SEMIGROUPS SATISFYING MINIMAL CONDITIONS II

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In this paper we continue the investigation of minimal conditions on semigroups begun by J. A. Green [**3**] and taken up by Munn [**5**]. A unified account of the results in [**3**] and [**5**], together with some additional material, is presented in the text-book by Clifford and Preston [**1**, §6.6]. All terminology and notation not introduced explicitly will be as in [**1**].

Let S be a semigroup. The relation $\leqslant$ defined on the set $S/\mathscr{L}$ of all $\mathscr{L}$-classes of S by the rule that

$$L_a \leqslant L_b \Leftrightarrow S^1a \subseteq S^1b \qquad (a, b \in S)$$

is a partial ordering. Similar partial orderings are defined on the sets $S/\mathscr{R}$ and $S/\mathscr{J}$. (No ambiguity will result from the use of the same symbol $\leqslant$ for all three partial orderings and for others introduced below.) Following Green [**3**], we say that S satisfies the condition $M_L[M_R, M_J]$ if and only if every nonempty collection of $\mathscr{L}$-classes [$\mathscr{R}$-classes, $\mathscr{J}$-classes] contains a minimal member. It is easy to verify that $M_L[M_R, M_J]$ is equivalent to the condition that every strictly descending chain of $\mathscr{L}$-classes [$\mathscr{R}$-classes, $\mathscr{J}$-classes] must be finite.

Now consider the relation $\leqslant$ defined on the set $S/\mathscr{H}$ of all $\mathscr{H}$-classes of S as follows:

$$H_a \leqslant H_b \Leftrightarrow L_a \leqslant L_b \text{ and } R_a \leqslant R_b \qquad (a, b \in S).$$

Evidently this is again a partial ordering. At the centre of our discussion is the corresponding minimal condition M_H: every nonempty set of $\mathscr{H}$-classes contains a minimal member (equivalently, every strictly descending chain of $\mathscr{H}$-classes must be finite).

It is convenient for our purpose to consider three further conditions on S, namely M_L^*, M_R^* and GB. As in [**5**; **1**, §6.6], we say that S satisfies $M_L^*[M_R^*]$ if and only if, for all $J \in S/\mathscr{J}$, the set of all $\mathscr{L}$-classes [$\mathscr{R}$-classes] of S contained in J contains a minimal member. An element $a \in S$ is said to be *group-bound* if and only if a^n lies in a subgroup of S for some positive integer n. Clearly, every periodic element is group-bound. We say that S itself is group-bound, or that S satisfies the condition GB, if and only if each of its elements is group-bound. It should be noted that in an earlier paper [**6**] (based, in turn, on [**2**]) group-bound elements of a semigroup were termed "pseudo-invertible".

The paper is in two sections. In the first of these we examine the interdependence of the seven conditions M_L, M_R, M_J, M_H, M_L^*, M_R^*, GB. Green [**3**, Theorem 4; **1**, Theorem 6.49] has shown that M_L and M_R together imply M_J: we extend this result by proving that the conjunction of M_L and M_R is logically equivalent to the conjunction of M_J and M_H, to the conjunction of M_J and GB, and to the conjunction of M_J, M_L^* and M_R^* (Corollary 1.3). Exactly thirteen pairwise inequivalent conditions can be formed from the given seven by taking conjunctions. A complete picture of their interrelationship is provided by a Hasse diagram (1.5).

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The second section concerns Schützenberger groups [**1**, §2.4]. We show that in a semigroup S satisfying M_H the Schützenberger group of an arbitrary $\mathcal{H}$-class must be a homomorphic image of a subgroup of S (Theorem 2.1): thus each $\mathcal{H}$-class of S has cardinal not exceeding that of some subgroup of S. We also prove that in a group-bound semigroup whose subgroups are all trivial the relation $\mathcal{H}$ must itself be trivial (Theorem 2.3). These theorems extend a result of Rhodes [**7**] for finite semigroups and examples show that they cannot be improved within the context of the conditions studied here.

1. Interdependence of the seven conditions. Let X and Y be semigroup conditions. We write $X \leqslant Y$ ("X implies Y") if and only if every semigroup satisfying X also satisfies Y: furthermore, we write $X = Y$ ("X is equivalent to Y") if and only if $X \leqslant Y$ and $Y \leqslant X$. With equality of conditions thus defined as logical equivalence, the relation $\leqslant$ is readily seen to be a partial ordering of any set of semigroup conditions. The conjunction of a finite family $(A_1, A_2, \ldots, A_n)$ of semigroup conditions will be denoted by $A_1 \wedge A_2 \wedge \ldots \wedge A_n$ and is defined as follows: a semigroup satisfies $A_1 \wedge A_2 \wedge \ldots \wedge A_n$ if and only if it satisfies each of the conditions A_i $(i = 1, 2, \ldots, n)$. Evidently if A, B, C are semigroup conditions such that $A \leqslant B$ then $A \wedge C \leqslant B \wedge C$.

Throughout the remainder of the paper we shall denote the family

$$(M_L, M_R, M_J, M_H, M_L^*, M_R^*, \mathrm{GB})$$

by Ω and the set of all conjunctions of nonempty subfamilies of Ω by $\Lambda(\Omega)$. Clearly $\Lambda(\Omega)$ is a finite lower semilattice with respect to $\leqslant$, the greatest lower bound of the pair (A, B) being the conjunction $A \wedge B$. This section is concerned with the structure of $\Lambda(\Omega)$.

We begin with a restatement of an elementary property of the conditions M_L^* and M_R^* [**5**, Lemma 2.2; **1**, Lemma 6.41].

Lemma 1.1. *Let S be a semigroup satisfying $M_L^*[M_R^*]$. Then, for all $a \in S$, $L_a[R_a]$ is minimal in the set of all $\mathcal{L}$-classes $[\mathcal{R}$-classes$]$ of S contained in J_a.*

The following theorem establishes various basic relationships between the members of $\Lambda(\Omega)$.

Theorem 1.2.
(i) $M_L \leqslant M_L^*$; $M_R \leqslant M_R^*$;
(ii) $M_L \wedge M_R \leqslant M_J$;
(iii) $M_J \wedge M_L^* \leqslant M_L$; $M_J \wedge M_R^* \leqslant M_R$;
(iv) $M_L \wedge M_R^* \leqslant M_H$; $M_R \wedge M_L^* \leqslant M_H$;
(v) $M_H \leqslant \mathrm{GB}$;
(vi) $\mathrm{GB} \leqslant M_L^* \wedge M_R^*$.

Proof. We note first that the assertions in (i) are immediate consequences of the definitions and that the result in (ii) is due to Green [**3**, Theorem 4; **1**, Theorem 6.49].

(iii) Let S be a semigroup satisfying M_J and M_L^*. Consider a nonempty set $\mathcal{C}$ of $\mathcal{L}$-classes of S. Since S satisfies M_J there exists $a \in S$ such that $L_a \in \mathcal{C}$ and, for all $x \in S$, if $L_x \in \mathcal{C}$ and $J_x \leqslant J_a$ then $J_x = J_a$. Suppose that $b \in S$ is such that $L_b \in \mathcal{C}$ and $L_b \leqslant L_a$. Then

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$J_b \leq J_a$ and so $J_a = J_b$. But since S satisfies M_L^* it follows from Lemma 1.1 that L_a is minimal in the set of all $\mathscr{L}$-classes contained in J_a. Hence $L_b = L_a$. Consequently L_a is minimal in $\mathscr{C}$. This shows that S satisfies M_L: thus $M_J \wedge M_L^* \leq M_L$. A similar argument shows that $M_J \wedge M_R^* \leq M_R$.

(iv) Let S be a semigroup satisfying M_L and M_R^*. Consider a sequence $a_1, a_2, a_3, \ldots$ of elements of S such that

$$H_{a_1} \geq H_{a_2} \geq H_{a_3} \geq \ldots .$$

We have that $L_{a_1} \geq L_{a_2} \geq L_{a_3} \geq \ldots$ and so, since S satisfies M_L, there exists a positive integer k such that the elements $a_k, a_{k+1}, a_{k+2}, \ldots$ are $\mathscr{L}$-equivalent. Thus $a_k, a_{k+1}, a_{k+2}, \ldots$ are $\mathscr{J}$-equivalent. But $R_{a_k} \geq R_{a_{k+1}} \geq R_{a_{k+2}} \geq \ldots$ and so, by Lemma 1.1, the elements $a_k, a_{k+1}, a_{k+2}, \ldots$ are $\mathscr{R}$-equivalent. It follows that $H_{a_k} = H_{a_{k+1}} = H_{a_{k+2}} = \ldots$. This shows that S satisfies M_H. Hence $M_L \wedge M_R^* \leq M_H$. By duality, $M_R \wedge M_L^* \leq M_H$.

(v) Let S be a semigroup satisfying M_H. Consider any element $a \in S$. Since $H_a \geq H_{a^2} \geq H_{a^3} \geq \ldots$ there exists a positive integer n such that $H_{a^n} = H_{a^{n+1}} = H_{a^{n+2}} = \ldots$. But this means that $(a^n, a^{2n}) \in \mathscr{H}$ and so, by [**1**, Theorem 2.16], H_{a^n} is a group. Thus S is group-bound. Hence $M_H \leq \mathrm{GB}$.

(vi) Let S be a group-bound semigroup. We shall show that S satisfies M_L^*. Let $a, b \in S$ be such that $(a, b) \in \mathscr{J}$ and $L_b \leq L_a$. Then there exist elements $u, v, c \in S^1$ such that $a = ubv$ and $b = ca$. Thus $a = (uc)av$ and so $a = (uc)^n av^n$ for all positive integers n. Now S^1 is group-bound and so we can choose n such that $n > 1$ and $(uc)^n$ lies in a subgroup of S^1, with identity element e, say. Write $g = (uc)^n$ and let g^{-1} denote the inverse of g in the subgroup H_e of S^1. Then

$$ea = e(gav^n) = (eg)av^n = gav^n = a$$

and so

$$g^{-1}(uc)^{n-1}ub = g^{-1}(uc)^{n-1}uca = g^{-1}ga = ea = a.$$

Hence $L_a \leq L_b$, from which it follows that $L_a = L_b$. Consequently, S satisfies M_L^*. By duality, S also satisfies M_R^*. Thus $\mathrm{GB} \leq M_L^* \wedge M_R^*$.

COROLLARY 1.3.

$$M_L \wedge M_R = M_J \wedge M_H = M_J \wedge \mathrm{GB} = M_J \wedge M_L^* \wedge M_R^*.$$

Proof. By (ii), $M_L \wedge M_R \leq M_J$ and, by (i) and (iv), $M_L \wedge M_R \leq M_L^* \wedge M_R \leq M_H$. Thus $M_L \wedge M_R \leq M_J \wedge M_H$. On the other hand, by (v), (vi) and (iii),

$$M_J \wedge M_H \leq M_J \wedge \mathrm{GB} \leq M_J \wedge (M_L^* \wedge M_R^*) = (M_J \wedge M_L^*) \wedge (M_J \wedge M_R^*) \leq M_L \wedge M_R.$$

This gives the result.

It is straightforward to check that, in view of Theorem 1.2, $\Lambda(\Omega)$ has at most thirteen elements We proceed to show by means of examples that it has exactly thirteen.

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First, we require some further notation. For an arbitrary semigroup S let S^{opp} denote the semigroup with the same set of elements as S but with the multiplication reversed. Also, for any two semigroups S and T let $S+T$ denote the 0-direct union of S^0 and T^0 [**1**, §6.3].

The next lemma is almost immediate.

Lemma 1.4. *Let S and T be semigroups and let X be a member of Ω. Then $S+T$ satisfies X if and only if both S and T satisfy X.*

We now consider four semigroups $S_i (i=1,2,3,4)$ defined as follows: S_1, S_2 and S_3 are, respectively, an infinite cyclic semigroup, an infinitely descending semilattice and a Croisot-Teissier semigroup of the form $CT(A, \mathscr{C}, p, p)$ [**1**, §8.2], while S_4 is the semigroup with zero 0, nonzero elements the ordered pairs (i, j) of positive integers i, j such that $i<j$, and multiplication of nonzero elements according to the rule that

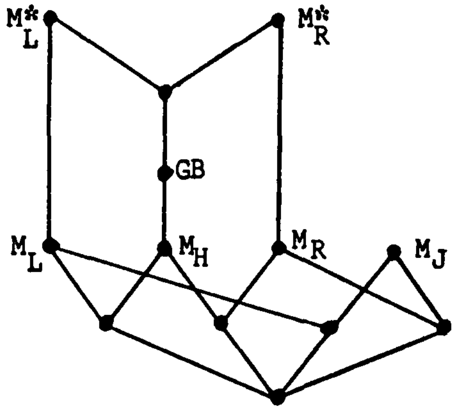
$$(i,j)(r,s)=\begin{cases}(i,s) & \text{if } j=r,\\ 0 & \text{if } j\neq r\end{cases}$$

[**5**; **1**, §6.6, Example 1].

These semigroups, and four others derived from them, label the rows of the following table, the columns of which are labelled by the members of Ω. The entry in the table corresponding to a semigroup S and a condition X is 1 or 0 according as S satisfies or fails to satisfy X. It is a routine matter to check the entries in the first four rows (see [**1**, Theorem 8.11] for S_3): the remaining entries are then easily obtained with the aid of Lemma 1.4 and duality.

	M_L	M_R	M_J	M_H	M_L^*	M_R^*	GB
S_1	0	0	0	0	1	1	0
S_2	0	0	0	0	1	1	1
S_3	0	1	1	0	0	1	0
S_4	1	0	0	1	1	1	1
S_1+S_3	0	0	0	0	0	1	0
$S_3+S_3^{\mathrm{opp}}$	0	0	1	0	0	0	0
$S_3+S_4^{\mathrm{opp}}$	0	1	0	0	0	1	0
$S_4+S_4^{\mathrm{opp}}$	0	0	0	1	1	1	1

From Theorem 1.2 and Corollary 1.3, together with the above table, duality and the observation that the trivial semigroup satisfies all the members of Ω, we see that $\Lambda(\Omega)$ has the Hasse diagram shown below.



(1.5)

Note that the seven members of Ω are distinct and that $\Lambda(\Omega)$ can be obtained from the free semilattice on Ω by imposing precisely the relationships listed in Theorem 1.2.

REMARK 1.6. There exist semigroups satisfying none of the conditions in Ω: for example, $S_1+(S_3+S_3^{opp})$.

REMARK 1.7. By [**1**, Theorem 6.45] the condition $M_L^* \wedge M_R^*$ on a semigroup S implies that $\mathcal{D}=\mathcal{J}$ on S. Now take the Croisot-Teissier semigroup S_3 to be such that $\mathcal{D}\neq\mathcal{J}$. Then the examples S_3 and S_3^{opp} show that $M_L^* \wedge M_R^*$ is the weakest conjunction of members of Ω to imply that $\mathcal{D}=\mathcal{J}$, in the sense that any other conjunction implying the condition $\mathcal{D}=\mathcal{J}$ also implies $M_L^* \wedge M_R^*$.

Furthermore, the proof of [**5**, Theorem 2.3] (see [**1**, Theorem 6.45]) shows that the condition $M_L^* \wedge M_R^*$ on a semigroup implies that each [0-] simple principal factor of the semigroup is completely [0-] simple. Again the examples S_3 and S_3^{opp} demonstrate that $M_L^* \wedge M_R^*$ is the weakest conjunction of members of Ω to imply this condition on principal factors.

REMARK 1.8. Let S be a regular semigroup. Then, for all a and b in S, $L_a \geqslant L_b$ if and only if to each idempotent $e \in L_a$ there corresponds an idempotent $f \in L_b$ such that $e \geqslant f$ [**4**, Remark 2]. Thus, on S, the conditions M_L, M_R and M_H are each equivalent to the condition that every nonempty set of idempotents of S contains a minimal member with respect to the usual partial ordering.

2. The Schützenberger group of an $\mathcal{H}$-class. Let S be a semigroup and H an $\mathcal{H}$-class of S. Write $T=\{x\in S^1 : Hx\subseteq H\}$. Then T is a subsemigroup of S^1 containing the identity 1. Corresponding to each $t\in T$ we define an element γ_t of $\mathcal{T}_H$ (the full transformation semigroup on H) by the rule that $h\gamma_t=ht$ for all $h\in H$. Next, we define $\gamma : T\to\mathcal{T}_H$ by setting $t\gamma=\gamma_t$ for all $t\in T$. Then γ is a homomorphism and the image $T\gamma$ is a group of permutations of H; moreover, $|T\gamma|=|H|$ and if H is a subgroup of S then $T\gamma\cong H$ [**1**, §2.4]. We call $T\gamma$ *the Schützenberger group of H.*

The following theorem generalises a result on finite semigroups due to Rhodes (see [**7**, Proposition 1.1, equivalence of (a) and (c)]).

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THEOREM 2.1. *Let S be a semigroup satisfying M_H and let H be an $\mathscr{H}$-class of S. Then the Schützenberger group of H is a homomorphic image of a subgroup of S.*

Proof. Let $T, \gamma_t(t \in T)$ and γ be defined as above. Evidently the $\mathscr{H}$-classes of S^1 are just those of S, together with $\{1\}$ in the case where $S \neq S^1$. Thus S^1 satisfies M_H and hence the set $\{H_t \in S^1/\mathscr{H} : t \in T\}$ contains a minimal member H', say. Consider an arbitrary element a in $H' \cap T$. Since $H' = H_a \geqslant H_{a^2}$ in S^1 and $a^2 \in T$, it follows from the minimality of H' that a and a^2 are $\mathscr{H}$-equivalent in S^1. Therefore H' is a subgroup of S^1, by [**1**, Theorem 2.16], and so $H' \cap T$ is a subsemigroup of S^1. Now let a^{-1} denote the inverse of a in H'. Then, since $Ht = H$ for all $t \in T$ [**1**, Lemma 2.21], we have that $Ha^{-1} = (Ha^2)a^{-1} = H(a^2a^{-1}) = Ha = H$. Hence $a^{-1} \in T$. Thus $H' \cap T$ is a subgroup of H'.

Denote the identity of H' by e. Then $e \in H' \cap T$. Now consider an arbitrary element t of T. Clearly $ete \in T$ and $H_{ete} \leqslant H_e = H'$ in S^1. Hence $ete \in H'$, by the minimality of H'. Thus $ete \in H' \cap T$. But since γ is a homomorphism and $e^2 = e$, γ_e must be the identity element of the group $T\gamma$. Hence $\gamma_t = \gamma_e\gamma_t\gamma_e = \gamma_{ete} \in (H' \cap T)\gamma$. Consequently $T\gamma \subseteq (H' \cap T)\gamma$ and so $T\gamma = (H' \cap T)\gamma$. The Schützenberger group $T\gamma$ of H is thus the image under γ of a subgroup of S^1, namely $H' \cap T$.

The stated result now clearly holds if $S^1 = S$. It also holds if $|T\gamma| = 1$: for S must contain at least one idempotent, by Theorem 1.2(v). We therefore assume that $S^1 \neq S$ and that $|T\gamma| > 1$. To complete the proof it is enough to show that, with these hypotheses, $H' \cap T \subseteq S$. Now $|H' \cap T| > 1$, since $T\gamma = (H' \cap T)\gamma$. Hence $H' \neq H_1 = \{1\}$ in S^1 and so $H' \subseteq S$. Thus $H' \cap T \subseteq S$, as required.

COROLLARY 2.2. *Let S be a semigroup satisfying M_H and let H be an $\mathscr{H}$-class of S. Then S has a maximal subgroup G such that $|G| \geqslant |H|$.*

It follows from Corollary 2.2 that if S is a semigroup satisfying M_H and if every subgroup of S is trivial then $\mathscr{H}$ is trivial on S. A better result can, however, be obtained directly:

THEOREM 2.3. *Let S be a group-bound semigroup in which every subgroup is trivial. Then $\mathscr{H}$ is trivial on S.*

Proof. Let H be an $\mathscr{H}$-class of S and let $T, \gamma_t(t \in T)$ and γ be defined as before. Let $t \in T$. Since S^1 is also a group-bound semigroup in which every subgroup is trivial there exists a positive integer n such that t^n is an idempotent of S^1. Hence $t^n . t^{n+1} = t^{n+1}$ and $t^{n+1} . t^{2n-1} = t^n$, from which it follows that $t^{n+1} \in H_{t^n}$. But H_{t^n} is a group and so $t^{n+1} = t^n$. Thus $\gamma_{t^n} = \gamma_{t^n}\gamma_t$ and so, since $T\gamma$ is a group, γ_t is the identity of $T\gamma$. Consequently $|T\gamma| = 1$. But $|T\gamma| = |H|$ and therefore $|H| = 1$.

We conclude by showing that, in a certain sense, the results of Theorems 2.1 and 2.3 are best possible.

EXAMPLES 2.4. Let $(T, \cdot)$ be a semigroup, let $(H, *)$ be a group and let $\phi : T \to H$ be a surjective homomorphism. We assume that the sets T, H and $\{0\}$ are pairwise disjoint and we write $S = T \cup H \cup \{0\}$. By means of the following rules we extend the binary operation on T to an operation on S:

$$s0 = 0s = 0, \qquad gh = 0, \qquad gt = g * t\phi, \qquad tg = (t\phi) * g$$

for all $s \in S$, all $g, h \in H$ and all $t \in T$. It is straightforward to verify that S is a semigroup with H and $\{0\}$ as two of its $\mathcal{H}$-classes, the remaining $\mathcal{H}$-classes being precisely those of T. Clearly H is also a $\mathcal{J}$-class of S.

By making particular choices for T, H and ϕ in this construction we obtain three examples ((a), (b), (c) below).

(a) Let K be a nontrivial finite group and let 1 denote its identity. For all positive integers n let $K^{(n)}$ denote the direct product of n copies of K and for all positive integers m, n with $m \leq n$ define a homomorphism $\phi_n^m : K^{(m)} \to K^{(n)}$ by the rule that

$$(k_1, \ldots, k_m)\phi_n^m = (k_1, \ldots, k_m, 1, \ldots, 1)$$

for all $k_1, \ldots, k_m \in K$. Take $T = \bigcup_{n=1}^{\infty} K^{(n)}$ and define a multiplication on T by setting

$$st = (s\phi_p^m)(t\phi_p^n),$$

where $s \in K^{(m)}$, $t \in K^{(n)}$ and $p = \max\{m, n\}$, the product on the right-hand side being computed in $K^{(p)}$. By [**1**, Theorem 4.11], T is a semilattice of groups. Its $\mathcal{H}$-classes are just the groups $K^{(n)}$, each of which is finite.

Next, take H to be the group consisting of all infinite sequences $(k_1, k_2, k_3, \ldots)$ of elements of K with at most finitely many entries different from 1, under componentwise multiplication. Finally, define $\phi : T \to H$ by the rule that

$$(k_1, \ldots, k_n)\phi = (k_1, \ldots, k_n, 1, 1, \ldots)$$

for all positive integers n and all $k_1, \ldots, k_n \in K$. Then ϕ is a surjective homomorphism (and $\phi \circ \phi^{-1}$ is the least group congruence on T).

In this case the semigroup S has only finite subgroups, but possesses an infinite $\mathcal{H}$-class, namely H. Clearly S is periodic and so satisfies the condition GB.

(b) Take H to be a nontrivial group, take T to be the free semigroup $\mathcal{F}_H$ on H and take $\phi : T \to H$ to be any surjective homomorphism. Then S has exactly one subgroup, namely $\{0\}$. Thus every subgroup of S is trivial but S has a nontrivial $\mathcal{H}$-class, namely H. In this case S satisfies M_L^* and M_R^*.

(c) Let H be a nontrivial group, let U be a Baer-Levi semigroup [**1**, §8.1] and let T denote the direct product $U \times H$ $[U^{\text{opp}} \times H]$. Define ϕ to be the projection of T onto $H : (u, h) \mapsto h$ for all $(u, h) \in T$. It is readily seen that T has no subgroups. Hence, as in (b), the only subgroup of S is the trivial subgroup $\{0\}$, while S has a nontrivial $\mathcal{H}$-class, namely H. It is also easy to verify that since $U\,[U^{\text{opp}}]$ satisfies M_R and M_J $[M_L$ and $M_J]$ the same is true for T and so also for S.

Remark 2.5. By Theorem 2.1, the condition M_H on a semigroup S implies that the Schützenberger group of each $\mathcal{H}$-class of S is a homomorphic image of a subgroup of S. Examples 2.4(a) and (c) show that M_H is the weakest member of $\Lambda(\Omega)$ to imply this condition on the Schützenberger groups of $\mathcal{H}$-classes.

By Theorem 2.3, the condition GB on a semigroup S with no nontrivial subgroups implies that $\mathcal{H}$ is trivial on S. Examples 2.4(b) and (c) show that GB is the weakest member of $\Lambda(\Omega)$ to imply that $\mathcal{H}$ is trivial on semigroups with no nontrivial subgroups.

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