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NONINNER AUTOMORPHISMS OF ORDER p IN FINITE p-GROUPS OF COCLASS 2, WHEN $p > 2$

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Abstract

It is shown that if G is a finite p-group of coclass 2 with $p > 2$, then G has a noninner automorphism of order p.

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1. Introduction

Let G be a finite nonabelian p-group. A longstanding conjecture asserts that G possesses at least one noninner automorphism of order p (see [13, Problem 4.13]). This is a sharpened version of a celebrated theorem of Gaschütz [9] which states that finite nonabelian p-groups have noninner automorphisms of p-power order. By a result of Deaconescu and Silberberg [7], if a p-group G satisfies $C_G(Z(\Phi(G))) \neq \Phi(G)$, then G admits a noninner automorphism of order p leaving $\Phi(G)$ elementwise fixed. However, the conjecture is still open. Various attempts have been made to find noninner automorphisms of order p in some classes of finite p-groups (see [2, 6, 7, 10, 16, 17]). In particular, the conjecture has been proved for finite p-groups of class 2, class 3 and of maximal class (see [1, 3, 12] and [17, Corollary 2.7]). In this paper, in light of the importance of classifying p-groups by coclass, we restrict our attention to p-groups with a certain coclass. The notion of coclass was introduced by Leedham-Green and Newman [11] and other authors have since investigated this topic (see for example [8, 14, 15]). In this paper we show the validity of the conjecture when G is a finite p-group of coclass 2 with $p > 2$ (see Theorem 2.5). Note that the nilpotency coclass of a p-group of order p^n is $n - c$, where c is the nilpotency class of G.

Throughout this paper the following notation is used. Let N be a normal subgroup of a group G. Then $\mathrm{Aut}^N(G)$ denotes the group of all automorphisms of G normalising N and centralising G/N, and $\mathrm{Aut}_N(G)$ denotes the group of all automorphisms of G

centralising N. Moreover $\mathrm{Aut}_N^N(G) = \mathrm{Aut}^N(G) \cap \mathrm{Aut}_N(G)$. All central automorphisms of G are denoted by $\mathrm{Aut}_c(G)$. The terms of the upper central series of G are denoted by $Z_i(G)$; note that $Z_1(G) = Z(G)$. Also, the terms of the lower central series of G are denoted by $\Gamma_i(G)$. The group of all derivations from G/N to $Z(N)$ is denoted by $Z^1(G/N, Z(N))$, where G/N acts on $Z(N)$ as $a^{Ng} = a^g$ for all $a \in Z(N)$ and $g \in G$. We use the notation $x \equiv y \pmod H$ to indicate that $Hx = Hy$, where H is a subgroup of a group G and $x, y \in G$. The minimal number of generators of G is denoted by $d(G)$ and C_n is the cyclic group of order n. All unexplained notation is standard. Also a nonabelian group G that has no nontrivial abelian direct factor is said to be *purely nonabelian*.

2. The main result

In this section, we prove that if G is a p-group of order p^n $(p > 2)$ and coclass 2, then G has a noninner automorphism of order p. To prove this, we find two noncentral automorphisms of order p and we show that one of these automorphisms is noninner. Moreover, to define these automorphisms we use derivations. First we may assume that $n \geq 7$ by [6] for $p > 3$, and for $p = 3$ by using GAP [18] we see that all groups of order 3^m for $m < 7$ have a noninner automorphism of order 3. Moreover, we have the following upper central series for G since G is of coclass 2:

$$1 < Z_1(G) < Z_2(G) < \cdots < Z_{n-3}(G) < G,$$

which indicates that $p^{n-3} \leq |Z_{n-3}(G)| \leq p^{n-2}$, $p \leq |Z(G)| \leq p^2$ and $p^2 \leq |Z_2(G)| \leq p^3$. We note that $C_G(Z(\Phi(G))) = \Phi(G)$ by [7]. Now by [17, Theorem (2)], if $Z_2(G)/Z(G)$ is cyclic, then G has a noninner automorphism of order p. Therefore, we may assume that $|Z(G)| = p$, $|Z_2(G)| = p^3$ and $Z_2(G)/Z(G) \cong C_p \times C_p$. Also by [17, Theorem (3)], we deduce that $Z_2(G) \leq Z(\Phi(G))$ and $d(G) = 2$ since in other cases G has a noninner automorphism of order p. Now by the above observation we state the following lemma and we use the assumption and notation of it throughout the paper.

Lemma 2.1. *Assume that G is a group of order p^n $(n \geq 7, p > 2)$ and coclass 2 with $|Z(G)| = p$, $Z_2(G)/Z(G) \cong C_p \times C_p$, $Z_2(G) \leq Z(\Phi(G))$ and $d(G) = 2$. Then:*

(i) *G is purely nonabelian;*
(ii) *$|Z_i(G)| = p^{i+1}$ for $2 \leq i \leq n-3$ and $Z_{n-3}(G) = \Phi(G)$;*
(iii) *$\exp(G/Z_{n-4}(G)) = p$;*
(iv) *$|\mathrm{Aut}_c(G)| = p^2$ and $\mathrm{Aut}_c(G) \leq \mathrm{Inn}(G)$;*
(v) *there exists a normal subgroup N of G such that $N < Z_2(G)$, $N \cong C_p \times C_p$ and $C_G(N)$ is a maximal subgroup of G.*

Proof. (i) and (ii) are obvious.

(iii) We set $G_1 = G/Z_2(G)$ and $G_2 = G_1/\Gamma_4(G_1)$. Then G_1 and G_2 are both of maximal class having orders p^{n-3} and p^4, respectively. Since $p + 1 \geq 4$ it follows that $\exp(G_2/\Gamma_3(G_2)) = p$ by [5, Theorem 3.2]. However, since $\Gamma_3(G_2) = \Gamma_3(G_1)/\Gamma_4(G_1)$,

$$G_2/\Gamma_3(G_2) \cong G_1/\Gamma_3(G_1) = (G/Z_2(G))/(Z_{n-4}(G)/Z_2(G)),$$

completing the proof.

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(iv) This follows from (i), [4, Theorem 1] and the fact that $\mathrm{Aut}_c(G) \cap \mathrm{Inn}(G) = Z(\mathrm{Inn}(G))$.

(v) We see that $Z_2(G)$ is a noncyclic abelian group of order p^3. If $Z_2(G) \cong C_p \times C_p \times C_p$ then we may choose N such that $N/Z(G)$ is a subgroup of order p in $Z_2(G)/Z(G)$ and we set $N = \Omega_1(Z_2(G))$ if $Z_2(G) \cong C_p^2 \times C_p$. Moreover, $G/C_G(N) \hookrightarrow GL(2, p)$, which completes the proof. □

Lemma 2.2. *Assume the same hypotheses as in Lemma 2.1. If $b \in G \setminus C_G(N)$, $a \in C_G(N) \setminus \Phi(G)$ and $w \in N \setminus Z(G)$, then:*

(i) $G = \langle a, b\rangle$;

(ii) $[a^r, b^s] \equiv [a, b]^{rs} \pmod{Z_{n-4}(G)}$, *where r and s are integers;*

(iii) *the map α defined by $\alpha(Nfa^jb^i) = w^i[w, b]^{i(i-1)/2}$ is a derivation from G/N to N, where $f \in \Phi(G)$ and $i, j \in \mathbb{Z}$;*

(iv) *the map β defined by $\beta(Nx[a, b]^ta^jb^i) = w^j[w, b]^{ij+t}$ is a derivation from G/N to N, where $x \in Z_{n-4}(G)$ and $i, j, t \in \mathbb{Z}$.*

Proof. (i) This is clear.

(ii) First assume that r and s are positive. By using induction on r we see that $[a^r, b] \equiv [a, b]^r \pmod{Z_{n-4}(G)}$ since $[a^{r+1}, b] = [a^r, [a, b]^{-1}][a, b][a^r, b]$ and $[a^r, [a, b]^{-1}] \in Z_{n-4}(G)$. Hence, using induction on s, $[a^r, b^s] \equiv [a, b]^{rs} \pmod{Z_{n-4}(G)}$. The rest follows from $[a^{-r}, b^{-s}] = [b^{-s}a^{-r}, [a^r, b^s]^{-1}][a^r, b^s]$.

(iii) Since $|G/C_G(N)| = |C_G(N)/\Phi(G)| = p$, any element of G can be written as fa^jb^i, where $f \in \Phi(G)$ and $i, j \in \mathbb{Z}$. First we prove that α is well defined. To see this, let $g_1 = f_1a^{j_1}b^{i_1}$ and $g_2 = f_2a^{j_2}b^{i_2}$. If $Ng_1 = Ng_2$, then $C_G(N)g_1 = C_G(N)g_2$ and so $b^{i_2-i_1} \in C_G(N)$ which implies that $i_2 = i_1 + kp$ for some $k \in \mathbb{Z}$. Therefore, $\alpha(Ng_2) = \alpha(Ng_1)$ since $|w| = |[w, b]| = p$ and p is odd. Now we have $\alpha(Ng_1)^{g_2}\alpha(Ng_2) = (w^{i_1})^{b^{i_2}}w^{i_2}[w, b]^{i_1(i_1-1)+i_2(i_2-1)/2}$ and $(w^{i_1})^{b^{i_2}} = w^{i_1}[w, b]^{i_1i_2}$ since $f_2a^{j_2} \in C_G(N)$ and $[w, b] \in Z(G)$. Hence $\alpha(Ng_1)^{g_2}\alpha(Ng_2) = w^{i_1+i_2}[w, b]^{(i_1+i_2-1)(i_1+i_2)/2}$. Moreover, $g_1g_2 \equiv a^{j_1+j_2}b^{i_1+i_2} \pmod{\Phi(G)}$ which completes the proof.

(iv) Since $|\Phi(G)/Z_{n-4}(G)| = p$ and $[a, b] \in \Phi(G) \setminus Z_{n-4}(G)$, any element of G can be expressed as $x[a, b]^ta^jb^i$, where $x \in Z_{n-4}(G)$ and $i, j, t \in \mathbb{Z}$. First we prove that β is well defined. To see this let $g_1 = x_1[a, b]^{t_1}a^{j_1}b^{i_1}$ and $g_2 = x_2[a, b]^{t_2}a^{j_2}b^{i_2}$. If $Ng_1 = Ng_2$, then $b^{i_2-i_1} \in C_G(N)$ and so $i_2 = i_1 + kp$ for some $k \in \mathbb{Z}$. This implies that $j_2 = j_1 + \ell p$ for some $\ell \in \mathbb{Z}$ since $\Phi(G)g_1 = \Phi(G)g_2$. Therefore, we see that $t_2 = t_1 + up$ for some $u \in \mathbb{Z}$ by the fact that $Z_{n-4}(G)g_1 = Z_{n-4}(G)g_2$ and Lemma 2.1(iii), which states that $\exp(G/Z_{n-4}(G)) = p$. Hence β is well defined. Now we have $\beta(Ng_1)^{g_2}\beta(Ng_2) = w^{j_1+j_2}[w, b]^{i_1j_1+t_1+i_2j_2+t_2+j_1i_2}$ since $x_2[a, b]^{t_2}a^{j_2} \in C_G(N)$ and $[w, b] \in Z(G)$. Moreover, $Z(G/Z_{n-4}(G)) = \Phi(G)/Z_{n-4}(G)$ by Lemma 2.1(ii), which yields that $g_1g_2 \equiv [a, b]^{t_1+t_2}a^{j_1}b^{i_1}a^{j_2}b^{i_2} \pmod{Z_{n-4}(G)}$. Furthermore, $a^{j_1}b^{i_1}a^{j_2}b^{i_2} = a^{j_1+j_2}[a^{j_2}, b^{-i_1}]b^{i_1+i_2}$. Therefore, $g_1g_2 \equiv [a, b]^{t_1+t_2-i_1j_2}a^{j_1+j_2}b^{i_1+i_2} \pmod{Z_{n-4}(G)}$ by (ii). Consequently, $\beta(Ng_1Ng_2) = \beta(Ng_1)^{g_2}\beta(Ng_2)$. □

We use the following theorem to complete the proof of Theorem 2.5.

Theorem 2.3. *Suppose that N is a normal subgroup of a group G. Then there is a natural isomorphism $\varphi : Z^1(G/N, Z(N)) \to \mathrm{Aut}_N^N(G)$ given by $g^{\varphi(\gamma)} = g\gamma(Ng)$ for $g \in G$ and $\gamma \in Z^1(G/N, Z(N))$.*

Proof. See for example [16, Result 1.1]. □

Corollary 2.4. *With the assumptions of Lemma 2.2, the maps α^* and β^* defined by $a^{\alpha^*} = a$, $b^{\alpha^*} = bw$ and $a^{\beta^*} = aw$, $b^{\beta^*} = b$ are noncentral automorphisms of order p lying in $\mathrm{Aut}_N^N(G)$.*

Proof. This is obvious by Lemma 2.2 and Theorem 2.3. □

Now we give our main theorem.

Theorem 2.5. *Let G be a finite p-group of coclass 2 with $p > 2$. Then G has a noninner automorphism of order p. Moreover, this noninner automorphism leaves either $\Phi(G)$ or $Z_{n-4}(G)$ fixed elementwise when $n \geq 7$.*

Proof. First we may assume that $n \geq 7$ by [6] for $p > 3$. Also, for $p = 3$ by using GAP [18] we see that all groups of order 3^m for $m < 7$ have a noninner automorphism of order 3. Moreover, we may assume that G satisfies the hypotheses of Lemma 2.1 according to the theorem stated in [17, Introduction]. We have $\mathrm{Aut}_c(G) \leq \mathrm{Aut}_N^N(G)$ since, if $\gamma \in \mathrm{Aut}_c(G)$, then γ is the inner automorphism induced by g for some $g \in G$ by Lemma 2.1(iv), which implies that $g \in Z_2(G)$ and so $\gamma \in \mathrm{Aut}_N(G)$. Therefore, $\mathrm{Aut}_c(G) \leq \mathrm{Aut}_N^N(G) \cap \mathrm{Inn}(G) \leq \mathrm{Aut}^{Z_2(G)}(G) \cap \mathrm{Inn}(G) \cong Z_3(G)/Z(G)$. Hence by Corollary 2.4, if $\alpha^* \in \mathrm{Inn}(G)$ then $\mathrm{Aut}_N^N(G) \cap \mathrm{Inn}(G) = \mathrm{Aut}_c(G)\langle\alpha^*\rangle$. Moreover, if $\beta^* \in \mathrm{Inn}(G)$ then $\beta^* \in \mathrm{Aut}_c(G)\langle\alpha^*\rangle$, which is impossible, by considering the image of β^* on a. Therefore, α^* or β^* is noninner. Furthermore, by Lemma 2.2(iii) and (iv) we see that α^* leaves $\Phi(G)$ and β^* leaves $Z_{n-4}(G)$ fixed elementwise, as desired. □

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