ON THE ABSOLUTE CESARO SUMMABILITY OF NEGATIVE ORDER OF A SERIES ASSOCIATED WITH THE CONJUGATE SERIES OF A FOURIER SERIES

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1. Definition. Let $\lambda \equiv \lambda(\omega)$ be continuous, differentiable, and monotonic increasing in $(0, \infty)$ and let it tend to infinity as $\omega \to \infty$. A series $\sum_1^\infty a_n$ is summable $|R, \lambda, r|$, where $r > 0$, if

$$\int_A^\infty \frac{r\lambda'(\omega)}{\{\lambda(\omega)\}^{r+1}} \left| \sum_{n\leq\omega} \{\lambda(\omega) - \lambda(n)\}^{r-1}\lambda(n)a_n \right| d\omega < \infty,$$

where A is a fixed positive number (**6**, Definition B).

Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$ and let

$$f(t) \sim \tfrac{1}{2}a_0 + \sum_{n=1}^\infty (a_n \cos nt + b_n \sin nt) \equiv \tfrac{1}{2}a_0 + \sum_{n=1}^\infty A_n(t). \tag{1.1}$$

The series conjugate to (1.1), at $t = x$, is

$$\sum_{n=1}^\infty (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^\infty B_n(x). \tag{1.2}$$

In what follows we use the following notation:

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\}, \qquad \psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\},$$

$$\psi_1(t) \log \frac{2\pi}{t} = \theta(t) = \frac{1}{2}\int_t^\pi \psi(u) \cot \tfrac{1}{2}u\, du,$$

$$h(t) = \psi(t)/\log\frac{2\pi}{t}, \quad \Psi(t) = \int_0^t \psi(u)\, du, \quad \bar{S}_n(x) = \sum_{k=1}^n B_k(x).$$

By $F(t) \in \mathrm{BV}(h, k)$ we mean that $F(t)$ is of bounded variation over (h, k).

2. Our aim in this paper is to obtain a criterion for the absolute Cesàro summability of negative order of the series

$$\sum_{n=1}^\infty \frac{\bar{S}_n(x)}{n \log(n+1)}. \tag{2.1}$$

In the last section of this note we shall deduce a criterion for the $|R, \log\omega, 1|$ summability of the series (1.2) from our main theorem. Our main theorem reads as follows.

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THEOREM. *If*
(i) $h(t) \in \mathrm{BV}(0, \pi)$,
(ii) $\int_0^\pi (|h(t)|/t)\, dt < \infty$, *and*
(iii) $\int_0^\pi (|\psi_1(t)|/t)\, dt < \infty$,
then the series (2.1) *is summable* $|C, K|$, $K > -1$.

3. For the proof of the theorem, we require a number of lemmas.

LEMMA 1 (**10**). *If* $\phi(t) \in \mathrm{BV}(0, \pi)$, *then the series* $\sum_1^\infty A_n(x)/\log(n + 1)$ *is absolutely harmonic summable.*

LEMMA 2 (**8**, Lemma 2). *If*

$$\int_0^\pi \frac{|\psi(t)|}{t \log(2\pi/t)}\, dt < \infty,$$

then $\psi_1(t) \in \mathrm{BV}(0, \pi)$.

LEMMA 3. *If* $\sum u_n$ *is summable* $|C|$ (*i.e., absolutely Cesàro summable of any unspecified order*), *then a necessary and sufficient condition that it should be summable* $|C, K|$, $K > -1$ *is that the sequence* $\{nu_n\}$ *is summable* $|C, K + 1|$.

This is a particular case of a well-known result (**5**).

LEMMA 4. *If*
(i) $h(t) \in \mathrm{BV}(0, \pi)$ *and*
(ii) $\int_0^\pi (|h(t)|/t)\, dt < \infty$,
then $\sum_{n=1}^\infty B_n(x)/\log(n + 1)$ *is summable* $|C, \delta|$, $\delta > 0$.

See (**9**, Theorem II).

LEMMA 5. *For* $0 \leqq \rho \leqq 1$, *the sequence* $\{\bar{S}_n(x)/\log n\}$ *is summable* $|C, \rho|$ *whenever the series* $\sum_{n=1}^\infty B_n(x)/\log(n + 1)$ *is summable* $|C, \rho|$.

Proof of Lemma 5. It is known (**4**) that $|C, 1| \sim |R, \omega, 1|$. Furthermore, by the second theorem of consistency for absolute summability (**2**), summability $|R, \omega, 1|$ implies summability $|R, \log \omega, 1|$. Thus, from the hypotheses, it follows that $\sum_1^\infty B_n(x)/\log(n + 1)$ is summable $|R, \log \omega, 1|$, i.e., by the definition

$$\int_2^\infty \frac{d\omega}{\omega(\log \omega)^2} \left| \sum_{n \leqq \omega} B_n(x) \right| < \infty;$$

from which it follows that

$$\sum_2^\infty \frac{|\bar{S}_n(x)|}{n(\log n)^2} < \infty. \tag{3.1}$$

We have that

$$\Delta\left(\frac{\bar{S}_n(x)}{\log n}\right) = \frac{\bar{S}_n(x)}{\log n} - \frac{\bar{S}_{n+1}(x)}{\log(n + 1)} = \frac{\bar{S}_n(x) \log(1 + n^{-1})}{\log n \cdot \log(n + 1)} - \frac{B_{n+1}(x)}{\log(n + 1)}. \tag{3.2}$$

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Since $\log(1 + n^{-1}) = O(n^{-1})$, the absolute convergence of the series

$$\sum_2^\infty \frac{\bar{S}_n(x) \log(1 + n^{-1})}{\log n \cdot \log(n + 1)}$$

follows immediately from (3.1). Now, by virtue of the identity (3.2), the series $\sum_2^\infty \Delta(\bar{S}_n(x)/\log n)$ is summable $|C, \rho|$ whenever $\sum_1^\infty B_n(x)/\log(n + 1)$ is summable $|C, \rho|$, $0 \leqq \rho \leqq 1$. This completes the proof of Lemma 5.

Combining the results of Lemmas 4 and 5 we at once obtain the following result.

LEMMA 6. *The hypotheses of Lemma* 4 *imply that the sequence* $\{\bar{S}_n(x)/\log n\}$ *is summable* $|C, \delta|$, $\delta > 0$.

We show that hypothesis (i) of Lemma 4 alone cannot imply summability $|C, \delta|$, $\delta > 0$, of the sequence $\{\bar{S}_n(x)/\log n\}$. To do so, we need the following asymptotic formula

$$\int_0^\pi \log \frac{2\pi}{t} \sin nt\, dt \sim \frac{\log n}{n}. \tag{3.3}$$

Proof of (3.3). Write

$$\int_0^\pi \log \frac{2\pi}{t} \sin nt\, dt = \int_0^{\pi/2n} + \int_{\pi/2n}^\pi = I_1 + I_2, \quad \text{say.} \tag{3.4}$$

$$I_1 = \frac{1}{n} \int_0^{\pi/2} \log \frac{2\pi n}{t} \sin t\, dt$$

$$= \frac{\log n}{n} \int_0^{\pi/2} \sin t\, dt + \frac{1}{n} \int_0^{\pi/2} \log \frac{2\pi}{t} \sin t\, dt.$$

Thus, we have that

$$I_1 = n^{-1} \log n + O(n^{-1}). \tag{3.5}$$

Integrating by parts, we have that

$$I_2 = \left[-\log \frac{2\pi}{t} \frac{\cos nt}{n} \right]_{\pi/2n}^\pi - \frac{1}{n} \int_{\pi/2n}^\pi \frac{\cos nt}{t}\, dt$$

$$= -\log 2 \frac{\cos n\pi}{n} - \frac{1}{n} \int_{\pi/2}^{n\pi} \frac{\cos t}{t}\, dt.$$

Thus,

$$I_2 = O(n^{-1}). \tag{3.6}$$

Collecting (3.4), (3.5), and (3.6) we obtain (3.3). We choose an odd function $\psi(t) = \log 2\pi/t$ $(0 < t < \pi)$, defined elsewhere by periodicity. Using (3.3), we find that

$$\frac{2}{\pi} \int_0^\pi \psi(t) \left(\sum_{k=1}^n \sin kt \right) dt \sim \frac{2}{\pi} \frac{(\log n)^2}{2} = \frac{(\log n)^2}{\pi}.$$

Thus, it follows that the partial sum of the series conjugate to the Fourier series of the same function at $t = 0$ (i.e., $\bar{S}_n(x)$) is asymptotic to $(\log n)^2/\pi$. Thus,

$$\frac{\bar{S}_n(x)}{\log n} \sim \frac{\log n}{\pi},$$

and hence the sequence $\{\bar{S}_n(x)/\log n\}$ is not summable $|C, \delta|$, $\delta > 0$.

LEMMA 7. *If $\psi(t) \in L(0, \pi)$, then the series $\sum_2^\infty \bar{S}_n(x)/n \log n$ is summable $|C, \delta|$, $\delta > 0$, if and only if the series $\sum_2^\infty c_n/\log n$ is summable $|C, \delta|$, where c_n is the coefficient of the Fourier sine series of*

$$\theta(t) = \frac{1}{2} \int_t^\pi \psi(u) \cot \tfrac{1}{2}u \, du.$$

Proof of Lemma 7. Since $\psi(t) \in L(0, \pi)$, it follows (**3**) that $\theta(t) \in L(0, \pi)$ and $t\theta(t) \to 0$ as $t \to +0$. Hence, integrating by parts, we obtain the identity

$$\bar{S}_n(x) - \tfrac{1}{2}B_n(x) = nc_n. \tag{3.7}$$

The corresponding formula for Fourier series is well known (**3**). Integration by parts yields

$$\frac{1}{2}\frac{B_n(x)}{n \log n} = \frac{1}{\pi} \int_0^\pi \Psi(t) \frac{\cos nt}{\log n} \, dt.$$

$\Psi(t)$ being an integral is absolutely continuous, and therefore by Lemma 1, $\sum_2^\infty B_n(x)/n \log n$ is absolutely harmonic summable, and hence *a fortiori* summable $|C, \delta|$, $\delta > 0$. Hence, by the identity (3.7), the series $\sum_2^\infty \bar{S}_n(x)/n \log n$ is summable $|C, \delta|$, $\delta > 0$, if and only if $\sum_2^\infty c_n/\log n$ is summable $|C, \delta|$.

LEMMA 8. *If*
(i) $\psi_1(t) \in \mathrm{BV}(0, \pi)$ *and*
(ii) $\int_0^\pi (|\psi_1(t)|/t) \, dt < \infty$,
then the series $\sum_2^\infty \bar{S}_n(x)/n \log n$ is summable $|C, \delta|$, $\delta > 0$.

Proof of Lemma 8. Since

$$c_n = \frac{1}{\pi} \int_0^\pi \psi_1(t) \log \frac{2\pi}{t} \sin nt \, dt,$$

we notice that by Lemma 4 (writing $\psi_1(t)$ in place of $h(t)$) the series $\sum_2^\infty c_n/\log n$ is summable $|C, \delta|$, $\delta > 0$, with the hypotheses. Hence, by Lemma 7, our result follows.

4. Proof of the theorem. By Lemma 2, condition (ii) of the theorem implies condition (i) of Lemma 8. Thus, by Lemma 8, (ii) and (iii) of the theorem ensure summability $|C, \delta|$, $\delta > 0$, of $\sum_2^\infty \bar{S}_n(x)/n \log n$. By Lemma 6, (i) and (ii) of the theorem ensure summability $|C, K + 1|$, $K > -1$, of the sequence $\{\bar{S}_n(x)/\log n\}$, that is, the sequence $\{n \cdot \bar{S}_n(x)/n \log n\}$. Thus, it

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follows that our hypotheses at the same time ensure the summability $|C, K+1|, K > -1$, of the sequence $\{\bar{S}_n(x)/\log n\}$, i.e., the sequence $\{n \cdot \bar{S}_n(x)/n \log n\}$, as also the summability $|C, \delta|, \delta > 0$, of the series $\sum_2^\infty \bar{S}_n(x)/n \log n$, and hence, by Lemma 3, the series $\sum_2^\infty \bar{S}_n(x)/n \log n$ is summable $|C, K|, K > -1$.

5. LEMMA 9 (**7**, lemma). *If the series* $\sum_{n=1}^\infty u_n$ *is summable* $|R, \log \omega, 1|$, *then the necessary and sufficient condition that it should be absolutely convergent is that the sequence* $\{n \log n \cdot u_n\}$ *is summable* $|R, \log \omega, 1|$.

The conclusion of the main theorem ensures the absolute convergence of the series $\sum_2^\infty \bar{S}_n(x)/n \log n$, and hence by Lemma 9 the sequence $\{n \log n \cdot \bar{S}_n(x)/n \log n\}$, i.e. the sequence $\{\bar{S}_n(x)\}$, is summable $|R, \log \omega, 1|$. Thus, we obtain the following result.

COROLLARY. *If*

(i) $h(t) \in \mathrm{BV}(0, \pi)$,

(ii) $\int_0^\pi (|h(t)|/t)\, dt < \infty$, *and*

(iii) $\int_0^\pi (|\psi_1(t)|/t)\, dt < \infty$,

then $\sum_1^\infty B_n(x)$ *is summable* $|R, \log \omega, 1|$.

Remark. Conditions (ii) and (iii) of the above corollary taken together cannot ensure the summability $|R, \log \omega, 1|$ of the series (1.2), which is a non-local property of the generating function. Our result can be compared with the following result of Bosanquet and Hyslop (**1**): If

(i) $\psi(t) \in \mathrm{BV}(0, \pi)$ *and*

(ii) $\int_0^\pi (|\psi(t)|/t)\, dt < \infty$,

then $\sum_1^\infty B_n(x)$ is summable $|C, \delta|, \delta > 0$.

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