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DIRECT SUMS OF INFINITELY MANY KERNELS

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Abstract

Let $\mathcal{K}$ be the class of all right R-modules that are kernels of nonzero homomorphisms $\varphi : E_1 \to E_2$ for some pair of indecomposable injective right R-modules E_1, E_2. In a previous paper, we completely characterized when two direct sums $A_1 \oplus \cdots \oplus A_n$ and $B_1 \oplus \cdots \oplus B_m$ of finitely many modules A_i and B_j in $\mathcal{K}$ are isomorphic. Here we consider the case in which there are arbitrarily, possibly infinitely, many A_i and B_j in $\mathcal{K}$. In both the finite and the infinite case, the behaviour is very similar to that which occurs if we substitute the class $\mathcal{K}$ with the class $\mathcal{U}$ of all uniserial right R-modules (a module is uniserial when its lattice of submodules is linearly ordered).

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1. Introduction

This paper is a first step in the study of infinite direct sums of modules each of which is the kernel of a morphism between two indecomposable injective modules. In our previous paper [8], we developed a complete theory in the case of finite direct sums of such kernels, getting a weak form of the Krull–Schmidt theorem similar to the weak Krull–Schmidt theorem for finite direct sums of uniserial modules proved in [5]. In the case of infinite direct sums of uniserial modules, the three major steps in the characterization of when two direct sums of uniserial modules are isomorphic were the three papers by Dung and Facchini [4], Puninski [15] and Příhoda [14]. In this paper, we prove results analogous to the results obtained by Dung and Facchini, introducing the notion of upper quasismall module. Here is our main result.

THEOREM 1.1. *Let $\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$ be two families of right modules over an arbitrary ring R. Assume that all the A_i and the B_j are kernels of noninjective morphisms between indecomposable injective modules and that $\bigoplus_{i\in I} A_i \cong \bigoplus_{j\in J} B_j$.*

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Denote by I' the set of all $i \in I$ such that A_i is upper quasismall, and by J' the set of all $j \in J$ such that B_j is upper quasismall. Then there exist a bijection $\sigma : I \to J$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ for every $i \in I$ and a bijection $\tau : I' \to J'$ such that $[A_i]_u = [B_{\tau(j)}]_u$ for every $i \in I'$.

The remaining two steps, still open, are the construction of a homomorphism between two indecomposable injective modules whose kernel is not upper quasismall (a corresponding example of a nonquasismall uniserial module was discovered by Puninski in [15]) and the proof that our Theorem 1.1 can be inverted, which was done for uniserial modules by Příhoda in [14, Theorem 2.6].

In this paper, all rings are associative rings with identity, all modules are unital, and $E(A_R)$ denotes the injective envelope of a module A_R.

2. Notation and first results

The notation that will be used throughout this paper is the same notation as in our previous paper [8]. Let E_1, E_2, E'_1, and E'_2 be indecomposable injective right modules over an arbitrary ring R, and let $\varphi : E_1 \to E_2$ and $\varphi' : E'_1 \to E'_2$ be noninjective morphisms. Any morphism $f : \ker\varphi \to \ker\varphi'$ extends to a morphism $f_1 : E_1 \to E'_1$, because E_1 and E'_1 are injective modules containing $\ker\varphi$ and $\ker\varphi'$ respectively. Hence f_1 induces a morphism $\widetilde{f}_1 : E_1/\ker\varphi \to E'_1/\ker\varphi'$, which extends to a morphism $f_2 : E_2 \to E'_2$. Thus we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & E_1 & \xrightarrow{\varphi} & E_2 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ 0 & \longrightarrow & \ker\varphi' & \longrightarrow & E'_1 & \xrightarrow{\varphi'} & E'_2 \end{array} \tag{2.1}$$

The homomorphisms f_1 and f_2 are not uniquely determined by f. Nevertheless, assume that we have another commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \ker\varphi & \longrightarrow & E_1 & \xrightarrow{\varphi} & E_2 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'_1} & & \downarrow{\scriptstyle f'_2} \\ 0 & \longrightarrow & \ker\varphi' & \longrightarrow & E'_1 & \xrightarrow{\varphi'} & E'_2 \end{array}$$

for the same $f : \ker\varphi \to \ker\varphi'$. It is proved in [8] that both $f_1 - f'_1$ and $f_2 - f'_2$ have nonzero kernels when $\varphi \neq 0$.

Let A and B be two modules. Following the terminology introduced in [5, 8]:

- we say that *A and B have the same monogeny class*, and write $[A]_m = [B]_m$, if there exist a monomorphism $A \to B$ and a monomorphism $B \to A$;
- we say that *A and B have the same epigeny class*, and write $[A]_e = [B]_e$, if there exist an epimorphism $A \to B$ and an epimorphism $B \to A$;

- we say that *A and B have the same upper part*, and write $[A]_u = [B]_u$, if there exist a homomorphism $\varphi : E(A) \to E(B)$ and a homomorphism $\psi : E(B) \to E(A)$ such that $\varphi^{-1}(B) = A$ and $\psi^{-1}(A) = B$.

The motivation for the terminology 'having the same upper part' lies in the fact that if $[A]_u = [B]_u$, then $[E(A)/A]_m = [E(B)/B]_m$, so that $E(E(A)/A) \cong E(E(B)/B)$ by Bumby's theorem [3]. By [8, Proposition 4.1], $\ker f$ and $\ker g$ have the same monogeny class if and only if the cyclically presented modules corresponding to $\ker f$ and $\ker g$ via an exact contravariant functor have the same epigeny class. Similarly, $\ker f$ and $\ker g$ have the same upper part if and only if the modules corresponding to $\ker f$ and $\ker g$ via the same contravariant functor have the same lower part in the sense of [1]. This fact was generalized in [9, Section 5].

It is clear that a module A has the same monogeny (epigeny) class as the zero module if and only if $A = 0$. We leave to the reader the easy verification of the fact that a module A has the same upper part as the zero module if and only if A is an injective module.

LEMMA 2.1 [8, Lemma 2.4]. *Let E_1, E_2, E'_1 and E'_2 be indecomposable injective right modules over an arbitrary ring R and let $\varphi : E_1 \to E_2$ and $\varphi' : E'_1 \to E'_2$ be arbitrary morphisms. Then $\ker \varphi \cong \ker \varphi'$ if and only if $[\ker \varphi]_m = [\ker \varphi']_m$ and $[\ker \varphi]_u = [\ker \varphi']_u$.*

LEMMA 2.2 [8, Lemma 2.6]. *Let $\varphi : E_1 \to E_2$, $\varphi' : E'_1 \to E'_2$ and $\varphi'' : E''_1 \to E''_2$ be noninjective morphisms between indecomposable injective modules. Suppose that $[\ker \varphi]_m = [\ker \varphi']_m$ and $[\ker \varphi]_u = [\ker \varphi'']_u$. Then:*

(a) $\ker \varphi \oplus D \cong \ker \varphi' \oplus \ker \varphi''$ *for some R-module D;*
(b) *the module D in (a) is unique up to isomorphism and is the kernel of a noninjective morphism between indecomposable injective modules;*
(c) $[D]_m = [\ker \varphi'']_m$ *and* $[D]_u = [\ker \varphi']_u$.

Kernels of noninjective morphisms between indecomposable injective modules have semilocal endomorphism rings [8, Theorem 2.1]. Hence they cancel from direct sums [6, Corollary 4.6].

LEMMA 2.3 [6, Lemma 6.26(a)]. *Let R be an arbitrary ring, let A, B and C be nonzero R-modules, B being uniform, and let $\alpha : A \to B$ and $\beta : B \to C$ be homomorphisms. Then the composite mapping $\beta\alpha$ is a monomorphism if and only if β and α are both monomorphisms.*

Let A and B be right modules over a ring R and $\alpha : A \to B$ be a homomorphism. We say that α is an *upper homomorphism* if $\alpha_1^{-1}(B) = A$ for any extension $\alpha_1 : E(A) \to E(B)$ to the injective envelopes. The next lemma shows that it is sufficient to check this condition on one arbitrary extension.

LEMMA 2.4. *Let A and B be modules over a ring R, let $\alpha : A \to B$ be a homomorphism, and let $\alpha_1 : E(A) \to E(B)$ be an extension of α to the injective envelopes. Assume that $\alpha_1^{-1}(B) = A$. Then α is an upper homomorphism.*

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PROOF. Let $\alpha_1' : E(A) \to E(B)$ be another extension of α to the injective envelopes, so that α_1' induces a homomorphism $\widetilde{\alpha_1'} : E(A)/A \to E(B)/B$. Then $\alpha_1 - \alpha_1' : E(A) \to E(B)$ is zero on the submodule A of $E(A)$, hence it induces a morphism $\beta : E(A)/A \to E(B)$. As B is essential in $E(B)$, it follows that $\beta^{-1}(B)$ is essential in $E(A)/A$. As $\alpha_1^{-1}(B) = A$, we see that α_1 induces a monomorphism $\widetilde{\alpha_1} : E(A)/A \to E(B)/B$. Now $\widetilde{\alpha_1}$ is a monomorphism and $\beta^{-1}(B) = \ker(\widetilde{\alpha_1} - \widetilde{\alpha_1'})$ is essential in $E(A)/A$, whence $\widetilde{\alpha_1'}$ is a monomorphism. Thus $(\alpha_1')^{-1}(B) = A$. □

EXAMPLE 2.5. Let A and B be modules over a ring R, A being injective. Then every homomorphism $\alpha : A \to B$ is an upper homomorphism.

To see this, notice that if $\iota : B \to E(B)$ is an injective envelope of B, then $\iota\alpha : A \to E(B)$ is an extension of α to the injective envelopes, and $(\iota\alpha)^{-1}(B) = A$.

REMARK 2.6. We leave to the reader the proof that if φ and φ' are nonzero noninjective morphisms, then a morphism $f : \ker\varphi \to \ker\varphi'$ is an upper homomorphism if and only if $f_2 : E_2 \to E_2'$ is a monomorphism. More precisely, let $P : \text{Mod-}R \to \text{Spec Mod-}R$ be the canonical functor of Mod-R into the spectral category Spec Mod-R [12], where Spec Mod-R is obtained from Mod-R by formally inverting all essential monomorphisms. Let $\mathcal{K}$ be the full subcategory of Mod-R whose objects are all finite direct sums of modules that are kernels of morphisms between indecomposable injective modules. Let $\mathcal{A}$ be the full subcategory of Spec Mod-R whose objects are all semisimple objects of finite length. Then P restricts to a functor from $\mathcal{K}$ to $\mathcal{A}$. For any indecomposable module K in $\text{Ob}(\mathcal{K})$, $P(K)$ is a simple object of $\mathcal{A}$. If A is an object of $\mathcal{K}$, then $P(A) \cong P(E(A))$, and is a direct sum of m simple objects, where m is the Goldie dimension of A.

Since $P : \text{Mod-}R \to \text{Spec Mod-}R$ is a left exact covariant functor, it has a first right derived functor $P^{(1)} : \text{Mod-}R \to \text{Spec Mod-}R$; see [7, Proposition 2.2]. The functor $P^{(1)}$ restricts to a functor from $\mathcal{K}$ to $\mathcal{A}$. For every module $A \in \text{Ob}(\mathcal{K})$ with minimal injective resolution

$$0 \to A \to E_0 \to E_1 \to E_2 \to \cdots,$$

$P^{(1)}(A) \cong P(E_1)$, so that $P^{(1)}(A) = 0$ if and only if A is injective, and $P^{(1)}(A)$ is a simple object of $\mathcal{A}$ if A is the kernel of a nonzero noninjective morphism between indecomposable injective modules. Moreover, a morphism f in $\mathcal{K}$ is an upper homomorphism if and only if $P^1(f)$ is a monomorphism.

LEMMA 2.7. *Let R be an arbitrary ring, let A, B and C be nonzero R-modules such that $E(B)/B$ is uniform and let $\alpha : A \to B$ and $\beta : B \to C$ be homomorphisms. Then the composite mapping $\beta\alpha$ is an upper homomorphism if and only if β and α are both upper homomorphisms.*

PROOF. Let $\alpha_1 : E(A) \to E(B)$ and $\beta_1 : E(B) \to E(C)$ be extensions of α and β respectively, so that $\beta_1\alpha_1 : E(A) \to E(C)$ is an extension of $\beta\alpha$. Then α_1 and β_1

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induce mappings $\widetilde{\alpha_1} : E(A)/A \to E(B)/B$ and $\widetilde{\beta_1} : E(B)/B \to E(C)/C$, whose composite mapping is the homomorphism induced by $\beta_1\alpha_1$. Now apply Lemma 2.3. □

Notice that in Lemma 2.7, the 'if' implication is true even without the hypothesis that $E(B)/B$ is uniform.

LEMMA 2.8. *Let A and B be modules over a ring R. Assume that A is the kernel of a homomorphism between indecomposable injective modules.*

(a) *If $f, g : A \to B$ are two homomorphisms such that f is injective but not an upper homomorphism and g is an upper homomorphism but not a monomorphism, then $f + g$ is an isomorphism.*
(b) *If $f_1, \ldots, f_n : A \to B$ are homomorphisms and $f_1 + \cdots + f_n$ is an isomorphism, then either one of the f_i is an isomorphism or there exist two distinct indices $i, j \in \{1, 2, \ldots, n\}$ such that f_i is injective but not an upper homomorphism, and f_j is an upper homomorphism that is not injective.*

PROOF. We prove (a). Since $\ker(f) \supseteq \ker(g) \cap \ker(f+g)$, if f is injective and g is not injective, then $f + g$ must be injective. Now apply what we have just seen to the morphisms $\widetilde{f_1}, \widetilde{g_1} : E(A)/A \to E(B)/B$ induced by two extensions $f_1, g_1 : E(A) \to E(B)$.

Now we prove (b). Since $\ker(f_1 + \cdots + f_n) \supseteq \bigcap_i \ker(f_i)$, if $f_1 + \cdots + f_n$ is an isomorphism, then there exists i such that $\ker(f_i) = 0$. Now apply what we have just seen also to the morphisms $\widetilde{f_i} : E(A)/A \to E(B)/B$ induced by the n extensions $f_i : E(A) \to E(B)$, obtaining an index j such that f_j is an upper homomorphism. If f_i or f_j is an isomorphism, we are done. Otherwise $i \neq j$, and the conclusion follows easily. □

PROPOSITION 2.9. *Let $A, B, C_1, \ldots, C_n$ (where $n \geq 2$) be modules such that $A \oplus B = C_1 \oplus \cdots \oplus C_n$. Suppose that A is the kernel of a noninjective homomorphism between two indecomposable injective modules. Then there are two distinct indices $i, j \in \{1, \ldots, n\}$ and a direct sum decomposition $A' \oplus B' = C_i \oplus C_j$ of $C_i \oplus C_j$ such that $A \cong A'$ and $B \cong B' \oplus (\bigoplus_{k \neq i,j} C_k)$. In particular, if $C_1, \ldots, C_n$ are also kernels of noninjective homomorphisms between indecomposable injective modules, there are two indices i and j, possibly equal, such that $[A]_m = [C_i]_m$ and $[A]_u = [C_j]_u$.*

PROOF. The proof of the first part of the statement is like that of [6, Proposition 9.5], with epimorphism replaced by upper homomorphism. The second part of the statement follows from [8, Lemma 2.6]. □

THEOREM 2.10. *Let $\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$ be two families of modules over a ring R. Assume that all the B_j are kernels of noninjective morphisms between indecomposable injective modules. Suppose that there exist two bijections $\sigma, \tau : I \to J$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ and $[A_i]_u = [B_{\tau(i)}]_u$ for every $i \in I$. Then all the modules A_i are kernels of noninjective morphisms between indecomposable injective*

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modules and

$$\bigoplus_{i\in I} A_i \cong \bigoplus_{j\in J} B_j.$$

PROOF. Since $[A_i]_m = [B_{\sigma(i)}]_m$ and $[A_i]_u = [B_{\tau(i)}]_u$, we see that $E(A_i) \cong E(B_{\sigma(i)})$ and $E(E(A_i)/A_i) \cong E(E(B_{\tau(i)})/B_{\tau(i)})$. The module A_i is the kernel of the canonical homomorphism from $E(A_i)$ to $E(E(A_i)/A_i)$. Now $E(A_i) \cong E(B_{\sigma(i)})$, and is indecomposable, while $E(E(A_i)/A_i) \cong E(E(B_{\tau(i)})/B_{\tau(i)})$, and is either indecomposable or zero. In the second case, A_i must be injective, hence indecomposable injective, and is the kernel of the zero morphism $A_i \to A_i$. Hence every A_i is the kernel of a noninjective morphism between indecomposable injective modules.

The proof of the last part of the theorem is like that of [4, Theorem 3.1] or [6, Theorem 9.11], with epimorphism replaced by upper homomorphism. □

PROPOSITION 2.11. *Let R be an arbitrary ring, let $\{C_j \mid j \in J\}$ be a family of kernels of noninjective morphisms between indecomposable injective R-modules and let $A_1, A_2, \dots, A_n$ be uniform R-modules. If*

$$A_1 \oplus \cdots \oplus A_n$$

is a direct summand of $\bigoplus_{j\in J} C_j$, then there exist n distinct elements $k_1, \dots, k_n \in J$ such that $[A_i]_m = [C_{k_i}]_m$ when $i = 1, 2, \dots, n$.

PROOF. The proof is like that of [6, Proposition 9.9], with biuniform module replaced by kernel of a noninjective morphism between indecomposable injective modules. □

THEOREM 2.12. *Let $\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$ be two families of right modules over an arbitrary ring R that are kernels of noninjective morphisms between indecomposable injective modules. Assume that*

$$\bigoplus_{i\in I} A_i \cong \bigoplus_{j\in J} B_j.$$

Then there exists a bijection $\sigma : I \to J$ such that $[A_i]_m = [B_{\sigma(i)}]_m$ for every $i \in I$.

PROOF. The proof is like that of [6, Theorem 9.12], with biuniform module replaced by kernel of a noninjective morphism between indecomposable injective modules. □

3. Upper quasismall modules

Recall that a family $\{f_i \mid i \in I\}$ of homomorphisms from a module A into a module B is said to be a *summable family* if, for every $x \in A$, there exists a finite subset $I(x)$ of I such that $f_i(x) = 0$ for every $i \in I \setminus I(x)$. If $\{f_i \mid i \in I\}$ is a summable family of homomorphisms of A into B, it is possible to define its sum $\sum_{i\in I} f_i : A \to B$, which is clearly a morphism of A into B. Modifying one of the characterizations of quasismall uniserial modules [4, Lemma 4.4], we give the following definition.

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Let A be a right R-module. We say that A is *upper quasismall* if for every summable family $\{f_i \mid i \in I\}$ of endomorphisms of A such that $\sum_{i\in I} f_i = 1_A$, at least one of the f_i is an upper homomorphism.

From Example 2.5, we deduce the following result.

LEMMA 3.1. *Every injective module is upper quasismall.*

LEMMA 3.2. *Every module with a local endomorphism ring is upper quasismall.*

PROOF. Let A be a module with a local endomorphism ring and let $\{f_i \mid i \in I\}$ be a family of endomorphisms of A whose sum is 1_A, that is, $\sum_{i\in I} f_i = 1_A$. Then the composite mapping of the homomorphisms $F : A \to A^{(I)}$, $F = (f_i)_{i\in I}$, and $\Sigma : A^{(I)} \to A$, $\Sigma : (x_i)_{i\in I} \mapsto \sum_{i\in I} x_i$, is the identity of A. Hence $F(A)$ is a direct summand of $A^{(I)}$. The module $F(A) \cong A$ has a local endomorphism ring, hence $F(A)$ has the exchange property [6, Theorem 2.8]. For every $i \in I$, let A_i be the ith copy of A in $A^{(I)}$, so that every A_i is canonically isomorphic to A and $A^{(I)}$ is the internal direct sum of the A_i. As $F(A)$ is a direct summand of $A^{(I)}$, for every $i \in I$ there are submodules B_i and C_i of A_i such that

$$A_i = B_i \oplus C_i \quad \text{and} \quad A^{(I)} = F(A) \oplus \left(\bigoplus_{i\in I} B_i\right).$$

It follows that $F(A) \cong \bigoplus_{i\in I} C_i$, and since $F(A) \cong A$ is indecomposable, there exists an index $j \in I$ such that $C_j = A_j$ and $C_i = 0$ for every $i \in I \setminus \{j\}$. Thus

$$A^{(I)} = F(A) \oplus \left(\bigoplus_{i\in I\setminus\{j\}} A_i\right).$$

By [6, Lemma 2.6], the restriction of the canonical projection $\pi_j : A^{(I)} \to A_j$ to $F(A)$ is an isomorphism. Equivalently, the composite mapping $\pi_j F : A \to A_j$ is an isomorphism. But $\pi_j F = f_j$. Thus f_j is an isomorphism, hence an upper homomorphism. □

The following characterization of upper quasismall modules is similar to the exchange property.

PROPOSITION 3.3. *The following conditions are equivalent for an R-module A.*

(a) *A is upper quasismall.*

(b) *For any R-module G and any two direct sum decompositions*

$$G = A' \oplus B = \bigoplus_{i\in I} A_i$$

such that $A' \cong A$, if $\varepsilon_k : A_k \to G$ and $e' : A' \to G$ denote the embeddings and $p_k : \bigoplus_{i\in I} A_i \to A_k$ and $\pi' : G \to A'$ denote the canonical projections, then there exists an index $k \in I$ such that $\pi' \varepsilon_k p_k e'$ is an upper homomorphism.

PROOF. Suppose that (a) holds. Let A be an upper quasismall module and let G be an R-module with two direct sum decompositions $G = A' \oplus B = \bigoplus_{i\in I} A_i$ such

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that $A' \cong A$. Let $\varphi : A \to A'$ be an isomorphism. Then $\{\varphi^{-1}\pi'\varepsilon_i p_i e'\varphi \mid i \in I\}$ is a summable family of endomorphisms of A whose sum is 1_A.

Now suppose that (b) holds and let $\{f_i \mid i \in I\}$ be a family of endomorphisms of A such that $\sum_{i\in I} f_i = 1_A$. Arguing as in the proof of Lemma 3.2, one finds two direct sum decompositions $A^{(I)} = F(A) \oplus B = \bigoplus_{i\in I} A_i$ such that $F = (f_i)_{i\in I}$, $F(A) \cong A$ and $A_i \cong A$ for every $i \in I$. By (b), there exists an index $k \in I$ such that $\pi'\varepsilon_k p_k e'$ is an upper homomorphism. Then $\pi'\varepsilon_k p_k F = \pi'\varepsilon_k f_k$ is an upper homomorphism. It follows that f_k is an upper homomorphism by Lemma 2.7 if A is not injective, and is an upper homomorphism as we have seen in Example 2.5 if A is injective. □

PROPOSITION 3.4. *Let A be the kernel of a homomorphism between injective modules. Then A is upper quasismall if and only if:*

(a) *either A has a local endomorphism ring;*
(b) *or, whenever $\{f_i \mid i \in I\}$ is a summable family of endomorphisms of A and $\sum_{i\in I} f_i$ is an upper homomorphism, at least one of the f_i is an upper homomorphism.*

PROOF. If $A = 0$, then A is upper quasismall and (b) holds trivially. Therefore the proposition holds in this case. Assume that $A \neq 0$.

Suppose that A is upper quasismall, with a nonlocal endomorphism ring, and let $\{f_i \mid i \in I\}$ be a summable family of endomorphisms of A such that $\sum_{i\in I} f_i$ is an upper homomorphism.

Assume that $\sum_{i\in I} f_i$ is an isomorphism. Let $h : A \to A$ be its inverse. It is easy to see that $\{hf_i \mid i \in I\}$ is a summable family of endomorphisms of A and that its sum $\sum_{i\in I} hf_i$ is 1_A. Since A is upper quasismall, there exists an index i such that hf_i is an upper homomorphism. Then f_i an upper homomorphism, as required in (b).

Thus we can assume that $\sum_{i\in I} f_i$ is not an isomorphism. Since it is an upper homomorphism, it must be noninjective. By [8, Theorem 2.1], the endomorphism ring of A has two maximal ideals and there exists an endomorphism g of A that is injective but not an upper homomorphism. Thus the family consisting of g and the f_i is a summable family of endomorphisms of A, and its sum $g + \sum_{i\in I} f_i$ is an automorphism of A. Let h be its inverse. Arguing as in the previous paragraph, one sees that the family consisting of hg and the hf_i is a summable family of endomorphisms of A and $hg + \sum_{i\in I} hf_i = 1_A$, and one concludes immediately.

Now we assume that (a) or (b) holds, and show that A is quasismall. When (a) holds, the implication was proved in Lemma 3.2. When (b) holds, the conclusion follows immediately from the trivial fact that the identity is an upper homomorphism. □

4. Cokernels of morphisms between projective modules

Propositions 4.1, 4.6 and 4.7 below are related to [10, Lemma 5.2]. To prove Proposition 4.7, we are forced to make a digression through a study of the cokernels A of morphisms between couniform projective modules, that is, the modules A for which

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there exists an exact sequence $Q \to P \to A \to 0$, where P and Q are couniform projective modules (projective modules with local endomorphism rings); see [9]. Usually, we will avoid considering the trivial case where $A = 0$, that is, the case in which the homomorphism $Q \to P$ is surjective. If A and B are the cokernels of two nonsurjective homomorphisms between couniform projective modules and $\varphi : A \to B$ is a homomorphism, it is always possible to construct a commutative diagram of the form.

$$\begin{array}{ccccccc} Q & \longrightarrow & P & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi} & & \\ Q' & \longrightarrow & P' & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

This construction is the dual of construction 2.1 that we saw in Section 2. We say that φ is a *lower homomorphism* if φ_2 is surjective. For further details, see [9].

PROPOSITION 4.1. *Let A and B be cokernels of nonsurjective homomorphisms between couniform projective modules such that $[A]_e = [B]_e$. Then A has a local endomorphism ring if and only if B has a local endomorphism ring.*

PROOF. It suffices to show that if the endomorphism ring of A is not local, then the endomorphism ring of B is not local. Hence suppose that $\mathrm{End}_R(A)$ is nonlocal.

Let $f : A \to B$ and $g : B \to A$ be two epimorphisms. If B is projective, then f splits, so that $A \cong B$, and therefore $\mathrm{End}_R(A) \cong \mathrm{End}_R(B)$. Thus we can suppose that B is nonprojective. As $\mathrm{End}_R(A)$ contains an epimorphism ψ that is not a lower homomorphism, it follows that $f\psi g \in \mathrm{End}_R(B)$ is an epimorphism that is not a lower homomorphism. We must check that $\mathrm{End}_R(B)$ also contains a lower homomorphism that is not an epimorphism. Without loss of generality, we can suppose that $B = A/X$ for some submodule X of A and $f : A \to B = A/X$ is the canonical projection. Let $\varphi : A \to A$ be a lower homomorphism that is not an epimorphism, and let $\widetilde{\varphi} : A/X \to A/\varphi(X)$ be the homomorphism induced by φ.

Step A. We prove that $A/\varphi(X)$ is the cokernel of a nonsurjective homomorphism between couniform projective modules.

Let $Q \xrightarrow{\alpha} P \xrightarrow{\pi} A \to 0$ be an exact sequence such that α is a nonsurjective homomorphism and P, Q couniform projective modules. We can suppose that A is the factor module $P/\alpha(Q)$ and that π is the canonical projection. The submodule X of $A = P/\alpha(Q)$ is therefore of the form $X = X'/\alpha(Q)$ for some submodule $X' \supseteq \alpha(Q)$ of P. The module $B = A/X \cong P/X'$ is the cokernel of a nonsurjective homomorphism α' between couniform projective modules P' and Q', that is, we have an exact sequence

$$Q' \xrightarrow{\alpha'} P' \xrightarrow{\pi'} B = A/X \to 0.$$

We also have the short exact sequence

$$0 \to X' \hookrightarrow P \xrightarrow{f\pi} B = A/X \to 0. \tag{4.1}$$

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By Schanuel's lemma, $P' \oplus X' \cong P \oplus \alpha'(Q')$. Couniform projective modules are the projective covers of all their nonzero factors [1, Lemma 8.7]. Therefore both P and P' are projective covers of $A = A/X$. Hence $P' \cong P$ and, since modules with local endomorphism rings cancel from direct sums, $X' \cong \alpha'(Q')$. Thus from the exact sequence (4.1), we can construct an exact sequence

$$Q' \xrightarrow{\beta} P \xrightarrow{f\pi} B = A/X \to 0$$

such that $\beta(Q') = \ker(f\pi) = X'$.

Let

$$\begin{array}{ccccccc}
Q & \xrightarrow{\alpha} & P & \xrightarrow{\pi} & A & \longrightarrow & 0 \\
\downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi} & & \\
Q & \xrightarrow{\alpha} & P & \xrightarrow{\pi} & A & \longrightarrow & 0
\end{array}$$

be the commutative diagram relative to φ, where φ_2 is onto because φ is a lower homomorphism, so that $\varphi_1(\alpha(Q)) = \alpha(Q)$. Then

$$\begin{aligned}
\varphi(X) &= \varphi(X'/\alpha(Q)) \\
&= (\varphi_1(X') + \alpha(Q))/\alpha(Q) \\
&= (\varphi_1(X') + \varphi_1(\alpha(Q)))/\alpha(Q) \\
&= \varphi_1(X' + \alpha(Q))/\alpha(Q) \\
&= \varphi_1(X')/\alpha(Q).
\end{aligned}$$

Thus $A/\varphi(X) \cong P/\varphi_1(X') = P/\varphi_1\beta(Q')$ has a projective resolution

$$Q' \xrightarrow{\varphi_1\beta} P \xrightarrow{\overline{\pi}\pi''\pi} A/\varphi(X) \to 0,$$

where we denote by $\pi'' : P/\alpha(Q) \to P/\varphi_1(X')$ the canonical projection and by $\overline{\pi} : P/\varphi_1(X') \to A/\varphi(X)$ the isomorphism induced by π. This concludes Step A.

Step B. The homomorphism $\widetilde{\varphi} : A/X \to A/\varphi(X)$ induced by $\varphi : A \to A$ is a lower homomorphism.

The diagram relative to $\widetilde{\varphi}$ is

$$\begin{array}{ccccccc}
Q' & \xrightarrow{\beta} & P & \xrightarrow{f\pi} & A/X & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \tilde{\varphi}} & & \\
Q' & \xrightarrow{\varphi_1\beta} & P & \xrightarrow{\overline{\pi}\pi''\pi} & A/\varphi(X) & \longrightarrow & 0
\end{array}$$

Hence $\widetilde{\varphi}$ is a lower homomorphism, which concludes Step B.

Now $\widetilde{\varphi} : A/X = B \to A/\varphi(X)$ is a lower homomorphism between the two cokernels B and $A/\varphi(X)$, and the composite mapping of the epimorphism $g : B \to A$ and the canonical projection $A \to A/\varphi(X)$ is an epimorphism $\nu : B \to A/\varphi(X)$. Therefore either $\widetilde{\varphi}$ or ν or $\widetilde{\varphi} + \nu$ is an isomorphism of B onto $A/\varphi(X)$ by the

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analogue of Lemma 2.8(a). In all these three cases, there is an isomorphism $\gamma : B \to A/\varphi(X)$. Now $\varphi : A \to A$ is not an epimorphism, and therefore the lower homomorphism $\widetilde{\varphi} : B \to A/\varphi(X)$ induced by φ is not an epimorphism. Thus the composite mapping $\gamma^{-1}\widetilde{\varphi}$ is an endomorphism of B that is a lower homomorphism but not an epimorphism. □

Consider the cokernel A of a homomorphism $\alpha : Q \to P$ between two couniform projective right modules P and Q, and assume that A is nonprojective, that is, that the homomorphism $\alpha : Q \to P$ is nonzero and nonsurjective. We can suppose that $P = eR$, $Q = fR$ for suitable nonzero idempotents $e, f \in R$ with eRe and fRf local rings [1, Lemma 8.7]. The homomorphism $\alpha : Q = fR \to P = eR$ is given by left multiplication by a unique element $erf \in eRf$. The cokernel A is the factor module $eR/erfR$. Applying the functor $\mathrm{Hom}_R(-, R)$ to α, we get the left R-module morphism

$$\mathrm{Hom}_R(\alpha, R) : \mathrm{Hom}_R(P, R) \cong Re \to \mathrm{Hom}_R(Q, R) \cong Rf,$$

given by right multiplication by the same element $erf \in eRf$. The cokernel of this left R-module morphism $\mathrm{Hom}_R(\alpha, R)$ is $Rf/Rerf$. (Notice that A is nonprojective, so that $eR \supset erfR$ and $erf \neq 0$. But $Rf/Rerf$ could be zero, that is, Rf might be $Rerf$.)

The endomorphism ring of the right R-module $eR/erfR$ is canonically isomorphic to $T/erfRe$, where

$$T := \{x \in eRe \mid xerf \in erfR\}$$

and $erfRe = eRe \cap erfR$. Similarly, the endomorphism ring of the left R-module $Rf/Rerf$ is canonically isomorphic to $T'/fRerf$, where

$$T' := \{y \in fRf \mid erfy \in Rerf\}.$$

Thus, for every element $x \in T$, there exists an element $y \in R$ with $xerf = erfy$, so that $fyf \in T'$. Dually, for every element $y \in T'$, there exists an element $x \in R$ with $erfy = xerf$, so that $exe \in T$. In other words, every endomorphism of $eR/erfR$ and $Rf/Rerf$ yields two commutative diagrams.

$$\begin{array}{ccc} fR & \xrightarrow{erf} & eR \\ {\scriptstyle fyf}\downarrow & & \downarrow{\scriptstyle x} \\ fR & \xrightarrow{erf} & eR \end{array} \quad \text{and} \quad \begin{array}{ccc} Re & \xrightarrow{erf} & Rf \\ {\scriptstyle exe}\downarrow & & \downarrow{\scriptstyle y} \\ Re & \xrightarrow{erf} & Rf \end{array} \tag{4.2}$$

Thus we get correspondences $T \to T'$, $x \mapsto fyf$ and $T' \to T$, $y \mapsto exe$, which are not well-defined mappings in general. (The two squares in (4.2) correspond to the square on the right in the commutative diagram (2.1), in which the mappings f_1 and f_2 are not uniquely determined as well.)

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According to [9, Theorem 2.5], the two completely prime ideals of the endomorphism ring $\mathrm{End}_R(eR/erfR)$ are the kernels of the two morphisms

$$\mathrm{End}_R(eR/erfR) \to eRe/J(eRe)$$
$$x + erfRe \mapsto x + J(eRe)$$

and

$$\mathrm{End}_R(eR/erfR) \to fRf/J(fRf)$$
$$x + erfRe \mapsto fyf + J(fRf).$$

It follows that $\mathrm{End}_R(eR/erfR)$ is local if and only if either, for every $x \in T$, $x \in J(eRe)$ implies that $fyf \in J(fRf)$, or, for every $y \in T'$, $y \in J(fRf)$ implies that $exe \in J(eRe)$. The situation is entirely similar and symmetric for $\mathrm{End}_R(Rf/Rerf)$, so that $\mathrm{End}_R(eR/erfR)$ is local if and only if $\mathrm{End}_R(Rf/Rerf)$ is local. We have thus proved the following lemma.

LEMMA 4.2. *Assume that $eR/erfR$ and $Rf/Rerf$ are nonzero, that is, $erfR \subset eR$ and $Rerf \subset Rf$. Then $\mathrm{End}_R(eR/erfR)$ is local if and only if $\mathrm{End}_R(Rf/Rerf)$ is local.*

(In the case in which either $eR/erfR$ or $Rf/Rerf$ is nonzero but projective, then $erf = 0$, so that the two modules are eR and Rf, and their endomorphism rings are both local.)

REMARK 4.3. We are considering the Auslander–Bridger transpose of the cyclically presented module $eR/erfR$. The Auslander–Bridger transpose gives a duality of the stable category $\underline{\mathrm{mod}}$-R of the category of finitely presented right R-modules into the stable category R-$\underline{\mathrm{mod}}$ [13]. It sends the cokernel of a right R-module morphism $Q \to P$ to the cokernel of the left R-module morphism $\mathrm{Hom}_R(P, R) \to \mathrm{Hom}_R(Q, R)$. In this paper, we decided to introduce our setting in an elementary way.

REMARK 4.4. The two ring morphisms of the ring $\mathrm{End}_R(eR/erfR)$ into the division rings $eRe/J(eRe)$ and $fRf/J(fRf)$ are the two morphisms induced by the two functors F and $F_{(1)}$ of [2, Example 6.5]. If the module $eR/erfR$ is projective, then $F_{(1)}(eR/erfR) = 0$, so that the corresponding ring morphism

$$\mathrm{End}_R(eR/erfR) \to \mathrm{End}_R(F_{(1)}(eR/erfR))$$

is the morphism into the zero ring.

Similarly, the following result is [1, Proposition 7.1].

LEMMA 4.5. *Assume that $eR/erfR$ and $Rf/Rerf$ are nonzero. Then the following pairs of equalities are equivalent:*

(a) $[eR/erfR]_e = [e'R/e'r'f'R]_e$ *and* $[Rf/Rerf]_l = [Rf'/Re'r'f']_l$;
(b) $[eR/erfR]_l = [e'R/e'r'f'R]_l$ *and* $[Rf/Rerf]_e = [Rf'/Re'r'f']_e$.

PROPOSITION 4.6. *Let A and B be cokernels of nonsurjective nonzero homomorphisms between couniform projective modules such that $[A]_l = [B]_l$. Then A has a local endomorphism ring if and only if B has a local endomorphism ring.*

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PROOF. Suppose that $[A]_l = [B]_l$. By Lemma 4.5(b), the Auslander–Bridger transposes of A and B are left modules with the same epigeny class. Now apply Proposition 4.1 to the Auslander–Bridger transposes. Notice that Proposition 4.1 was proved for right modules, but, passing to the opposite ring, one gets that it trivially holds for left modules as well. □

PROPOSITION 4.7. *Let A and B be kernels of noninjective homomorphisms between indecomposable injective modules. Assume that either $[A]_m = [B]_m$ or $[A]_u = [B]_u$. Then A has a local endomorphism ring if and only if B has a local endomorphism ring.*

PROOF. Let A and B be kernels of noninjective homomorphisms between injective modules. Assume that either $[A]_m = [B]_m$ or $[A]_u = [B]_u$. Apply the duality in [9, Theorem 5.1] to the two right R-modules A and B, to get two left modules $H(A)$ and $H(B)$ over a suitable ring S. These modules are cokernels of morphisms between couniform projective left S-modules. By [9, Proposition 5.2], the two modules $H(A)$ and $H(B)$ have either the same epigeny class or the same lower part. By Propositions 4.1 and 4.6, $H(A)$ has a local endomorphism ring if and only if $H(B)$ has. Modules corresponding via a duality have anti-isomorphic endomorphism rings. This allows us to conclude. □

5. Factor categories

There is a relation between our upper quasismall modules and the theory developed in [11]. Throughout this section, let $\mathcal{A}$ be the full subcategory of Mod-R whose objects are all right R-modules that are direct sums of (possibly infinitely many) kernels of morphisms between indecomposable injective modules. Let A be an indecomposable module in $\mathrm{Ob}(\mathcal{A})$, that is, the kernel of a noninjective homomorphism between two indecomposable injective modules, and let I be a completely prime ideal in $\mathrm{End}_R(A)$. Let $\mathcal{I}$ be *the ideal of $\mathcal{A}$ associated to I* [11]. It is the ideal in the category $\mathcal{A}$ defined as follows: a morphism $f : X \to Y$ is in $\mathcal{I}(X, Y)$ if and only if $\beta f \alpha \in I$ for every $\alpha : A \to X$ and every $\beta : Y \to A$. The ideal $\mathcal{I}$ is the greatest among the ideals $\mathcal{I}'$ of $\mathcal{A}$ such that $\mathcal{I}'(A, A) \subseteq I$, and, in this case, $\mathcal{I}(A, A) = I$. Let $F : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ denote the canonical functor. Then $F(A)$ is an indecomposable object of $\mathcal{A}/\mathcal{I}$ [11, Lemma 2.1]. In particular, $F(A)$ is a nonzero object of $\mathcal{A}/\mathcal{I}$.

Let I be the completely prime ideal of $\mathrm{End}_R(A)$ consisting of all the endomorphisms of A that are not upper homomorphisms. Then A satisfies condition (b) of Proposition 3.4 if and only if it is an I-small module in the sense of [11]; see [11, Lemma 2.3(b)]. Until the end of the proof of Proposition 5.2, we will apply the results of [11] assuming that A is the kernel of a noninjective homomorphism between two indecomposable injective modules, $\mathrm{End}_R(A)$ is not a local ring, so that the ideal I of all the endomorphisms of A that are not upper homomorphisms is a maximal ideal of $\mathrm{End}_R(A)$, and A satisfies condition (b) of Proposition 3.4. Thus $\mathcal{I}(X, Y)$ is the group of all homomorphisms $f : X \to Y$ for which $\beta f \alpha \in \mathrm{End}_R(A)$

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is not an upper homomorphism for every $\alpha : A \to X$ and every $\beta : Y \to A$. We have already remarked that, under our hypotheses, A is an I-small module in the sense of [11]. The canonical functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ sends direct sums of modules in $\mathcal{A}$ to the corresponding coproducts in $\mathcal{A}/\mathcal{I}$ [11, Corollary 2.7].

PROPOSITION 5.1. *Let B be the kernel of a noninjective homomorphism between indecomposable injective right R-modules. Then the following hold.*

(a) *If $[A]_u = [B]_u$, then B is not injective, $F(B)$ is an indecomposable object of $\mathcal{A}/\mathcal{I}$, $\mathcal{I}(B, B)$ consists of all the endomorphisms of B that are not upper homomorphisms, and $\mathrm{End}_R(B)/\mathcal{I}(B, B)$ is a division ring.*

(b) *If $[A]_u \neq [B]_u$, then $F(B) = 0$ in $\mathcal{A}/\mathcal{I}$ and $\mathcal{I}(B, B) = \mathrm{End}_R(B)$.*

PROOF. To prove (a), assume that $[A]_u = [B]_u$, so that there exist upper homomorphisms $\alpha : A \to B$ and $\beta : B \to A$. If B is injective, then $[B]_u = 0 \neq [A]_u$, which is a contradiction. Hence B is not injective. If $F(B) = 0$, then $1_B \in \mathcal{I}(B, B)$, and $\beta\alpha \in I$. Thus $\alpha_1^{-1}(B) = A$, $\beta_1^{-1}(A) = B$ and $(\beta_1\alpha_1)^{-1}(A) \supset A$, which is another contradiction. This proves that $F(B) \neq 0$ and that $\beta\alpha$ is an upper homomorphism. In particular, the ideal $\mathcal{I}(B, B)$ is a proper ideal of $\mathrm{End}_R(B)$. The endomorphism ring of $F(B)$ is isomorphic to $\mathrm{End}_R(B)/\mathcal{I}(B, B)$. An endomorphism f of B is in $\mathcal{I}(B, B)$ if and only if $\beta' f \alpha' \in \mathrm{End}_R(A)$ is not an upper homomorphism for any choice of $\alpha' : A \to B$ and $\beta' : B \to A$. By Lemma 2.7, $\mathcal{I}(B, B)$ turns out to be the set of all endomorphisms of B that are not upper homomorphisms. Hence $\mathcal{I}(B, B)$ is a completely prime ideal of $\mathrm{End}_R(B)$. If $F(B) = X \oplus Y$, where X and Y are nonzero, then there are nonzero orthogonal idempotents in $\mathrm{End}(F(B))$, which is not possible because $\mathcal{I}(B, B)$ is completely prime. In order to show that $\mathrm{End}_R(B)/\mathcal{I}(B, B)$ is a division ring, we will show that, for any upper homomorphism $f : B \to B$, the element $f + \mathcal{I}(B, B)$ has a right inverse in $\mathrm{End}_R(B)/\mathcal{I}(B, B)$. Since $\beta f \alpha \notin I$, there exists $g : A \to A$ such that $1_A - \beta f \alpha g$ is not an upper homomorphism. Then $\alpha(1_A - \beta f \alpha g)\beta = \alpha\beta(1_B - f\alpha g\beta)$ is also not an upper homomorphism, so $1_B - f\alpha g\beta$ is not an upper homomorphism. In other words, $\alpha g\beta + \mathcal{I}(B, B)$ is a right inverse for $f + \mathcal{I}(B, B)$ in $\mathrm{End}_R(B)/\mathcal{I}(B, B)$.

Now we prove (b). If $[A]_u \neq [B]_u$, then either there are no upper homomorphisms $A \to B$ or there are no upper homomorphisms $B \to A$. If B is not injective, it follows that $1_B \in \mathcal{I}(B, B)$, whence $F(B) = 0$ and $\mathcal{I}(B, B) = \mathrm{End}_R(B)$. If B is injective, then there is no upper homomorphism $A \to B$, so that every endomorphism of A that factors through B is not an upper homomorphism. It follows that $1_B \in \mathcal{I}(B, B)$, $F(B) = 0$ and $\mathcal{I}(B, B) = \mathrm{End}_R(B)$. □

From [11, Lemma 3.1, Proposition 3.2], we deduce the following result.

PROPOSITION 5.2. *The two categories $\mathcal{A}/\mathcal{I}$ and* Mod-($\mathrm{End}_R(A)/I$) *are equivalent.*

Therefore there is a direct summand preserving functor from $\mathcal{A}$ into the category of vector spaces over the field $\mathrm{End}_R(A)/I$ with the property that, for every object $X = \bigoplus_{i \in I} A_i$ of $\mathcal{A}$, the dimension of the vector space corresponding to X is equal to

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the cardinality of the set $\{i \in I \mid [A_i]_u = [A]_u\}$. Hence this cardinality depends only on X and not on the direct sum representation $X = \bigoplus_{i \in I} A_i$ of X as a direct sum of kernels A_i of nonzero morphisms between indecomposable injective modules.

Now apply the results of [11] assuming that A is the kernel of a noninjective homomorphism between two indecomposable injective modules, $\mathrm{End}_R(A)$ is a local ring and I is its maximal ideal. Now the group $\mathcal{I}(X, Y)$ consists of all homomorphisms $f : X \to Y$ for which $\beta f \alpha \in \mathrm{End}_R(A)$ is not an automorphism for any choice of $\alpha : A \to X$ and $\beta : Y \to A$. The module A is an I-small module in the sense of [11, Section 6]. The canonical functor $F : \mathcal{A} \to \mathcal{A}/\mathcal{I}$ sends direct sums of modules in $\mathcal{A}$ to the corresponding coproducts in $\mathcal{A}/\mathcal{I}$ [11, Corollary 2.7]. For every kernel B of morphisms between indecomposable injective modules, $F(B) \cong F(A)$ is an indecomposable object of $\mathcal{A}/\mathcal{I}$ if $A \cong B$, and $F(B) = 0$ if $A \ncong B$ [11, Proposition 6.2]. (To see this, notice that if B is the kernel of a homomorphism between two indecomposable injective modules and $A \ncong B$, then $\beta 1_B \alpha \in \mathrm{End}_R(A)$ is not an automorphism for every $\alpha : A \to B$ and every $\beta : B \to A$.) Moreover, property (*) of [11, Section 3] holds, and the categories $\mathcal{A}/\mathcal{I}$ and Mod-($\mathrm{End}_R(A)/I$) are equivalent [11, Lemma 3.1]. It follows that there is a direct summand preserving functor of $\mathcal{A}$ into the category of vector spaces over the field $\mathrm{End}_R(A)/I$, with the property that, for every object $X = \bigoplus_{i \in I} A_i$ of $\mathcal{A}$, the dimension of the vector space corresponding to X is equal to the cardinality of the set $\{i \in I \mid A_i \cong A\}$. Hence this cardinality depends only on X and not on the direct sum representation $X = \bigoplus_{i \in I} A_i$ of X as a direct sum of kernels A_i of nonzero morphisms between indecomposable injective modules. Taking the union of the isomorphism classes of kernels of nonzero morphisms between indecomposable injective modules that are in the same upper class as A, we find that the cardinality of the set $\{i \in I \mid [A_i]_u = [A]_u\}$ depends only on X and not on the direct sum representation $X = \bigoplus_{i \in I} A_i$ of X as a direct sum of kernels of nonzero morphisms between indecomposable injective modules. We have proved the following theorem.

THEOREM 5.3. *Let $\{A_i \mid i \in I\}$ and $\{B_j \mid j \in J\}$ be two families of right modules over a ring R. Assume that all the A_i and the B_j are kernels of noninjective morphisms between indecomposable injective R-modules. Suppose that $\bigoplus_{i \in I} A_i \cong \bigoplus_{j \in J} B_j$. Denote by I' the set of all $i \in I$ such that A_i is upper quasismall, and by J' the set of all $j \in B$ such that B_j is upper quasismall. Then there is a bijection $\tau : I' \to J'$ such that $[A_i]_u = [B_{\tau(j)}]_u$ for every $i \in I'$.*

Theorem 1.1 is now just a combination of Theorems 2.12 and 5.3.

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References

[1] B. Amini, A. Amini and A. Facchini, 'Equivalence of diagonal matrices over local rings', *J. Algebra* **320** (2008), 1288–1310.

[2] B. Amini, A. Amini and A. Facchini, 'Direct summands of direct sums of modules whose endomorphism rings have two maximal right ideals', manuscript (2010).

[3] R. T. Bumby, 'Modules which are isomorphic to submodules of each other', *Arch. Math.* **16** (1965), 184–185.

[4] N. V. Dung and A. Facchini, 'Weak Krull–Schmidt for infinite direct sums of uniserial modules', *J. Algebra* **193** (1997), 102–121.

[5] A. Facchini, 'Krull–Schmidt fails for serial modules', *Trans. Amer. Math. Soc.* **348** (1996), 4561–4575.

[6] A. Facchini, *Module Theory. Endomorphism Rings and Direct Sum Decompositions in Some Classes of Modules*, Progress in Mathematics, 167 (Birkhäuser, Basel, 1998).

[7] A. Facchini, 'Injective modules, spectral categories, and applications', in: *Noncommutative Rings, Group Rings, Diagram Algebras and Their Applications*, Contemporary Mathematics, 456 (eds. S. K. Jain and S. Parvathi) (American Mathematical Society, Providence, RI, 2008), pp. 1–17.

[8] A. Facchini, Ş. Ecevit and M. Tamer Koşan, 'Kernels of morphisms between indecomposable injective modules', *Glasgow Math. J.* (2010), to appear.

[9] A. Facchini and N. Girardi, 'Couniformly presented modules and dualities', in: *Advances in Ring Theory*, Trends in Mathematics (eds. Dinh Van Huynh and Sergio R. López Permouth) (Birkhäuser, Basel, 2010), pp. 149–163.

[10] A. Facchini and P. Příhoda, 'Representations of the category of serial modules of finite Goldie dimension', in: *Models, Modules and Abelian Groups* (eds. R. Göbel and B. Goldsmith) (de Gruyter, Berlin, 2008), pp. 463–486.

[11] A. Facchini and P. Příhoda, 'Factor categories and infinite direct sums', *Int. Electron. J. Algebra* **5** (2009), 1–34.

[12] P. Gabriel and U. Oberst, 'Spektralkategorien und reguläre Ringe im Von-Neumannschen Sinn', *Math. Z.* **82** (1966), 389–395.

[13] I. Herzog, 'Contravariant functors on the category of finitely presented modules', *Israel J. Math.* **167** (2008), 347–410.

[14] P. Příhoda, 'A version of the weak Krull–Schmidt theorem for infinite direct sums of uniserial modules', *Comm. Algebra* **34** (2006), 1479–1487.

[15] G. Puninski, 'Some model theory over a nearly simple uniserial domain and decompositions of serial modules', *J. Pure Appl. Algebra* **163** (2001), 319–337.

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