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Correction to 'Equivariant spectral decomposition for flows with a $\mathbb{Z}$-action'

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The statement and proof of the $\mathbb{Z}$-spectral decomposition theorem for a pseudo-Anosov flow ϕ on a 3-manifold M are in error (see [**1**]) There are counter-examples which show that the theorem as stated is false We remark that the results of § 9 of [**1**], concerning analogues of the $\mathbb{Z}$-spectral decomposition theorem for basic sets of Axiom *A* flows, are unaffected by this error

To correct the statement of the theorem, we shall define a 'dynamic blowup' of a singular periodic orbit of ϕ There are several possible ways to dynamically blow up a singular orbit, and we shall show how to parameterize them below Given $\alpha \in H^1(M, \mathbb{Z})$ as in the statement of the theorem, it will only be necessary to blow up those singular orbits γ such that $\langle \alpha, \gamma \rangle = 0$, we refer to such a γ as an α-null singular orbit After the first sentence of the theorem [**1**, p 334], insert the following

> There is a way to dynamically blow up each α-null singular orbit of ϕ, such that if $\phi^{\#}$ is the resulting flow, then the following hold

For the remainder of the theorem, replace the symbol ϕ with the symbol $\phi^{\#}$

The introduction to [**1**] also mis-states the main result of [**2**], concerning the existence of a surface transverse to ϕ and Poincaré dual to α, such a surface exists only after blowing up α-null singular orbits Also, these methods are not sufficient to settle Oertel's conjecture, although partial results can still be obtained (see [**2**])

First we define dynamic blowups in the context of pseudo-Anosov maps Let s be a singular fixed point of a pseudo-Anosov map $f\ S \to S$, and consider first the case where f does not rotate the separatrices To obtain a dynamic blowup of s, replace s by a finite set of pseudo-Anosov fixed points which are connected in a tree pattern by invariant paths Here is a more precise description Let D be a coordinate disc centered on s List the stable and unstable separatrices in circular order as $\{\ell_n\ n \in \mathbb{Z}/2N\}$, where $N \geq 3$ Let $p_n = \ell_n \cap \partial D$. Choose an embedded tree $T = T_s \subset D$, such that T intersects ∂D transversely in the set $\{p_n\}$, and every interior vertex of T is of even valence ≥ 4 Let $\ell_n^{\#}$ be the edge of T incident on p_n, and let $T^\circ = \mathrm{cl}\,(T - \bigcup \{\ell_n^{\#}\})$ With these conditions on T, the map f can be replaced by a map $f^{\#}$ which is semi-conjugate to f, by a semi-conjugacy $\rho\ S \to S$ which collapses T° to the point s, so that $f^{\#}$ has a prong singularity at each interior vertex of T, $f^{\#}$ leaves T° invariant, and each edge E of T° is an invariant path for $f^{\#}$, with $f^{\#}$ acting as a translation on $\mathrm{int}\,(E)$ We say that $f^{\#}$ is obtained by *dynamically blowing up s* The set $\{\ell_n\}$ is partitioned in such a way that ℓ_n and ℓ_m are in the same partition element if and only if $\ell_n^\# \cap \ell_m^\# \neq \varnothing$, the blowup is determined up to isotopy by the partition Not all partitions occur, it is a simple matter to describe which partitions are allowable Notice that the tree T is a directed graph, i e each edge E is naturally oriented according to the direction that points on E are moved under $f^\#$ For each interior vertex v of T, the edges incident on v point alternately toward and away from v, going around v in circular order

When f rotates the separatrices at s through a fraction K/N of a complete rotation, a dynamic blowup is similarly defined with the additional proviso that T is invariant under a K/N rotation of D

If γ is a singular periodic orbit of a pseudo-Anosov flow ϕ, a dynamic blowup of γ is defined as follows Choose a local cross-section near γ, having a pseudo-Anosov singular fixed point s, and choose a dynamic blowup of s by picking a tree T as above This can be suspended, to obtain a dynamic blowup of γ The result is determined up to conjugacy by a partition of the set whose elements are the stable and unstable manifolds of γ The effect is to introduce several annuli, each of which is invariant under the blown up flow $\phi^\#$, one annulus for each orbit of edges of T° under the rotation action

The mathematical error in the proof of the theorem first occurs in § 3 If ζ is a quasi-orbit, the intersection number $\langle \alpha, \zeta \rangle$ is assumed to take values in $\mathbb{Z}_{\geq} \cap \{+\infty\}$ This assumption is unjustified $\langle \alpha, \zeta \rangle$ takes values in $\mathbb{Z} \cup \{+\infty\}$ Counter-examples show that negative values can occur, in which case the theorem fails The error recurs in § 4, in which the terms of the generalized splice equation are assumed to take values in $\mathbb{Z}_{\geq} \cup \{+\infty\}$, rather than $\mathbb{Z} \cup \{+\infty\}$ The error is manifested in the following incorrect statement, from the proof of Lemma 4 2 'Note that each directed loop of Γ'_A corresponds uniquely to a symbolic quasi-loop $\underline{m}$ of Γ_A such that $0 \leq U_\alpha(\underline{m}) < +\infty$' The only restriction is $U_\alpha(\underline{m}) \in \mathbb{Z}$ The proof of Proposition 7 1 contains another manifestation of the error One effect of the error is that the invariant sets $L(\alpha)$, $R(\alpha)$, and $L^q(\alpha)$ as defined in the paper are inutile Theorem 3 8 is incorrect with these definitions Also, the auxiliary graph Γ'_A used in Lemma 4.2 is inutile We shall construct new auxiliary graphs Γ^1_A and Γ^2_A below, to take over various tasks previously performed by Γ'_A

The following corrections in the proof are needed First of all, recall that § 1 reduces to the case when ϕ is the suspension flow of a pseudo-Anosov map f which fixes all singularities and does not rotate the separatrices This reduction no longer seems necessary or appropriate, so we shall henceforth abandon it, and deal directly with a general pseudo-Anosov map f

For notational convenience, we shall drop the subscript A from the notation Γ_A, Γ^1_A and Γ^2_A, denoting these as Γ, Γ^1, and Γ^2

The contents of § 4 starting with Lemma 4 2 should be replaced with the following discussion, whose aim is to show how to choose the blowups needed to define $\phi^\#$, and to give the correct versions of $R(\alpha)$, $L(\alpha)$ and $L^q(\alpha)$

Let γ be an N-pronged α-null singular orbit Choose a point $s = s_\gamma \in \gamma \cap S$ Let $\{m_n \in \mathcal{M} \mid n \in \mathbb{Z}/2N\}$ be the list of Markov rectangles containing the point s, listed in circular order around s Choose a $2N$-pronged star Σ_γ, and glue the endpoints

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$\{v_n \mid n \in \mathbb{Z}/2N\}$ of Σ_γ in a 1-1 manner to the vertices $\{m_n \mid n \in \mathbb{Z}/2N\}$ of the digraph Γ Doing this for each α-null singular orbit γ, we obtain a graph Γ^1, having Γ as a subgraph Although Γ is a directed graph, Γ^1 is not, since no orientations are assigned to the edges of each star Σ_γ A closed, oriented edge loop L in Γ^1 is *semi-directed* if it passes over each directed edge of Γ in the positive sense Each semi-directed loop L of Γ^1 determines in a natural manner a symbolic quasi-loop of Γ, and thus a periodic quasi-orbit denoted $O(L)$ The generalized splice equation holds for semi-directed loops, and from this it easily follows that U_α extends to a cohomology class on Γ^1, denoted U^1_α The non-negative cocycle u_α constructed in proposition 3 2 can then be extended to a cocycle u^1_α on Γ^1 representing U^1_α

For each α-null singular orbit γ, we now specify how γ is to be blown up Choose a zero-dimensional-cochain f_γ on Σ_γ whose coboundary is $u^1_\alpha|\Sigma_\gamma$ Note that f_γ is well-defined on the endpoints $\{v_n\}$ of Σ_γ, up to an additive constant Let $\{\ell_n \mid n \in \mathbb{Z}/2N\}$ be the list of separatrices at $s = s_\gamma$ as above, rotated by ϕ through K/N of a complete rotation Choose the notation so that ℓ_n is stable when n is even and unstable when n is odd Choose a small coordinate disc $D = D_\gamma \subset S$ centered on s, such that the sector of D between ℓ_n and ℓ_{n+1} is contained in the Markov rectangle m_n Let $y_n = \partial D \cap \ell_n$ Choose a point x_n contained in the interior of the arc $[y_n, y_{n+1}]$ of ∂D Let F_γ $\{x_n\} \to \mathbb{Z}$ be defined by $F_\gamma(x_n) = f_\gamma(v_n)$ Since $\langle \alpha, \gamma \rangle = 0$, it is easy to check that f_γ is rotationally invariant, under a K/N rotation on $\{v_n\}$ Thus, F_γ can be extended to a K/N rotationally invariant real-valued continuous function on ∂D, still denoted F_γ, such that on the arc $[y_n, y_{n+1}]$, if n is even then F_γ is increasing, and if n is odd then F_γ is decreasing Thus, at the point y_n, F_γ has a local minimum on ∂D if n is even and a local maximum if n is odd Collapse ∂D to a K/N rotationally invariant tree $T_\gamma \subset D$, with endpoint set $\{y_n\}$, in such a way that the following conditions are satisfied

(i) two points on ∂D are identified only if they have the same F_γ value,

(ii) for each n, the shorter of the two intervals $[x_{n-1}, y_n]$, $[y_n, x_n]$ is identified with a sub-interval of the other

The tree T_γ can be used in the definition of a dynamic blowup of γ Note that by (i), F_γ induces a function on T_γ, still denoted F_γ The orientation on each edge of T_γ agrees with the direction of the gradient of F_γ

Applying the construction in the previous paragraph to each α-null singular orbit γ, we have defined the blown up flow $\phi^\#$ The suspension of T°_γ is a union of invariant annuli of $\phi^\#$, denoted Susp (T°_γ) The semi-conjugacy ρ $M \to M$ from $\phi^\#$ to ϕ collapses all invariant annuli, and takes each quasi-orbit $\zeta^\#$ of $\phi^\#$ to a quasi-orbit $\zeta = \rho(\zeta^\#)$ of ϕ, preserving $\langle \alpha, \ \rangle$ We must prove that $\langle \alpha, \zeta^\# \rangle \in \mathbb{Z}_\geq \cap \{+\infty\}$ for each quasi-orbit $\zeta^\#$ of $\phi^\#$ To do this, we must study some properties of $\phi^\#$

We introduce a new auxiliary graph Γ^2, which will be a directed graph Consider an α-null singular orbit γ Adopting the notation above, let $\bar{x}_n, \bar{y}_n \in T_\gamma$ be the images of x_n, y_n under the collapsing $\partial D_\gamma \to T_\gamma$ Of the two points $\bar{x}_{n-1}$, $\bar{x}_n$, let z_n be the one closest to $\bar{y}_n$ Let $\bar{T}_\gamma$ be the smallest sub-tree of T_γ containing each $\bar{x}_n$, or equivalently the smallest sub-tree containing each z_n Notice that $T^\circ_\gamma \subset \bar{T}_\gamma$ Glue $\bar{T}_\gamma$ to Γ by *identifying* $\bar{x}_n$ *with the vertex* m_n *of* Γ, *this may result in identification of* vertices of Γ Doing this for each α-null orbit γ, the result is a directed graph denoted Γ^2 By construction, the 1-cocycle u_α on Γ and the 1-cocycle δF_γ on $\bar{T}_\gamma$ combine to yield a non-negative 1-cocycle u^2_α on Γ^2_α, and u^2_α is positive on each directed edge of $\bar{T}_\gamma$

Each directed loop L in Γ^2 determines a periodic quasi-orbit $O(L)$ of $\phi^\#$ such that $u^2_\alpha(L) = \langle \alpha, O(L) \rangle$, as follows Each portion of L restricted to Γ determines an orbit of $\phi^\#$ which is not contained in any invariant annulus, each portion of L restricted to $\bar{T}_\gamma$ determines a sequence of orbits in Susp(T°_γ) These orbits piece together to give $O(L)$

Conversely, we must show that for each periodic quasi-orbit $\zeta^\#$, either $\langle \alpha, \zeta^\# \rangle = +\infty$, or there is a directed loop L in Γ^2 with $\zeta^\# = O(L)$, for then it will follow that $\langle \alpha, \zeta^\# \rangle = u^2_\alpha(L) \geq 0$ We shall give the argument in the case where ϕ does not permute separatrices, the other case is left to the reader Assume $\langle \alpha, \zeta^\# \rangle < +\infty$ Let $\zeta^\# = (\zeta^\#_k)_{k \in \mathbf{Z}/J}$, and consider $\zeta^\#_k$ not contained in any invariant annulus $\rho(\zeta^\#_k)$ approaches some α-null singular orbit γ in positive time Consider the coordinate disc $D = D_\gamma$ around $s = s_\gamma \in \gamma \cap S$ The point set $\rho(\zeta^\#_k) \cap D$ accumulates on s along some stable separatrix ℓ_n In constructing a symbolic path L_k in Γ for the orbit $\zeta^\#_k$, as L_k approaches $+\infty$ there are two possibilities L_k will cycle infinitely around a loop in Γ representing γ, and this loop will pass through either the symbol m_{n-1} or the symbol m_n, since these are the two Markov rectangles incident on s and ℓ_n In the tree $\bar{T}_\gamma$, at least one of the two points $\bar{x}_{n-1}, \bar{x}_n$ is identified with z_n Choose L_k to cycle through m_{n-1} if $\bar{x}_{n-1} = z_n$, and to cycle through m_n if $\bar{x}_n = z_n$ In either case, under the identification map $\Gamma \to \Gamma^2$, L_k should then be truncated at z_n A similar construction is made for the negative direction of L_k Do this for each $\zeta^\#_k$ not contained in an invariant annulus of ζ Each remaining portion of $\zeta^\#$ is contained in Susp(T°_γ) for some α-null orbit γ, and consists of a sequence of orbits $\zeta^\#_{k+1}, \ldots, \zeta^\#_{k'-1}$, yielding a directed path $E_{k+1}, \ldots, E_{k'-1}$ in T°_γ Note that L_k ends at some vertex $z_n \in \bar{T}_\gamma$, and $L_{k'}$ starts at some other vertex $z_{n'} \in \bar{T}_\gamma$ Condition (ii) in the construction of T_γ guarantees that the edge-path $\mathscr{E}$ from z_n to z_n in $\bar{T}_\gamma$ intersects T°_γ in the directed path $E_{k+1}, \ldots, E_{k'-1}$ Thus, $\mathscr{E}$ is directed Now concatenate $\mathscr{E}$ between L_k and $L_{k'}$. Doing this for each appropriate portion of $\zeta^\#$ results in the desired directed loop in Γ^2 representing $\zeta^\#$

Now we say how to define the sets $R(\alpha)$, $L(\alpha)$, and $L^q(\alpha)$, which are invariant sets of $\phi^\#$ $R(\alpha)$ is defined as the chain kernel of $\phi^\#$ with respect to α, i e the set of all points x such that for all ε, T, there exists an ε, T cycle X through x such that $\langle \alpha, X \rangle = 0$ $L(\alpha)$ is defined as the closure of all periodic orbits γ of $\phi^\#$ such that $\langle \alpha, \gamma \rangle = 0$ $L^q(\alpha)$ is defined as the closure of all quasi-periodic orbits ζ of $\phi^\#$ such that $\langle \alpha, \zeta \rangle = 0$ Observe that $L(\alpha)$ and $L^q(\alpha)$ do not intersect the interior of any invariant annulus of $\phi^\#$ This is a consequence of the fact that u^2_α is positive on each directed edge of $\bar{T}_\gamma$, for each α-null singular orbit γ

The statement of Proposition 3 7 is true with the new definition of $L(\alpha)$ An analogue of Proposition 4 3 holds, characterizing the subgraph of Γ^2 which is the union of all simple loops L for which $u^2(L) = 0$ Proposition 4 7 is proven exactly as before

The pseudo-Anosov shadowing theory presented in § 5 needs the following changes After proving Lemma 5 1 *Visitors enter and leave through corridors*, an addendum to the lemma needs to be proven for the invariant tree T_s constructed by blowing up a singular fixed point s of a pseudo-Anosov map The addendum says that if ε is small enough in terms of the diameter of T_s, then in an appropriately constructed neighbourhood $N(T_s)$, an ε-chain which visits $N(T_s)$ enters through the stable corridor corresponding to some endpoint v_0 of T_s, and leaves through the unstable corridor corresponding to some endpoint v_1 of T_s, and there is a directed path in T_s leading from v_0 to v_1 Using this addendum, a version of Lemma 5 3, general pseudo-Anosov shadowing, should be proven for $f^{\#}$, stating that arbitrary chains of $f^{\#}$ are shadowed by quasi-orbits, and stating the appropriate version of uniqueness The remainder of § 5 is unchanged, and in particular we have recovered the proof of Theorem 3 8, that $R(\alpha) = L^q(\alpha)$

To adapt the construction given in § 6 of an isolating block N for $R(\alpha)$, as before one starts with a pseudo-Markov partition $\mathcal{M}^p$ for f such that for each $P \in \mathcal{M}^p$, and for each α-null periodic orbit γ of ϕ, $\gamma \cap P \subset \operatorname{int}(P)$ In particular, if γ is an n-pronged singular orbit and $\gamma \cap P \neq \phi$, then P is a $2n$-gon In § 6, we produced a certain subset $\mathcal{M}^p(\alpha) \subset \mathcal{M}^p$, which was a pseudo-Markov partition for the invariant set $R(\alpha) \cap S$ In the present context we must follow a more involved procedure in order to obtain a pseudo-Markov partition $\mathcal{M}^p(\alpha)$ for $R(\alpha) \cap S$ Consider a $2n$-gon $P \in \mathcal{M}^p$, $n \geq 3$, with n-pronged singular point $x \in P$ P decomposes into $2n$ quadrants, each bounded by one stable and one unstable separatrix For each quadrant $Q \subset P$, consider $\hat{Q} = \operatorname{cl}(\rho^{-1}(\operatorname{int}(Q)))$ Observe that if $\hat{Q}$ intersects the interior of an invariant path of $f^{\#}$, then $\operatorname{int}(\hat{Q})$ is disjoint from $R(\alpha)$, this follows from the fact that $R(\alpha)$ is disjoint from the interior of each invariant annulus of $\phi^{\#}$, together with a simple splicing argument Thus, for each k-pronged periodic point s of $f^{\#}$ in $\rho^{-1}(P)$ obtained from the blowup of x, there is a Markov $2k$-gon $P_s \subset \rho^{-1}(P)$ containing s, such that if $s \neq s'$ then $P_s \cap P_{s'} = \varnothing$, and $\cup_s\{P_s\} \supset R(\alpha) \cap \rho^{-1}(P)$ Hence we obtain a pseudo-Markov partition $\mathcal{M}^p(\alpha)$ for $R(\alpha) \cap S$, as follows For each $2n$-gon $P \in \mathcal{M}^p$ with $n \geq 3$, and for each singularity s of $f^{\#}$ in $\rho^{-1}(P)$, P_s is an element of $\mathcal{M}^p(\alpha)$ And for each rectangular $P \in \mathcal{M}^p$ such that $R(\alpha) \cap \rho^{-1}(P) \neq \varnothing$, $\rho^{-1}(P)$ is an element of $\mathcal{M}^p(\alpha)$ The isolating block N for $R(\alpha)$ can now be constructed from $\mathcal{M}^p(\alpha)$ exactly as in § 6

In § 7, the shadowing proof of Proposition 7 1 goes through as stated, with $\phi^{\#}$ in place of ϕ

The proof of property (E) in § 8 is as before, except that in the final paragraph of the proof, the graph Γ^2 and the class $U^2 \in H^1(\Gamma^2, \mathbb{Z})$ are used, in place of Γ^p and $U^p \in H^1(\Gamma^p, \mathbb{Z})$

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