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GROWTH SEQUENCES OF FINITE GROUPS IV

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For B. H. Neumann on his Seventieth Birthday

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Abstract

The main result is that $d(S^{s^n}) = n+2$ for every finite non-abelian two-generator simple group S of order s and every integer $n \geqslant 0$. This is applied to give a very close estimate on $d(G^n)$ for any finite group G whose simple images are two-generator. The article is based on the author's previous papers with similar titles.

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In the three previous papers in this series, I have given a prescription for determining the minimum number $d(G^n)$ of generators of the nth direct power G^n of a finite group G. I wish to point out here just how tight the prescription is if one assumes that all (non-abelian) simple images of G are 2-generator. This will then give the picture one can expect if the famous conjecture about such things is true.

Firstly, the general set-up. Let G be any finite group, and $S_1, S_2, \ldots, S_r$ the different non-abelian simple groups (if any) that are images of G, and let $S_i^{\lambda_i}$ be the highest power of S_i that is an image of G. Then, by Wiegold (1975), and heavily based on Gaschütz (1955),

$$\text{(i)}\quad d(G^n) = \max(d(G), nd(G/G'), d(S_1^{\lambda_1 n}), \ldots, d(S_r^{\lambda_r n})),$$

with the obvious interpretation when $S_1, S_2, \ldots, S_r$ are absent.

Thus, all depends on the behaviour of the $d(S_i^{\lambda_i n})$. Since Hall (1936) has given a method for calculating the growth sequences of finite simple groups, the problem can be regarded as being completely solved. However, Hall's result depends in a fairly complicated way on knowledge of the Möbius function of the subgroup lattice of the group in question, and as such is not immediately accessible to the imagination. The following very simple consequence of Hall's theorem provides something a bit more digestible:

THEOREM. *Let S be a 2-generator non-abelian finite simple group of order s. Then $d(S^{s^n}) = n+2$ for all $n \geqslant 0$.*

PROOF. By Lemma 2 of Wiegold (1978), $d(S^{s^n}) \leqslant n+d(S) = n+2$. For the opposite inequality, let $\varphi(n+1)$ denote the number of generating $(n+1)$-vectors of S. Then $\varphi(n+1) < s^{n+1}$ since the trivial $(n+1)$-vector is not generating. Thus

$$\frac{\varphi(n+1)}{|\operatorname{Aut} S|} < s^n,$$

and so $d(S^{s^n}) > n+1$ by Hall (1936). Thus $d(S^{s^n}) = n+2$, as required.

We are really concerned with $d(S^{\lambda n})$, for fixed λ and arbitrary $n \geqslant 1$. Though we cannot pin it down with such accuracy as in the theorem, we can do nearly as well. Choose the positive integer m such that

$$s^{m-1} < \lambda n \leqslant s^m.$$

Then

$$m+1 = d(S^{s^{m-1}}) \leqslant d(S^{\lambda n}) \leqslant d(S^{s^m}) = m+2,$$

so that

$$\log_s n + \log_s \lambda + 1 \leqslant d(S^{\lambda n}) < \log_s n + \log_s \lambda + 3.$$

Thus $d(S^{\lambda n})$ must take one of two integer values near $\log_s n + \log_s \lambda + 2$. By looking at powers of A_5, or any other simple group where the growth sequence is known in detail, one sees easily that sometimes the lower of the two possibilities is correct, sometimes the higher. I thank Dr. C. H. Houghton for pointing out this possibility. Assuming now that G is a finite group with all its non-abelian simple images 2-generator, (1) tells us that $d(G^n)$ must be the largest of not more than $2r+2$ specified values. The applications to the perfect case ($d(G/G') = 0$) and the imperfect case ($d(G/G') > 0$) are obvious, and the reader is invited to compare these with the results of Wiegold (1978). What makes this note so much preferable is that we have obviated the difficulty of the precise nature of the growth sequence of a simple group for small n, which caused really all the significant problems in the three previous papers.

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