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A GELFAND-PHILLIPS SPACE NOT CONTAINING l_1 WHOSE DUAL BALL IS NOT WEAK* SEQUENTIALLY COMPACT

BENGT JOSEFSON

Department of Mathematics, Linköpings University, S-581 83 Linköping, Sweden
e-mail: bejos@mai.lin.se

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Abstract. A set D in a Banach space E is called *limited* if pointwise convergent sequences of linear functionals converge uniformly on D and E is called a *GP-space* (after Gelfand and Phillips) if every limited set in E is relatively compact. Banach spaces with weak* sequentially compact dual balls (W*SCDB for short) are GP-spaces and $l_1(A)$ is a GP-space without W*SCDB. Disproving a conjecture of Rosenthal and inspired by James tree space, Hagler and Odell constructed a class of Banach spaces ([HO]-spaces) without both W*SCDB and subspaces isomorphic to l_1. Schlumprecht has shown that there is a subclass of the [HO]-spaces which are also GP-spaces. It is not clear however if any [HO]-construction yields a GP-space—in fact it is not even clear that W*SCDB$\Leftrightarrow$GP-space is false in general for the class of Banach spaces containing no subspace isomorphic to l_1. In this note the example of Hagler and Odell is modified to yield a GP-space without W*SCDB and without an isomorphic copy of l_1.

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A set D in a Banach space E is called *limited* if $\lim_{j\to\infty}\sup_{z\in D}\varphi_j(z) = 0$ for every weak* null sequence $(\varphi_j)_{j\in\mathbf{N}} \subset E^\star$, where $E^\star$ is the dual space; i.e. D is limited if pointwise convergent sequences of linear functionals converge uniformly on D. Obviously relatively compact sets are limited and a Banach space is called a *GP-space* (after Gelfand and Phillips) if all limited sets are relatively compact. If a bounded sequence $(\varphi_j)_{j\in\mathbf{N}} \subset E^\star$ separates a limited set $(a_j)_{j\in\mathbf{N}} \subset E$, i.e. $\varphi_j(a_j) = 1$ but $\lim_{j\to\infty}\varphi_j(a_k) = 0$ for every k (in particular $(a_j)_j$ cannot be relatively compact), then $(\varphi_j)_{j\in\mathbf{N}}$ has no weak* converging subsequence. Thus Banach spaces with weak* sequentially compact dual balls (W*SCDB for short) are GP-spaces. Hence l^∞ is an example of a Banach space without W*SCDB, since the set of unit vectors of c_0 is a limited set in l^∞. Another well known example of a Banach space without W*SCDB, and perhaps the most natural one, is $l_1(A)$, with A uncountable (for a survey of the topic, see [**3**]). In some sense $l_1(A)$ is the opposite extreme, compared with Banach spaces not being GP-spaces, regarding the W*SCDB-property, since there is hardly any limitedness in constructing sequences of bounded linear functionals on $l_1(A)$. Actually this richness of bounded linear functionals also explains why $l_1(A)$ is a GP-space and even more—no non relatively compact subset of l_1 can be limitedly embedded in any Banach space. In [**9**] H. P. Rosenthal asked if every Banach space without W*SCDB also contained an isomorphic copy of some $l_1(A)$. J. Hagler and E. Odell [**6**] disproved this by constructing a space (or a class of spaces) without both W*SCDB and subspaces isomorphic to l_1. Their space is related to a nonseparable analogue of JT, the James tree [**7**], which in turn disproved the conjecture that a separable Banach space with a nonseparable dual contained l_1. By a special choice

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of the sets in the construction, T. Schlumprecht showed in [**10**] that there is a subclass of the [HO]-spaces which are not GP-spaces either. However it is not clear if any [HO]-construction yields a GP-space—in fact it is not even clear that in general W*SCDB $\Leftrightarrow$ GP-space is false in general for Banach spaces without l_1. In this note we modify the example in [**6**] to yield a GP-space without W*SCDB and without isomorphic copies of l_1. Note that limited sets in Banach spaces without l_1 are relatively weakly compact, according to [**2**], because of a convergence property for certain sequences of linear functionals.

We recall the construction of [**6**]. There is a well ordered set $I, <$ and a collection of infinite subsets of $\mathbf{N}$, $(M_\alpha)_{\alpha\in I}$, such that (1) and (2) hold.

(1) If $\alpha < \beta$ then either $M_\beta \subset^a M_\alpha$ or $M_\beta \cap M_\alpha =^a \emptyset$.

(2) If $M \subset \mathbf{N}$, $|M| = \infty$, then there is an $\alpha \in I$ such that $|M \cap M_\alpha| = |M \setminus M_\alpha| = \infty$.

Here $|M|$ denotes the cardinality of M, $L \subset^a M$ means that $|L \setminus M| < \infty$ and $L \cap M =^a \emptyset$ means that $|L \cap M| < \infty$.

Define a new partial ordering $\prec$ on I as follows: $\alpha \prec \beta$ if $\alpha < \beta$ and $M_\beta \subset^a M_\alpha$. Note that $(I, \prec)$ is a tree and that every nonempty subset of $(I, \prec)$ has at least one minimal element.

A subset $C = [\gamma, \beta] = \{\alpha \in I : \gamma \prec \alpha \prec \beta\}$ is called a *segment in I*. Let $(g_\alpha)_{\alpha\in I}$ be a linearly independent set of vectors in some vector space. If $(t_\alpha)_{\alpha\in I}$ is a finite set of non-zero scalars, we define

($\star$) $\|\sum_{\alpha\in I} t_\alpha g_\alpha\| = \sup\{\sum_{i=1}^{k}(\sum_{\alpha\in C_i} t_\alpha)^2|^{1/2} : C_1, \ldots; C_k$ are pairwise disjoint segments$\}$.

Let Y be the completion of the linear span of the set $(g_\alpha)_{\alpha\in I}$. For each $\alpha \in I$, let 1_{M_α} be the indicator function of M_α in l_∞ and let

$$h_\alpha = (1_{M_\alpha}, g_\alpha) \in (l_\infty \oplus Y)_\infty.$$

Then

$$\|\sum t_\alpha h_\alpha\| = \max\{\|\sum t_\alpha 1_{M_\alpha}\|_\infty, \|\sum t_\alpha g_\alpha\|\}.$$

Finally X, the closed subspace of $(l_\infty \oplus Y)_\infty$ generated by $(h_\alpha)_{\alpha\in I}$, is the space constructed in [**6**].

Construction of E. Let $\{A_\alpha^n : \alpha \in I, n \in \mathbf{N}\}$ be a collection of sets such that $A_\alpha^n \cap \mathbf{N} = \emptyset$, for all $\alpha \in I$ and $n \in \mathbf{N}$, $A_\beta^n \subset A_\alpha^n$ if $\alpha < \beta$ and $n \in \mathbf{N}$ but $A_\beta^n \cap A_\alpha^m = \emptyset$, for all $\alpha, \beta \in I$ if $m \neq n$. Put $A^n = \cup_{\alpha\in I} A_\alpha^n$ and $A = \cup_{n=1}^{\infty} A^n$. Let $U_\alpha = \cup_{n\in M_\alpha} A_\alpha^n$ and 1_{U_α} be the indicator function of U_α in $l_\infty(A)$ and let

$$u_\alpha = (1_{U_\alpha}, 1_{M_\alpha}, g_\alpha) \in (l_\infty(V) \oplus Y)_\infty,$$

where $V = A \cup \mathbf{N}$. Thus

$$\|\sum t_\alpha u_\alpha\| = \max\{\|\sum t_\alpha 1_{V_\alpha}\|_\infty, \|\sum t_\alpha g_\alpha\|\},$$

where $V_\alpha = U_\alpha \cup M_\alpha$. Let E be the closed subspace of $(l_\infty(V) \oplus Y)_\infty$ generated by $(u_\alpha)_{\alpha\in I}$.

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E has no W⋆SCDB. Define a linear mapping $P : E \to X$ by $P(u_\alpha) = h_\alpha$. Obviously P is a norm one projection of E onto X. Thus E has no W⋆SCDB since X has not.

The main difference between E and the Hagler-Odells space X is pointed out in the following Lemma.

Lemma 1. *Let* $B = \{\alpha \in I : \gamma < \alpha < \beta\}$ *and define* $P_B : E \to E_B$ *by* $P_B(\sum t_\alpha u_\alpha) = \sum_{\alpha\in B} t_\alpha u_\alpha$, *where* E_B *is the subspace of* E *generated by* $\{u_\alpha\}_{\alpha\in B}$. *Then* P_B *is a norm two projection of* E *onto* E_B.

Proof. Note that $A^n_\beta \subset A^n_\alpha \subset A^n_\gamma$ if $\gamma < \alpha < \beta$; that is if $\alpha \in B$. Thus

$$\begin{aligned}\|\sum t_\alpha 1_{V_\alpha}\|_\infty &\geq \|\sum_{\alpha<\beta} t_\alpha 1_{V_\alpha}\|_\infty \\ &\geq \max\{\|\sum_{\alpha<\gamma} t_\alpha 1_{V_\alpha}\|_\infty, \frac{1}{2}\|\sum_{\alpha\in B} t_\alpha 1_{V_\alpha}\|_\infty\} \geq \frac{1}{2}\|\sum_{\alpha\in B} t_\alpha 1_{V_\alpha}\|_\infty\}.\end{aligned}$$

Since pairwise disjoint segments of B are also pairwise disjoint segments of I we have $\|\sum_{\alpha\in B} t_\alpha g_\alpha\| \leq \|\sum t_\alpha g_\alpha\|$. Altogether we get $\|\sum_{\alpha\in B} t_\alpha u_\alpha\| \leq 2\|\sum t_\alpha u_\alpha\|$. QED

Remark. The corresponding projections in the Hagler-Odells space are bounded if B is a segment but not in general; (there are different ways to generate c_0-vectors in the l_∞-part of X).

E is a GP-space. Following an argument in [**7**], this is almost proved in [**1**].

We shall prove that a bounded, non-relatively compact sequence $(a_j)_{j\in\mathbf{N}} \subset E$ is not limited. Assume that we have found an infinite set $M \subset \mathbf{N}$, $\epsilon > 0$ and $B_j = \{\alpha \in I : \gamma_j < \alpha < \beta_j\}$ such that $B_i \cap B_j = \emptyset$ if $i, j \in M$, $i \neq j$ and $\|b_j\| > \epsilon$, for every $j \in M$, where $b_j = P_{B_j}(a_j)$.

Then there exists, for every $j \in M$, $\varphi_j \in E^\star_{B_j}$ such that $\|\varphi_j\| < 1/\epsilon$ but $\varphi_j(b_j) = 1$. Extend φ_j to $\psi_j \in E^\star$ by setting $\psi_j(u_\alpha) = 0$ when $\alpha \notin B_j$. Then $\psi_j(b_j) = \varphi_j(b_j) = 1$ and $\|\psi_j\| < 2/\epsilon$ for every $j \in M$ according to Lemma 1. Thus $\psi_j(a_j) = 1$ because $b_j = P_{B_j}(a_j)$. Further $(\psi_j)_{j\in M}$ is a weak⋆ null sequence because $B_i \cap B_j = \emptyset$ if $i \neq j$ in M and because finitely generated vectors are dense in E.

To prove the assumption made above there is no loss of generality in assuming that $a_j = \sum_{\alpha\in U_j} t_{\alpha,j} u_\alpha$, where U_j is finite.

If $\beta \in I$ we put $U^\beta_j = \{\alpha \in U_j : \alpha < \beta\}$ and $a^\beta_j = \sum_{\alpha\in U^\beta_j} t_{\alpha,j} u_\alpha$. Let ω be the smallest $\beta \in I$ such that $(a^\beta_j)_{j\in\mathbf{N}}$ is not relatively compact. ω exists since $(I, <)$ is well ordered and each U_j is a finite set. We also have, because U_j finite, that $\beta_i < \omega$, where β_i is the smallest $\beta \in I$ such that $\cup^i_{k=1} U^\omega_i \subset \{\alpha \in I : \alpha < \beta\}$. In particular we get that $(a^{\beta_i}_j)_{j\in\mathbf{N}}$ is relatively compact since $\beta_i < \omega$. But then it is clear that there exist a subsequence $M \subset \mathbf{N}$, $\epsilon > 0$ and, for every $j \in M$, $B_j = \{\alpha \in I : \gamma_j < \alpha < \beta_j\}$ such that $B_i \cap B_j = \emptyset$ if $i, j \in M$, $i \neq j$, that is $\gamma_j > \beta_{j-1}$, and such that $\|b_j\| > \epsilon$ for every $j \in M$ where $b_j = P_{B_j}(a_j)$.

Thus the assumption is proved and E is a GP-space.

E contains no subspace isomorphic to l_1. Let F be the closed linear span of $(1_{A^n_\alpha} : \alpha \in I, n \in \mathbf{N})$ in $l_\infty(A)$ and note that $F = (\sum^\infty_{n=1} \oplus F_n)_{c_0}$, where F_n is the closed

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linear span of $(1_{A_\alpha^n} : \alpha \in I)$, and hence is isomorphic to $c_0(I)$ because $A_\beta^n \subset A_\alpha^n$ if $\alpha < \beta$. Thus F does not contain an isomorphic copy of l_1. Let Z be the closed subspace of $(l_\infty(V) \oplus Y)_\infty$ generated by E and F. The quotient space Z/F is isometric to X. Since X, according to [**6**], does not contain an isomorphic copy of l_1, the Lemma below shows that neither Z nor E contains an isomorphic copy of l_1.

LEMMA 2. *Let Z and F be Banach spaces such that the quotient space Z/F and F contain no isomorphic copy of l_1. Then Z does not contain an isomorphic copy of l_1.*

Proof. Assume that the Lemma is false and that $(a_j)_{j\in\mathbf{N}} \subset Z$ is a sequence isomorphic to the unit vectors of l_1. Let $q : Z \to Z/F$ be the quotient map and put $b_j = q(a_j)$. Since $(b_j)_{j\in\mathbf{N}}$ does not contain any isomorphic copy of the basis of l_1, there exist a subsequence $(j_k)_{k\in\mathbf{N}}$ and, for every $j \in \mathbf{N}$, $t_j \in \mathbf{R}$ such that $\sum_{r=j_{k-1}+1}^{j_k} |t_r| = 1$ but $\| \sum_{r=j_{k-1}+1}^{j_k} t_r b_r \| = 2^{-k}$, for every $k \in \mathbf{N}$. Put $c_k = \sum_{r=j_{k-1}+1}^{j_k} t_r a_r$ and take $d_k \in F$ such that $\|c_k - d_k\| < 2^{-k}$. Note that $(c_k)_{k\in\mathbf{N}}$ is isomorphic to the unit vector basis of l_1 and hence also to $(d_k)_k$, which gives a contradiction. Thus $(a_j)_j$ is not isomorphic to the unit vector basis of l_1. QED

Thus E is a GP-space without an isomorphic copy of l_1 whose dual ball is not weak* sequentially compact.

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