An Expansion for $x^n + y^n$.

By A. WATERSON.

In this note an expansion is found for the expression $x^n + y^n$ in terms of $x + y$ and of xy. Two illustrative applications are appended.

Theorem. If n be a positive integer, and x and y be any two numbers, then

$$x^n + y^n = \sum_{r=0}^{N} (-)^r \frac{n}{n-r} \binom{n-r}{r} (x+y)^{n-2r} (xy)^r$$

where $N = \frac{n}{2}$ *if n is even, and* $N = \frac{n-1}{2}$ *if n is odd.*

Proof. Suppose the theorem to be true for two successive values of n, say $k-1$ and k.

$$\therefore\ x^k + y^k = (x+y)^k - \frac{k}{k-1}\binom{k-1}{1}(x+y)^{k-2}(xy) + \ldots$$

$$+ (-) \frac{k}{k-r}\binom{k-r}{r}(x+y)^{k-2r}(xy)^r + \ldots$$

$$x^{k-1} + y^{k-1} = (x+y)^{k-1} - \frac{k-1}{k-2}\binom{k-2}{1}(x+y)^{k-3}(xy) + \ldots$$

$$+ (-1)^{r-1}\frac{k-1}{k-r}\binom{k-r}{r-1}(x+y)^{k-2r+1}(xy)^{r-1} + \ldots$$

But $x^{k+1} + y^{k+1} = (x^k + y^k)(x+y) - (x^{k-1} + y^{k-1})\,xy$

$$= \left[(x+y)^k - \frac{k}{k-1}\binom{k-1}{1}(x+y)^{k-2}(xy) + \ldots\right](x+y)$$

$$- \left[(x+y)^{k-1} - \frac{k-1}{k-2}\binom{k-2}{1}(x+y)^{k-3}(xy) + \ldots\right]xy.$$

The coefficient of $(x+y)^{k-2r+1}(xy)^r$ is therefore

$$(-)^r \frac{k}{k-r}\binom{k-r}{r} - (-)^{r-1}\frac{k-1}{k-r}\binom{k-r}{r-1}$$

$$= (-)^r\left[\frac{k}{k-r}\binom{k-r}{r} + \frac{k}{k-r}\binom{k-r}{r-1} - \frac{1}{k-r}\binom{k-r}{r-1}\right]$$

$$= (-)^r\left[\frac{k}{k-r}\binom{k-r+1}{r} - \frac{1}{k-r}\binom{k-r}{r-1}\right]$$

$$= (-)^r\left[\frac{k}{k-r} - \frac{r}{(k-r)\,(k-r+1)}\right]\binom{k-r+1}{r}$$

$$= (-)^r \frac{k+1}{k+1-r}\binom{k+1-r}{r},$$

which is exactly what it would be using the theorem.

But the theorem is true for $n = 1$ and $n = 2$, so it is true for $n = 3$, and hence similarly for all positive integral values of n.

Applications. (i) *Expansion for $2 \cos n\,\theta$ in terms of $2 \cos \theta$.* Let $x = \cos\theta + i \sin\theta$, $y = \cos\theta - i\sin\theta$; then $x + y = 2\cos\theta$ and $xy = 1$. Also $x^n = \cos n\,\theta + i \sin n\,\theta$; $y^n = \cos n\,\theta - i \sin n\,\theta$ so that $x^n + y^n = 2 \cos n\,\theta$. Substituting these in the theorem, we obtain

$$2 \cos n\,\theta = \sum_{r=0}^{N} (-)^r \frac{n}{n-r} \binom{n-r}{r} (2\cos\theta)^{n-2r}.$$

(ii) *To find the equation whose roots are the n-th powers of those of $ax^2 + bx + c = 0$.* Let the roots of the given equation be α and β. Then the required equation is $x^2 - (\alpha^n + \beta^n)\,x + (\alpha\beta)^n = 0$

$$i.e. \quad x^2 - \left[\sum_{r=0}^{N} (-)^r \frac{n}{n-r}\binom{n-r}{r}(\alpha+\beta)^{n-2r}(\alpha\beta)^r\right]x + (\alpha\beta)^n = 0$$

or, since $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$,

$$a^n x^2 - \left[\sum_{r=0}^{N} (-)^{n-r} \frac{n}{n-r}\binom{n-r}{r} a^r b^{n-2r} c^r\right] x + c^n = 0.$$

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Dirichlet's Integrals.

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The integrals referred to here are those given by the following *Theorem*:

$$\text{Let } I = \int\int\int f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz,$$

where the integration is extended over

$$x \geqq 0,\ y \geqq 0,\ z \geqq 0, \quad a \leqq x + y + z \leqq b,$$

and $l > 0$, $m > 0$, $n > 0$, $0 \leqq a \leqq b < \infty$.

$$\text{Let } J = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} \int_a^b t^{l+m+n-1} f(t)\, dt.$$

Then $I = J$ when either,