SEMIGROUPS CO-ORDINATIZING ORTHOMODULAR GEOMETRIES

D. J. FOULIS

1. Introduction. In (**2**, **3**, **4**, and **5**), the author has established a connection between orthomodular lattices and Baer $*$-semigroups. In brief, the connection is as follows. The lattice of closed projections of any Baer $*$-semigroup forms an orthomodular lattice. Conversely, if L is any orthomodular lattice, there exists a Baer $*$-semigroup S which co-ordinatizes L in the sense that L is isomorphic to the lattice of closed projections in S. In this note we shall assume that the reader is familiar with the results and the notation of the quoted papers.

Suppose that the Baer $*$-semigroup S co-ordinatizes the orthomodular lattice L. We set forth the following natural question: *What connections exist between lattice theoretic properties which L might possess and semigroup theoretic properties which S might possess?*

Finding specific answers to this general question is complicated by the fact that although S determines L up to isomorphism, L does not determine S up to isomorphism. Answers are known only in certain favourable cases. For example, it is known (**3**, Theorem 10, p. 894) that modularity of L is equivalent to the $*$-regularity of some co-ordinatizing S. A "local" version of this same result has been proved in (**5**, Theorem 28, p. 81). A second example is provided by (**2**, Lemma 2, p. 650), which shows that L is a complete lattice if and only if it can be co-ordinatized by a complete Baer $*$-semigroup S.

We shall use a slight modification of the terminology of Wright (**8**, Definition 2.2, p. 476) in connection with Loomis' version of a dimension lattice (**7**, p. 4).

Definition 1. Let L be a complete orthomodular lattice. We call L an *orthomodular geometry* if L is equipped with a distinguished equivalence relation $\sim$ satisfying the following:

(i) If $e \sim 0$, then $e = 0$.

(ii) If $\{e_\alpha\}$ is an orthogonal family in L, and if $f \sim \vee_\alpha e_\alpha$, then there exists an orthogonal family $\{f_\alpha\}$ in L such that $f = \vee_\alpha f_\alpha$ and such that $f_\alpha \sim e_\alpha$ for every α.

(iii) If $\{e_\alpha\}$ is an orthogonal family in L and if $\{f_\alpha\}$ is a second orthogonal family in L with the same indices such that $e_\alpha \sim f_\alpha$ for every α, then $\vee_\alpha e_\alpha \sim \vee_\alpha f_\alpha$.

(iv) If e and f have a common complement in L, then $e \sim f$.

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Following (**6**, p. 19) we introduce the notion of *$*$-equivalent projections* in an involution semigroup.

Definition 2. Let S be an involution semigroup, i.e., a multiplicative semigroup equipped with an anti-automorphic involution $*:S \to S$. Let $e = e^2 = e^* \in S$ and $f = f^2 = f^* \in S$. If there exists an element $x \in S$ such that $x = exf$, $x^*x = f$, $xx^* = e$, then we say that e and f are *$*$-equivalent* and we write $x:e \sim_* f$. If there exist $e, f \in S$ such that $x:e \sim_* f$, we say that x is a *partially unitary element* of S. The notation $e \sim_* f$ means that there exists a partially unitary $x \in S$ such that $x:e \sim_* f$.

It is quite easy to see that $*$-equivalence is a *bona fide* equivalence relation on the set of elements $e \in S$ with $e = e^2 = e^*$. If S is the Baer $*$-semigroup of bounded operators on a Hilbert space, two projections are $*$-equivalent if and only if they project onto subspaces of the same dimension.

We now formulate the following special case of our basic natural question: *What semigroup theoretic property of the Baer $*$-semigroup S is equivalent to the condition that the orthomodular lattice $P'(S)$ of closed projections in S is an orthomodular geometry under the relation of $*$-equivalence?* In the present note we partially answer the latter question by giving a *sufficient* condition on S in terms of five postulates (Definitions 9 and 10) abstracted from the postulates given by Loomis in (**7**, pp. 25–32).

2. Basic definitions. In what follows the symbol S will always represent an *involution semigroup with* 0; i.e., S is a multiplicative semigroup with a zero and S is equipped with a mapping $*:S \to S$ such that for all $x, y \in S$ we have $(xy)^* = y^*x^*$ and $x^{**} = x$. An element $e \in S$ with $e = e^2 = e^*$ is called a *projection* and the set of all projections in S is denoted by $P = P(S)$. Note that $0 \in P$. We partially order P by defining $e \leqslant f$ to mean $e = ef$, $e, f \in P$.

Definition 3. Two elements $x, y \in S$ are said to be *orthogonal* (**7**, p. 26), in symbols $x \perp y$, in case $x^*y = xy^* = 0$.

Clearly, $x \perp y \Leftrightarrow y \perp x \Leftrightarrow x^* \perp y^*$. Also, two projections are orthogonal if and only if their product is zero.

Definition 4. We shall say that S satisfies the *$*$-cancellation law* or that S is a *$*$-cancellation semigroup* if $x, y \in S$ with $xx^* = yx^* = yy^*$ implies that $x = y$.

The notion of a $*$-cancellation semigroup was introduced in (**5**, p. 74). If S satisfies the $*$-cancellation law, then $xx^* = 0$ implies that $x = 0$. The proof of (**5**, Theorem 13, pp. 74–75) shows that if S is an involution ring, then the $*$-cancellation law is equivalent to the condition $xx^* = 0 \Rightarrow x = 0$.

LEMMA 1. *Suppose that S satisfies the $*$-cancellation law. Then $x, y, z \in S$ and $x^*xy = x^*xz$ imply that $xy = xz$.*

Proof. Put $a = y^*x^*$, $b = z^*x^*$. It is enough to prove that $a = b$. Taking $*$ on

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both sides of $x^*xy = x^*xz$ gives $ax = bx$. It follows that $aa^* = axy = bxy = ba^*$. Also, $ba^* = z^*x^*xy = z^*x^*xz = bb^*$. By the $*$-cancellation law, $a = b$.

Definition 5. A projection $e \in P(S)$ is said to be *closed* if there exists a (necessarily unique) projection $e' \in P(S)$ such that for every $x \in S$
(i) $ex = 0 \Leftrightarrow x = e'x$ and
(ii) $e'x = 0 \Leftrightarrow x = ex$.

It is easy to see that if e is a closed projection in S, then the projection e' of Definition 5 is also a closed projection and that $e'' = e$. Consequently, Definition 5 generalizes the notion of a closed projection in a Baer $*$-semigroup (**5**, p. 66).

We now set forth a definition which will play a central role in the remainder of the paper. The motivation for this definition occurs in (**7**, pp. 27–28).

Definition 6. Let $\{a_\alpha\}$ be any family of mutually orthogonal elements of S. We write $a \in \text{sum}_\alpha\, a_\alpha$ in case $a \in S$ and
(i) $aa_\alpha^* = a_\alpha\, a_\alpha^*$ for all α and
(ii) for $b, c \in S$, $a_\alpha\, b = a_\alpha\, c$ for all $\alpha \Rightarrow ab = ac$.

An interesting example of Definition 6 is obtained as follows. Let L be a complete orthomodular lattice and let $S(L)$ be the co-ordinizating Baer $*$-semigroup as defined in (**2**, p. 651). Let $\{\phi_\alpha\}$ be an orthogonal family of elements of $S(L)$, and define $\phi: L \rightarrow L$ by $e\phi = \vee_\alpha(e\phi_\alpha)$ for all $e \in L$. Then $\phi \in S(L)$ and $\phi \in \text{sum}_\alpha\, \phi_\alpha$.

Definition 7. If M is any non-empty subset of S, we define the *centralizer* of M, in symbols $Z(M)$, by

$$Z(M) = \{s \in S | sx = xs \text{ for all } x \in M\}.$$

We define $ZZ(M) = Z(Z(M))$.

The facts concerning $Z(M)$ given in the following lemma are both well known and easy to verify; hence we shall feel free to make use of these facts below without explicit reference to the lemma.

LEMMA 2. *Let M and N be non-empty subsets of S. Then*
(i) $M \subset N \Rightarrow Z(N) \subset Z(M)$;
(ii) $M \subset ZZ(M)$;
(iii) $Z(M) = ZZZ(M)$;
(iv) *if $M = M^*$, then $Z(M)$ is a subsemigroup of S containing* 0 *and closed under the involution* $*$;
(v) *if $x = x^* \in S$, then $x \in ZZ(x) \subset Z(x)$ and $ZZ(x)$ is a commutative subsemigroup of S which is closed under the involution* $*$.

In (**6**, p. 30) Kaplansky has introduced the property (EP) (existence of projections), which we shall need below. For a while, we shall make do with a weaker property (WEP).

Definition 8. The involution semigroup S *has the property* (EP) if, given any non-zero element $x \in S$, there exists an element $y \in S$ such that $y = y^* \in ZZ(x^*x)$ and x^*xy^2 is a non-zero closed projection. If, given any non-zero element $x \in S$, there exists an element $z \in S$ such that $z = z^* \in ZZ(x^*x)$ and x^*xz is a non-zero closed projection, then we shall say that S *has property* (WEP).

LEMMA 3. *Suppose that the involution semigroup S has the property* (WEP). *Let x be a non-zero element of S. Then there exists a non-zero closed projection $e \in ZZ(x^*x)$ and there exists an element $h = h^* \in ZZ(x^*x)$ such that $x^*xh = e$ and $h = ehe$.*

Proof. By (WEP) there exists $z = z^* \in ZZ(x^*x)$ such that $x^*xz = e$, a non-zero closed projection. Put $h = ze$.

3. The postulates. In **(7**, pp. 25–32**)**, L. H. Loomis gives an elegant system of postulates, verified in any weakly closed self-adjoint ring of bounded operators on a Hilbert space. In Definitions 9 and 10 below we generalize these postulates in such a way that they become applicable to an involution semigroup.

Definition 9. We call S a *weak Loomis $*$-semigroup* if S satisfies the following postulates:

(i) S is an involution semigroup with 0;
(ii) all projections in $P = P(S)$ are closed;
(iii) S satisfies the $*$-cancellation law;
(iv) if $\{a_\alpha\}$ is an orthogonal family of partially unitary elements of S, then $\mathrm{sum}_\alpha\, a_\alpha$ is not empty;
(v) S satisfies (WEP).

Definition 10. A *Loomis $*$-semigroup* is a weak Loomis $*$-semigroup satisfying the (EP) property.

We remark that the multiplicative semigroup of two-by-two matrices over Z_3 (equipped with the involution $X \to X^* =$ transpose of Z) is a weak Loomis $*$-semigroup but fails to satisfy property (EP).

THEOREM 4. *Let S be the multiplicative involution semigroup of an involution ring with unit. Then S is a weak Loomis $*$-semigroup if and only if S satisfies Conditions* (iv) *and* (v) *of Definition* 9.

Proof. Let e be a projection in S, and put $e' = 1 - e$. Then, e' is a projection and Conditions (i) and (ii) of Definition 5 are satisfied; hence, all projections in S are closed. Clearly, property (WEP) implies that $xx^* = 0 \Rightarrow x = 0$; so (by the remarks following Definition 4) S satisfies the $*$-cancellation law.

The results in **(7**, pp. 25–32**)** show that the multiplicative involution semigroup of a von Neumann algebra is a Loomis $*$-semigroup.

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4. Basic properties of weak Loomis *-semigroups. In the present section we assume, once and for all, that S is a weak Loomis $*$-semigroup. Our main result in this section will be Theorem 15, which states that *a weak Loomis $*$-semigroup is a Baer $*$-semigroup.*

LEMMA 5. *Let $\{a_\alpha\}$ be an orthogonal family of elements of S and suppose that $a, b \in \mathrm{sum}_\alpha\, a_\alpha$. Then $a = b$.*

Proof. By Definition 6 we have $aa_\alpha{}^* = a_\alpha\, a_\alpha{}^* = ba_\alpha{}^*$ for all α; hence, $a_\alpha\, a^* = a_\alpha\, b^*$ for all α. By Definition 6 again, $aa^* = ab^*$ and $ba^* = bb^*$, so $aa^* = ab^* = bb^*$. By $*$-cancellation, $a = b$.

Definition 11. If $\{a_\alpha\}$ is an orthogonal family of elements of S, and if $\mathrm{sum}_\alpha\, a_\alpha$ is not empty, we shall write $\sum_\alpha a_\alpha$ for the (necessarily unique) element in $\mathrm{sum}_\alpha\, a_\alpha$. Henceforth, the notation $a = \sum_\alpha a_\alpha$ will be taken to mean that $\{a_\alpha\}$ is an orthogonal family of elements of S and that $\sum_\alpha a_\alpha$ exists and equals a.

LEMMA 6. *Let $a = \sum_\alpha a_\alpha$. If $b \in S$ with $ba_\alpha = 0$ for all α except possibly for $\alpha = \beta$, then $ba = ba_\beta$.*

Proof. For $\alpha \neq \beta$ we have

$$a_\alpha\, a^*b^* = a_\alpha\, a_\alpha{}^*\, b^* = a_\alpha(ba_\alpha)^* = 0 = a_\alpha\, a_\beta{}^*\, b^*.$$

For $\alpha = \beta$ we have

$$a_\alpha\, a^*b^* = a_\beta\, a^*b^* = a_\beta\, a_\beta{}^*\, b^* = a_\alpha\, a_\beta{}^*\, b^*.$$

Hence, for every α we have $a_\alpha\, a^*b^* = a_\alpha\, a_\beta{}^*\, b^*$. It follows that

$$aa^*b^* = aa_\beta{}^*b^* = a_\beta\, a_\beta{}^*b^* = a_\beta\, a^*b^*.$$

Consequently,

$$(ba)(ba)^* = baa^*b^* = ba_\beta\, a^*b^* = (ba_\beta)(ba)^*$$

and

$$(ba)(ba)^* = baa^*b^* = ba_\beta\, a_\beta{}^*b^* = (ba_\beta)(ba_\beta)^*.$$

The $*$-cancellation law now yields $ba = ba_\beta$ as desired.

COROLLARY 7. *Let $a = \sum_\alpha a_\alpha$. Then*
(i) *$a^*a_\alpha = a_\alpha{}^*a_\alpha$ for all α, and*
(ii) *if $ba_\alpha = 0$ for all α, then $ba = 0$.*

Proof. To prove (i), put $b = a_\beta{}^*$ in Lemma 6 and conclude that $a_\beta{}^*a = a_\beta{}^*a_\beta$. Taking $*$ on both sides of the latter equation and rewriting β as α yields (i). Conclusion (ii) is evident from Lemma 6.

THEOREM 8. *If $a = \sum_\alpha a_\alpha$, then $a^* = \sum_\alpha a_\alpha{}^*$.*

Proof. We must prove (i) $a^*a_\alpha = a_\alpha{}^*a_\alpha$ for all α and (ii) $a_\alpha{}^*b = a_\alpha{}^*c$ for all α implies $a^*b = a^*c$. Corollary 7 takes care of (i). To prove (ii), suppose that

$a_\alpha{}^*b = a_\alpha{}^*c$ for all α. Then, for all α, $a_\alpha a^*b = a_\alpha a_\alpha{}^*b = a_\alpha a_\alpha{}^*c = a_\alpha a^*c$; whence $aa^*b = aa^*c$. An application of Lemma 1 yields $a^*b = a^*c$.

THEOREM 9. *Let $a = \sum_\alpha a_\alpha$ and suppose that $x \in S$ is such that $\{a_\alpha x\}$ is an orthogonal family. Then $ax = \sum_\alpha a_\alpha x$.*

Proof. Fix any index β. Put $u = (ax)(a_\beta x)^*$, $v = (a_\beta x)(a_\beta x)^*$. Our first task is to show that $u = v$. For $\alpha \neq \beta$ we have

$$u^*a_\alpha = a_\beta xx^*a^*a_\alpha = a_\beta xx^*a_\alpha{}^*a_\alpha = 0$$

since $(a_\beta x)(a_\alpha x)^* = a_\beta xx^*a_\alpha{}^* = 0$. It follows from Lemma 6 that $u^*a = u^*a_\beta$. Also,

$$v^*a_\beta = a_\beta xx^*a_\beta{}^*a_\beta = a_\beta xx^*a^*a_\beta = u^*a_\beta.$$

It follows that

$$u^*u = u^*ax(a_\beta x)^* = u^*a_\beta x(a_\beta x)^* = u^*v$$

and

$$v^*v = v^*a_\beta x(a_\beta x)^* = u^*a_\beta x(a_\beta x)^* = u^*v.$$

The $*$-cancellation law now yields $u = v$. This gives us Condition (i) of Definition 6. To prove Condition (ii), suppose that $a_\alpha xb = a_\alpha xc$ for all α. Then $axb = axc$. The proof is complete.

COROLLARY 10. *Let $a = \sum_\alpha a_\alpha$.*
(i) *If $x, y \in S$ are such that $a_\alpha x = ya_\alpha$ for all α, then $ax = ya$.*
(ii) *If $x \in S$ is such that $a_\alpha x = xa_\alpha$ for all α, then $ax = xa$.*

Proof. Clearly, it is enough to prove (i). For $\alpha \neq \beta$ we have

$$(a_\alpha x)(a_\beta x)^* = ya_\alpha a_\beta{}^*y^* = 0 \quad \text{and} \quad (a_\alpha x)^*(a_\beta x) = x^*a_\alpha{}^*a_\beta x = 0,$$

so $\{a_\alpha x\}$ is an orthogonal family and $ax = \sum_\alpha a_\alpha x$. Since $a_\alpha{}^*y^* = x^*a_\alpha{}^*$, the result just obtained together with Theorem 8 gives $a^*y^* = \sum_\alpha a_\alpha{}^*y^*$; hence,

$$ya = \textstyle\sum_\alpha ya_\alpha = \sum a_\alpha x = ax.$$

COROLLARY 11. *If $a = \sum_\alpha a_\alpha$, then $aa^* = \sum_\alpha a_\alpha a_\alpha{}^*$.*

Proof. In Theorem 9, put $x = a^*$.

LEMMA 12. *Let e, f be projections in S and let $x \in S$. Then*
(i) $e \leqslant f \Rightarrow f' \leqslant e'$,
(i) $e, e' \leqslant f \Rightarrow f = 1$,
(iii) $ex = xe \Rightarrow e'x = xe'$.

Proof. To prove (i), suppose that $e \leqslant f$, i.e., $e = ef$. It follows that $ef' = 0$, hence that $f' = e'f'$, i.e., $f' \leqslant e'$. To prove (ii), suppose that $e, e' \leqslant f$. By (i), $f' \leqslant e'' = e$, so $f' \leqslant f$. Hence, $f' = ff' = 0$, so $f = 0' = 1$. To prove (iii), suppose that $ex = xe$. Then $exe' = 0$, so $xe' = e'xe'$. Since $ex = xe$, then

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$x^*e = ex^*$, so by the above argument (applied to x^* rather than x) $x^*e' = e'x^*e'$. Taking $*$ on both sides of the latter equation, $e'x = e'xe' = xe'$.

LEMMA 13. *Let $\{e_\alpha\}$ be any orthogonal family of projections in S. Then $\{e_\alpha\}$ has a supremum in the partially ordered set $P(S)$, and this supremum is given by $e = \sum_\alpha e_\alpha$.*

Proof. Setting $e = \sum_\alpha e_\alpha$, which exists because of part (iv) of Definition 9, and invoking Theorem 8 and Corollary 11, we have $e = e^* = ee^* = e^2 \in P$. The remainder of the proof is the same as the proof of **(7**, Lemma 53, p. 30**)**, provided that we write g' rather than $1 - g$ for a projection $g \in P$.

THEOREM 14. *The partially ordered set P of projections in a weak Loomis $*$-semigroup forms a complete lattice.*

Proof. The proof follows the lines of Loomis' proof in **(7**, Theorem 9, p. 30**)**. Let $\{e_\alpha\}$ be any family of projections in S. Let $\{f_\beta\}$ be an orthogonal family of projections in S that is maximal with respect to the property that every e_α is orthogonal to every f_β. Let $f = \sum_\beta f_\beta = \sup\{f_\beta\}$, and let $e = f'$. Since $f_\beta \leqslant e'_\alpha$ for every β and every α, then $f \leqslant e'_\alpha$ and $e_\alpha \leqslant f' = e$ for every α. It remains only to show that e is the least upper bound in P for the family $\{e_\alpha\}$.

Let $g \in P$ with $e_\alpha \leqslant g$ for every α. If $eg' = 0$, then $e \leqslant g$ and our argument is finished. Hence, we suppose that $eg' \neq 0$, from which it follows by $*$-cancellation that $eg'(eg')^* = eg'e \neq 0$. By (WEP), there exists $x = x^* \in ZZ(eg'e)$ such that $0 \neq eg'ex = h \in P$. Now

$$he_\alpha = eg'exe_\alpha = xeg'ee_\alpha = xeg'e_\alpha = xeg'ge_\alpha = 0,$$

so h is orthogonal to every e_α. Since $f_\beta \leqslant f = e'$, then $hf_\beta = xeg'ef_\beta = 0$; hence, h is orthogonal to every f_β. This contradicts the maximality of $\{f_\beta\}$.

THEOREM 15. *If S is a weak Loomis $*$-semigroup, then S is a complete Baer $*$-semigroup.*

Proof. We need only show that S is a Baer $*$-semigroup, since its completeness will then follow from Theorem 14. Let $a \in S$ with $a \neq 0$. We must show that there exists a projection $f = a' \in P(S)$ such that for all $y \in S$, $ay = 0 \Leftrightarrow y = fy$. Let $\{x_\alpha\}$ be a maximal orthogonal family of elements of $ZZ(a^*a)$ such that

$$0 \neq a^*ax_\alpha = e_\alpha \in P(S) \quad \text{and} \quad x_\alpha = e_\alpha x_\alpha e_\alpha = x_\alpha^* \qquad \text{for all } \alpha.$$

By Lemma 3, there exists at least one such x_α. Evidently, $\{e_\alpha\}$ is an orthogonal family of projections, so we can form the sum $e = \sum_\alpha e_\alpha$. Set $f = e'$.

If $ay = 0$, then $e_\alpha y = xa^*ay = 0$ for all α; hence, by Theorem 9, $ey = 0$ and $y = fy$. To show that $y = fy \Rightarrow ay = 0$, it is enough to show that $af = 0$. Suppose that $af \neq 0$, so that $(af)^*af \neq 0$ by $*$-cancellation. By Lemma 3 there exists a non-zero projection $k \in ZZ(fa^*af)$ and there exists $x = x^* \in ZZ(fa^*af)$ such that $fa^*afx = k$ and $x = kxk$. We have

$$x_\alpha x = x_\alpha e_\alpha kx = x_\alpha e_\alpha fa^*afx = x_\alpha e_\alpha efa^*afx = 0$$

for all α, since $ef = ee' = 0$. Thus, x is orthogonal to every x_α. Since every e_α belongs to $ZZ(a^*a)$, then $e \in ZZ(a^*a)$ by Corollary 10 and $f \in ZZ(a^*a)$ by part (iii) of Lemma 12. It follows that $fa^*af \in ZZ(a^*a)$, and hence that $x \in ZZ(a^*a)$. Since $x = kx = fkx = fx$, then $a^*ax = a^*afx = fa^*afx = k$; hence, x can be adjoined to the supposedly maximal family $\{x_\alpha\}$. This contradiction completes the proof.

Because of Theorem 15, all of the machinery of Baer *-semigroups is now at our disposal and we shall make use of this machinery in our subsequent study of the weak Loomis $*$-semigroup S.

LEMMA 16. *If* $a = \sum_\alpha a_\alpha$, *then* $a'' = \sum_\alpha (a_\alpha)''$.

Proof. By **(5**, Theorem 1, part (vii)**)** we see that for $x, y \in S$, $x \perp y \Leftrightarrow x'' \perp y''$ and $(x^*)'' \perp (y^*)''$. From this and the orthogonality of $\{a_\alpha\}$, it follows that $\{(a_\alpha)''\}$ is an orthogonal family. Let $e = \sum_\alpha (a_\alpha)''$. By Lemma 13, $e = \vee_\alpha (a_\alpha)''$, hence, $e' \leqslant (a_\alpha)'$ holds for every α. Consequently, $a_\alpha e' = a_\alpha(a_\alpha)'e' = 0$ for every α, so (Theorem 9) $ae' = 0$. On the other hand, if $ax = 0$, then $a_\alpha{}^*ax = a_\alpha{}^*a_\alpha x = 0$ and $(a_\alpha)'' = (a_\alpha{}^*a_\alpha)'' \leqslant (x^*)'$ holds for every α. It follows that $e \leqslant (x^*)'$, $(x^*)'' \leqslant e'$, $x^* = x^*e'$, $x = e'x$. These results imply that $a' = e'$, i.e., $a'' = e$.

5. The *-equivalence relation in weak Loomis *-semigroups. In the present section we maintain the convention that S is a weak Loomis $*$-semigroup. We shall show that the lattice $P(S)$, equipped with the equivalence relation $\sim_*$ of Definition 2, satisfies all conditions of Definition 1 for an orthomodular geometry except possibly for Condition (iv).

LEMMA 17. *Let* $x \in S$ *with* $xx^* = e \in P(S)$. *Then* $f = x^*x \in P(S)$ *and* x *is partially unitary with* $x{:}e \sim_* f$.

Proof. Since $f^2 = f^3 = f^4 = x^*ex$, we have $ff^* = f^2f^* = f^2(f^2)^*$; hence, $f = f^2$ by $*$-cancellation. Since $xx^* = (ex)x^* = (ex)(ex)^*$, we have $x = ex$ by $*$-cancellation. Similarly, $x = xf$, so $x = exf$ and $x{:}e \sim_* f$.

LEMMA 18. *Let* $e, f \in P(S)$ *with* $a{:}e \sim_* f$. *Then, for* $g \in P(S)$, $(ga)'' = [(g' \wedge e)a]' \wedge f$.

Proof. The proof follows immediately from **(5**, Theorem 12, p. 73**)** as soon as we note that a is $*$-regular in S and the relative inverse of a is a^*.

THEOREM 19. *Let* $\{e_\alpha\}$ *be an orthogonal family of projections in* S *and suppose that* $f \sim_* \vee_\alpha e_\alpha$, *where* f *is a projection in* S. *Then, there exists a decomposition* $f = \vee_\alpha f_\alpha$ *of* f *into orthogonal projections* $\{f_\alpha\}$ *such that* $f_\alpha \sim_* e_\alpha$ *for all* α.

Proof. Suppose that $a{:}e \sim_* f$, where $e = \vee_\alpha e_\alpha$. For each α, let $f_\alpha = (e_\alpha a)''$. By **(5**, Theorem 1, part (xvii)**)**, $\vee_\alpha f_\alpha = f$. By Lemma 18, $f_\alpha = [(e_\alpha' \wedge e)a]' \wedge f$. For $\beta \neq \alpha$, $e_\beta \leqslant e'_\alpha \wedge e$, so $f_\beta \leqslant [(e'_\alpha \wedge e)a]''$ and $f_\alpha \leqslant f_\beta'$. Let $a_\alpha = e_\alpha a$. Since

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$a_\alpha a_\alpha^* = e_\alpha e e_\alpha = e_\alpha$, then by Lemma 17, a_α is partially unitary with $a_\alpha{:}e_\alpha \sim_* (a_\alpha)'' = f_\alpha$.

THEOREM 20. *The relation of $*$-equivalence on $P(S)$ is completely additive, i.e., it satisfies condition* (iii) *of Definition* 1.

Proof. The proof is the same as the proof of (**7**, Theorem 10, p. 31).

The proof of the following theorem is suggested by Tarski's well-known proof of the Cantor-Schroeder-Bernstein theorem. The fixed-point theorem quoted in the proof can be found in (**1**, Theorem 8, p. 54).

THEOREM 21. *Let $e, f \in P(S)$ and let $a, b \in S$ be such that $(a^*)'' = e \geqslant b''$, $(b^*)'' = f \geqslant a''$. Then there exist projections $m, n \in P(S)$ such that*
(i) $m \leqslant e$, $n \leqslant f$,
(ii) $n = (ma)' \wedge f$, $m = (nb)' \wedge e$.

Proof. Let the mapping $\phi{:}P(S) \to P(S)$ be defined by

$$g\phi = e \wedge \{[f \wedge (ga)']b\}'$$

for $g \in P(S)$. Since ϕ is a monotone mapping and $P(S)$ is a complete lattice, there exists a projection $m \in P(S)$ such that

$$m = m\phi = e \wedge \{[(ma)' \wedge f]b\}'.$$

Set $n = (ma)' \wedge f$. Conditions (i) and (ii) now obtain.

THEOREM 22. *Let e, f be projections in $P(S)$. If e is $*$-equivalent to a subelement of f and if f is $*$-equivalent to a subelement of e, then $e \sim_* f$.*

Proof. Let $a{:}e \sim_* f_1 \leqslant f$, $b{:}f \sim_* e_1 \leqslant e$. By Theorem 21, there exist projections $m, n \in P(S)$ with $m \leqslant e$, $n \leqslant f$, $n = (ma)' \wedge f$, $m = (nb)' \wedge e$. Routine computation shows that $ma{:}m \sim_* (f \wedge n')$ and $b^*n{:}(e \wedge m') \sim_* n$. An application of Theorem 20 completes the proof.

We have already mentioned that the Baer $*$-semigroup S of two-by-two matrices over Z_3, equipped with the involution $X \to X^* =$ transpose of X, is a weak Loomis $*$-semigroup, but not a Loomis $*$-semigroup. One easily verifies that its lattice of projections $P(S)$, equipped with the equivalence relation $\sim_*$, does not satisfy Condition (iv) of Definition 1; hence it fails to be an orthomodular geometry.

We shall go on to prove that if S is a Loomis $*$-semigroup, i.e., if the (EP) property holds in S, then $P(S)$ is an orthomodular geometry under $*$-equivalence.

THEOREM 23. *Let S be a Loomis $*$-semigroup. Then, for $a \in S$, $a'' \sim_* (a^*)''$.*

Proof. We can assume $a \neq 0$. Let $\{x_\alpha\}$ be a maximal orthogonal family of elements of $ZZ(a^*a)$ such that

$$x_\alpha = x_\alpha^*, \qquad 0 \neq a^*ax_\alpha^2 = e_\alpha \in ZZ(a^*a) \qquad \text{and } (x_\alpha)'' = e_\alpha$$

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for all α. The existence of at least one such x_α follows from (EP) and the same argument given in Lemma 3 in connection with (WEP). Obviously, $\{e_\alpha\}$ is an orthogonal family of projections, so we can form $e = \sum_\alpha e_\alpha = \bigvee_\alpha e_\alpha$. Since, for every α, $e_\alpha = (x_\alpha{}^2 a^* a)'' \leqslant a''$, then $e \leqslant a''$. We claim that $e = a''$; for if not, then $a^*ae' = e'a^*ae' \neq 0$, and by (EP) there exists

$$x = x^* \in ZZ(e'a^*ae') \subset ZZ(a^*a)$$

such that $a^*ae'x^2 = f \neq 0, f \in P$, and $f = x''$. Clearly, $f \leqslant e'$, so x is orthogonal to every x_α and $a^*ax^2 = f$, contradicting the maximality of $\{x_\alpha\}$.

Put $w_\alpha = ax_\alpha$ for every α. Then for every α, $w_\alpha{}^* w_\alpha = a^*ax_\alpha{}^2 = e_\alpha$; hence, $w_\alpha: (w_\alpha{}^*)'' \sim_* e_\alpha$. Also, $(w_\alpha{}^*)'' = ax_\alpha{}^2a^*$ for every α, so for $\alpha \neq \beta$ we have

$$(w_\alpha{}^*)''(w_\beta{}^*)'' = ax_\alpha{}^2a^*ax_\beta{}^2a^* = aa^*ax_\alpha{}^2x_\beta{}^2a^* = 0.$$

Hence, the projections $\{(w_\alpha{}^*)''\}$ form an orthogonal family. By Theorem 20, $\bigvee_\alpha(w_\alpha{}^*)'' \sim_* e = a''$. Since, for every α, $(w_\alpha{}^*)'' = (x_\alpha a^*)'' \leqslant (a^*)''$, then $a'' \sim_* \bigvee_\alpha(w_\alpha{}^*)'' \leqslant (a^*)''$. Reversing the roles of a and a^*, we see that $(a^*)''$ is also $*$-equivalent to a subelement of a''. Theorem 22 can now be applied to complete the proof.

THEOREM 24. *If S is a Loomis $*$-semigroup, then the complete orthomodular lattice $P(S)$ of projections in S is an orthomodular geometry under the relation of $*$-equivalence.*

Proof. We have only to establish the validity of Condition (iv) of Definition 1. The proof in (**7**, Theorem 12, Corollary, p. 32) does the job.

6. The hull operation. In the present section the symbol S will always refer to a Loomis $*$-semigroup. Loomis makes extensive use of the so-called hull operation $e \to |e|$ in his study of dimension lattices (**7**, p. 13). In Theorem 25 we shall obtain an interesting formula for the hull in the orthomodular geometry $P(S)$.

Definition 12. If $e \in P(S)$, then the *hull* of e, in symbols $|e|$, is defined by

$$|e| = \bigvee\{f \in P(S) | \exists\, e_1 \leqslant e, f \sim_* e_1\}.$$

If L is any orthomodular lattice, then the *center* of L, in symbols $C(L)$, was defined in (**5**, p. 66). By (**7**, Theorem 2, p. 13), for $e \in P(S)$ we always have $|e| \in C(P(S))$. In addition, it is easy to verify that the hull operation has the following properties:

(i) $e \leqslant |e| \in C(P(S))$,

(ii) $|0| = 0$,

(iii) $|e \wedge |f|| = |e| \wedge |f|$ for $e, f \in L$,

(iv) the mapping $\eta: P(S) \to P(S)$ defined by $e\eta = |e|$ for $e \in P(S)$ is a projection in the Baer $*$-semigroup $S(P(S))$ defined in (**2**, p. 651).

THEOREM 25. *For $e \in P(S)$, we have $|e| = \bigvee\{(ex)'' | x \in S\}$.*

Proof. Put $g = \bigvee\{(ex)''|x \in S\}$. Let x be any element of S and use Theorem 23 to conclude that $(ex)'' \sim_* (x^*e)'' \leqslant e$. It follows that $(ex)'' \leqslant |e|$, and consequently that $g \leqslant |e|$. On the other hand, suppose that $f \in P(S)$ with $a{:}f \sim_* e_1 \leqslant e$. Using parts (iv) and (xiii) of **(5**, Theorem 1, p. 66**)** we have

$$a'' = (a^*a)'' = e_1'' = e_1 \leqslant e;$$

hence, $a = ae$, $a^* = ea^*$, and

$$f = f'' = (aa^*)'' = (a^*)'' = (ea^*)'' \leqslant g.$$

Consequently, $|e| \leqslant g$ and the proof is complete.

THEOREM 26. *For $e \in P(S)$, $|e| \in Z(S)$. In fact, for $e \in P(S)$, $|e| = \bigwedge\{f|f \in Z(S) \cap P(S)$ and $e \leqslant f\}$.*

Proof. By Theorem 25 we have $(ex)'' \leqslant |e|$ for every $x \in S$; hence, $ex = ex|e|$ for every $x \in S$. It follows that, for every $x \in S$, $|e|x = |e|x||e|| = |e|x|e|$. Taking $*$ on both sides of the latter equation and replacing x by x^* gives $x|e| = |e|x|e| = |e|x$. This shows that $|e| \in Z(S)$. Suppose that $f \in Z(S) \cap P(S)$. Using Theorem 25 we have $|f| = \bigvee\{(fx)''|x \in S\}$. But, since $f \in Z(S)$, $(fx)'' = (xf)'' \leqslant f$ for all $x \in S$; hence, $|f| \leqslant f, f = |f|$. Now, if $f \in Z(S) \cap P(S)$ with $e \leqslant f$, we have $|e| \leqslant |f| = f$; hence, $|e|$ is a lower bound for the set $\{f|f \in Z(S) \cap P(S), e \leqslant f\}$. Since $|e| \in Z(S) \cap P(S)$ and $e \leqslant |e|$, the theorem is proved.

If we compare Theorem 26 with **(6**, Definition 3, p. 5**)**, we see that the central cover, as defined by Kaplansky, of an element $a \in S$ is just the hull, as defined by Loomis, of the projection a''. In particular, then, we can adapt Kaplansky's proof of **(6**, Lemma, p. 30**)** to Loomis $*$-semigroups to prove the following:

THEOREM 27. *For $x, y \in S$, $xSy = 0$ if and only if $|x''|$ is orthogonal to $|y''|$.*

Using Theorem 27, and following the proof in **(6**, Lemma 1, p. 32**)**, we obtain the following result:

THEOREM 28. *If e is a projection in S, then the central projections in the subsemigroup eSe of S are those of the form eh with $h \in Z(S)$.*

7. Open questions. The results obtained in the preceding sections suggest many questions, most of which are open at present. In this concluding section we shall list some of the more provocative questions which thus arise.

1. Can every orthomodular geometry be co-ordinatized by a Loomis $*$-semigroup?
2. Can every Loomis $*$-semigroup be embedded in a suitable Baer $*$-ring?
3. Which orthomodular lattices can be co-ordinatized by Baer $*$-semigroups S satisfying the $*$-cancellation law?

4. Which orthomodular lattices can be co-ordinatized by Baer $*$-semigroups S in which all projections are closed?

5. If the orthomodular geometry L is modular, can it be co-ordinatized by a Loomis $*$-semigroup which is $*$-regular in the sense of (**5**, p. 71)?

6. Say that an orthomodular lattice L has the *relative centre property* in case for every $e \in L$ the centre of the interval sublattice $L(0, e)$ is the set of all elements of the form $e \wedge h$ with h in the centre of L. Using Theorem 28, it can be shown that if L can be co-ordinatized by a Loomis $*$-semigroup, then L has the relative centre property. Does every orthomodular geometry have the relative centre property?

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Wayne State University and the University of Florida

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