ON THE SQUARE OF A HOMOLOGICAL MONOID

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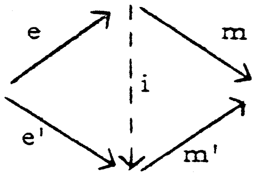
Homological monoids, as first defined by Hilton and Ledermann [1], are a generalization of abelian categories. It is known that if $\mathcal{Q}$ is an abelian category, so is $\mathcal{Q}^2$; here we prove the more general theorem that if $\mathcal{Q}$ is a homological monoid, so is $\mathcal{Q}^2$. Our definition differs from that originally given by Hilton and Ledermann by the addition of a uniqueness condition in Axiom 1.

If $\mathcal{Q}$ is any category, a map m of $\mathcal{Q}$ is called mono if $m \circ x = m \circ y$ implies $x = y$; a map e is called epi if $x \circ e = y \circ e$ implies $x = y$. A map with an inverse is called an isomorphism.

A category $\mathcal{Q}$ is called a homological monoid if it satisfies the following axioms:

(A0) $\mathcal{Q}$ has a zero object; i.e., there exists an object 0 of $\mathcal{Q}$ such that for each object A of $\mathcal{Q}$ there exists a unique map $0 \to A$ and a unique map $A \to 0$.

(A1) Every map f of $\mathcal{Q}$ can be written $f = m \circ e$, where m is mono and e is epi, and such a representation is unique up to isomorphism; i.e., if $f = m \circ e$ and also $f = m' \circ e'$ then there exists an isomorphism i such that $e' = i \circ e$ and $m = m' \circ i$.



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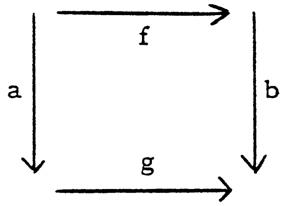
(A2) Every map f of $\mathcal{A}$ has a kernel and a cokernel; i.e., if f is any map of $\mathcal{A}$, there exists a mono k, called a kernel of f, such that $f \circ k = 0$, and $f \circ x = 0$ implies $x = k \circ y$ for some y; dually we have a cokernel c of f for any f.

(A3) If e is a normal epi (i.e., $e = \text{cok ker } e$) and m is a normal mono (i.e., $m = \text{ker cok } m$) then $e \circ m$ is normal; i.e., $e \circ m = m' \circ e'$ where m' is a normal mono and e' is a normal epi.

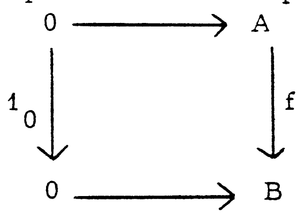
Given a category $\mathcal{A}$, we define a new category $\mathcal{A}^2$ whose objects are the maps of $\mathcal{A}$ and whose maps are commutative squares; i.e., a map from an object a of $\mathcal{A}^2$ to an object b of $\mathcal{A}^2$ is a pair (f, g) of maps of $\mathcal{A}$ such that the square commutes: $b \circ f = g \circ a$.



It can be seen that the composition of two commutative squares yields another commutative square, and in fact $\mathcal{A}^2$ is a category.

We now prove that if $\mathcal{A}$ is a homological monoid, so is $\mathcal{A}^2$.

(A0) We claim that $1_0 : 0 \to 0$ is a zero object of $\mathcal{A}^2$.
If $f : A \to B$ is any other object of $\mathcal{A}^2$, we know that there are unique maps $0 \to A$ and $0 \to B$ and the following square commutes, by the uniqueness of the map $0 \to B$.



Thus we have a unique map from 1_0 to any object f of $\mathcal{A}^2$ and similarly a unique map from any f to 1_0. Hence 1_0 is a zero object of $\mathcal{A}^2$.

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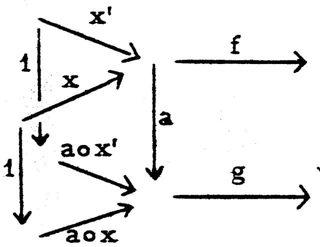
(A1) In order to prove that this axiom holds in $\mathcal{Q}^2$, we must first investigate which maps of $\mathcal{Q}^2$ are mono and which are epi. We claim that $(f, g) : a \to b$ in $\mathcal{Q}^2$ is mono if and only if both f and g are mono in $\mathcal{Q}$.

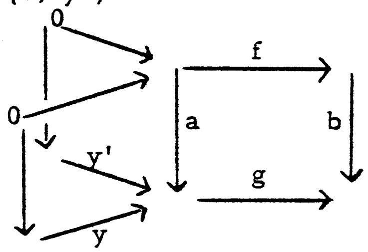
For, let f and g be mono in $\mathcal{Q}$, and let $(f, g)\circ(x, y) = (f, g)\circ(x', y')$. Then $(f\circ x, g\circ y) = (f\circ x', g\circ y')$; i.e., $f\circ x = f\circ x'$ and $g\circ y = g\circ y'$. Therefore $x = x'$ and $y = y'$, hence $(x, y) = (x', y')$. Therefore (f, g) is mono in $\mathcal{Q}^2$.

Conversely, suppose (f, g) is mono in $\mathcal{Q}^2$, and let $f\circ x = f\circ x'$. Then in $\mathcal{Q}^2$, $(x, a\circ x)$ is a map $1 \to a$ and $(x', a\circ x')$ is a map $1 \to a$.



Here $(f, g)\circ(x, a\circ x) = (f\circ x, g\circ a\circ x) = (f\circ x, b\circ f\circ x) = (f\circ x', b\circ f\circ x') = (f\circ x', g\circ a\circ x') = (f, g)\circ(x', a\circ x')$. Since (f, g) is mono, we conclude $(x, a\circ x) = (x', a\circ x')$; i.e., $x = x'$. Therefore f is mono.

To see that g is mono, suppose $g\circ y = g\circ y'$. Then $(0, y) : 0 \to a$ and $(0, y') : 0 \to a$, with $(f, g)\circ(0, y) = (f\circ 0, g\circ y) = (f\circ 0, g\circ y') = (f, g)\circ(0, y')$.



Hence $(0, y) = (0, y')$ and $y = y'$. Therefore g is mono.

Dually, (f, g) is epi in $\mathcal{Q}^2$ if and only if both f and g are epi in $\mathcal{Q}$. It is easily seen that (f, g) is an isomorphism of $\mathcal{Q}^2$ if and only if both f and g are isomorphisms of $\mathcal{Q}$.

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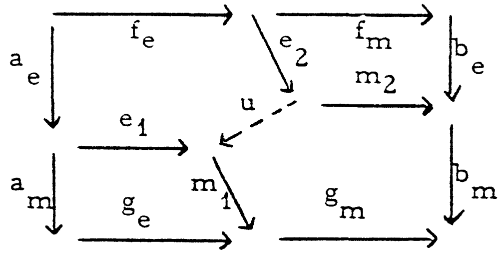
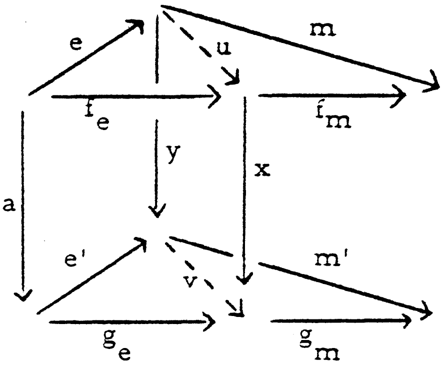
Returning now to the proof of (A1), let $(f, g) : a \to b$ be any map of $\mathcal{Q}^2$. We can write $f = f_m \circ f_e$, $g = g_m \circ g_e$, $a = a_m \circ a_e$, $b = b_m \circ b_e$ in $\mathcal{Q}$. Moreover we can write $g_e \circ a_m = m_1 \circ e_1$ and $b_e \circ f_m = m_2 \circ e_2$, say. Then $b \circ f = g \circ a$

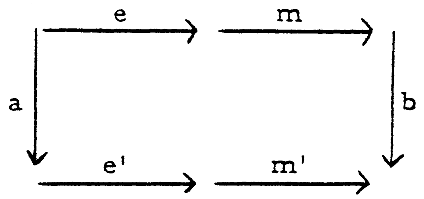


can be factored in two ways as $(g_m \circ m_1) \circ (e_1 \circ a_e)$ and $(b_m \circ m_2) \circ (e_2 \circ f_e)$. By uniqueness of such factorization, there exists an isomorphism u such that $g_m \circ m_1 \circ u = b_m \circ m_2$ and $u \circ e_2 \circ f_e = e_1 \circ a_e$. Now $g_m \circ m_1 \circ u \circ e_2 = b_m \circ m_2 \circ e_2 = b_m \circ b_e \circ f_m = b \circ f_m$ and $m_1 \circ u \circ e_2 \circ f_e = m_1 \circ e_1 \circ a_e = g_e \circ a_m \circ a_e = g_e \circ a$, so $(f_m, g_m) : m_1 \circ u \circ e_2 \to b$ and $(f_e, g_e) : a \to m_1 \circ u \circ e_2$ in $\mathcal{Q}^2$. Moreover, (f_m, g_m) is mono and (f_e, g_e) is epi, according to what was proved before. Thus we have factored (f, g) as $(f_m, g_m) \circ (f_e, g_e)$ in $\mathcal{Q}^2$.

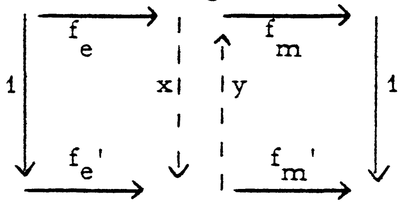
To show that such factorization in $\mathcal{Q}^2$ is unique, suppose that we have also $(f, g) = (m, m') \circ (e, e')$, where $(m, m') : y \to b$ is mono and $(e, e') : a \to y$ is epi. Let $m_1 \circ u \circ e_2 = x$, so $(f_e, g_e) : a \to x$ and $(f_m, g_m) : x \to b$. Then $(f, g) = (m, m') \circ (e, e') = (m \circ e, m' \circ e')$ implies $f = m \circ e$ and $g = m' \circ e'$; hence there exist isomorphisms u and v such that $f_e = u \circ e$, $g_e = v \circ e'$, $f_m \circ u = m$, $g_m \circ v = m'$. Now $x \circ u \circ e = x \circ f_e = g_e \circ a = v \circ e' \circ a = v \circ y \circ e$, hence $x \circ u = v \circ y$ since e is epi. Moreover u and v are isomorphisms in $\mathcal{Q}$, hence $(u, v) : y \to x$ is an isomorphism in $\mathcal{Q}^2$, such that $(f_m, g_m) \circ (u, v) = (m, m')$ and $(f_e, g_e) = (u, v) \circ (e, e')$.



<u>Remark.</u> The following observation is due to Professor P. Hilton. We have seen that the uniqueness of factoring $f = m \circ e$ in $\mathcal{Q}$ was necessary to obtain the isomorphism u and hence the map $x = m_1 \circ u \circ e_2$ of $\mathcal{Q}$ required for Axiom 1. In fact, we have seen that if we have the following situation



with m, m' mono and e, e' epi, then there exists a map x with $b \circ m = m' \circ x$ and $x \circ e = e' \circ a$. Conversely, if such a map x always exists in such a situation, then factoring $f = f_m \circ f_e$ in $\mathcal{Q}$ is unique; for if also $f = f'_m \circ f'_e$, taking $a = 1$ and $b = 1$, we have the following:



Hence by assumption there exist maps x and y such that $x \circ f_e = f'_e$, $f'_m \circ x = f_m$, $y \circ f'_e = f_e$, $f_m \circ y = f'_m$. Thus $y \circ x \circ f_e = y \circ f'_e = f_e$, and $y \circ x = 1$; also $f'_m \circ x \circ y = f_m \circ y \circ f'_m$, and $x \circ y = 1$. Therefore x is an isomorphism.

Returning to the proof that $\mathcal{Q}^2$ is a homological monoid, to prove (A2), we want to show that (ker f, ker g) is a kernel of (f, g) in $\mathcal{Q}^2$, where $(f, g) : a \to b$ say.

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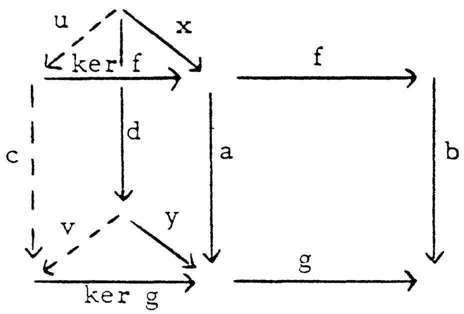

Now $g \circ a \circ (\ker f) = b \circ f \circ (\ker f) = 0$, hence there exists a map c such that $a \circ (\ker f) = (\ker g) \circ c$. Thus $(\ker f, \ker g)$ is a map from c to a in $\mathcal{Q}^2$. It is mono, since $\ker f$ and $\ker g$ are mono, and $(f, g) \circ (\ker f, \ker g) = (f \circ \ker f, g \circ \ker g) = (0, 0)$. Suppose now that $(f, g) \circ (x, y) = 0$ where $(x, y) : d \to a$; i.e. $f \circ x = 0$ and $g \circ y = 0$. Then there exist maps u and v such that $x = (\ker f) \circ u$ and $y = (\ker g) \circ v$. Now $(\ker g) \circ c \circ u = a \circ (\ker f) \circ u = a \circ x = y \circ d = (\ker g) \circ v \circ d$, hence $c \circ u = v \circ d$ since $\ker g$ is mono. Thus (u, v) is a map $d \to c$ in $\mathcal{Q}^2$ such that $(x, y) = (\ker f, \ker g) \circ (u, v)$. Therefore $(\ker f, \ker g)$ is a kernel of (f, g) in $\mathcal{Q}^2$.

Dually, $(\text{cok } f, \text{cok } g)$ is a cokernel of (f, g) in $\mathcal{Q}^2$, and hence every map (f, g) of $\mathcal{Q}^2$ has a kernel and a cokernel.

(A3) First we claim that (m, m') is a normal mono in $\mathcal{Q}^2$ if and only if m and m' are normal monos in $\mathcal{Q}$. For, (m, m') is normal in $\mathcal{Q}^2 \iff (m, m') = \ker \text{cok } (m, m') \iff (m, m') = (\ker \text{cok } m, \ker \text{cok } m') \iff m = \ker \text{cok } m$ and $m' = \ker \text{cok } m' \iff m$ and m' are normal in $\mathcal{Q}$. Similarly, (e, e') is a normal epi in $\mathcal{Q}^2$ if and only if e and e' are normal epis in $\mathcal{Q}$, and in general (f, g) is normal in $\mathcal{Q}^2$ if and only if f and g are normal in $\mathcal{Q}$.

Now suppose (e, e') is a normal epi $b \to c$ and (m, m') is a normal mono $a \to b$ in $\mathcal{Q}^2$. Then $e \circ m$ and $e' \circ m'$ are normal in $\mathcal{Q}$, so we can write $e \circ m = m_1 \circ e_1$ and $e' \circ m' = m'_1 \circ e'_1$ where m_1 and m'_1 are normal monos and e_1 and e'_1 are normal epis. From (A1) we know there exists a map x of $\mathcal{Q}$ such that $(e_1, e'_1) : a \to x$ and $(m_1, m'_1) : x \to b$. Thus $(e, e') \circ (m, m') = (e \circ m, e' \circ m') =$

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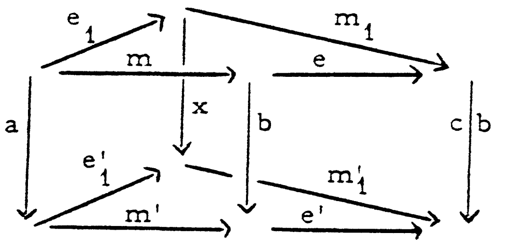
$= (m_1 \circ e_1, m'_1 \circ e'_1) = (m_1, m'_1) \circ (e_1, e'_1)$ where (m_1, m'_1) is a normal mono and (e_1, e'_1) is a normal epi, so $(e, e') \circ (m, m')$ is normal in $\mathcal{Q}^2$.



Therefore $\mathcal{Q}^2$ is also a homological monoid.

Remark. By very similar methods one may show that if $\mathcal{Q}$ is a homological monoid and $\mathcal{B}$ is a partially ordered set, then $\mathcal{Q}^{\mathcal{B}}$ is a homological monoid, a partially ordered set being regarded as a small category whose objects are the elements of the set and whose maps are pairs (i, j) where $i \leq j$.

REFERENCE

1. P. J. Hilton and W. Ledermann, On the Jordan-Hölder Theorem in Homological Monoids, Proc. London Math. Soc., 10 (1960), pp. 321-334.

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