehat{O_{-1}(2)}) = \mathrm{Gr}(\widehat{S_4^\tau})$ as subgroups of $S_4 = \mathrm{Gr}(\widehat{S_4^+})$ (we know from Theorem 2.12 that it is possible to find such a copy of $O_{-1}(2)$). But [3, Theorem 7.1], establishing the full list of subgroups of $O_{-1}(2)$, ensures us that $S_4^\tau \not\subseteq O_{-1}(2)$ and thus $\mathbb{G} = \overline{\langle O_{-1}(2), S_4^\tau\rangle} = S_4^+$, as this group as strictly bigger than $O_{-1}(2)$ and there are no intermediate groups between $O_{-1}(2)$ and S_4^+. But at the same time,

$$S_4 = \mathrm{Gr}(\widehat{\mathbb{G}}) \neq \langle D_4, D_4\rangle = D_4,$$

hence the inclusion in Theorem 1.1 can be proper.

An additional consequence of our considerations is the following proposition.

Proposition 2.14 $S_4^+ = \overline{\langle S_4^\tau \cup S_4\rangle}$.

Proof Let $\mathbb{G} = \overline{\langle S_4^\tau \cup S_4\rangle}$. As $\mathrm{Gr}(\widehat{\mathbb{G}}) = S_4$ and $\mathbb{G} \neq S_4$ (because $S_4^\tau \not\subseteq S_4$), we can check on the list of [3, Theorem 6.1] that the only remaining quantum subgroup of S_4^+ with $\mathrm{Gr}(\mathbb{G}) = S_4$ is S_4^+ itself. ■

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Remark 2.15 Recall $\widehat{\mathbb{G}}$ is hyperlinear if and only if $L^\infty(\mathbb{G}, h_\mathbb{G})$ can be embedded into R^ω, where R is the hyperfinite II_1 factor and ω is a free ultrafilter. In other words, $L^\infty(\mathbb{G}, h_\mathbb{G})$ satisfies the Connes' embedding property.

Corollary 2.16 *$\widehat{S_4^+}$ is hyperlinear.*

Proof This follows immediately from the fact that S_4 and S_4^τ are finite (and hence their duals are hyperlinear) and [8, Theorem 3.6]. ∎

Let us mention here that this result can also be proved by employing the fact that $C(S_4^+)$ is nuclear ($\widehat{S_4^+}$ is amenable). However, the above proof, as the proof of [8, Theorem 3.6], is elementary, unlike the first proof of nuclearity of $C(S_4^+)$, contained in [1]. However, faithfulness of Pauli matrix representation, discussed in [3], can also be employed to prove nuclearity of $C(S_4^+)$ and is still pretty elementary.

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