



On L -derivatives and biextensions of Calabi–Yau motives

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Abstract

We prove that certain differential operators of the form DLD with L hypergeometric and $D = z \frac{\partial}{\partial z}$ are of Picard–Fuchs type. We give closed hypergeometric expressions for minors of the biextension period matrices that arise from certain rank 4 weight 3 Calabi–Yau motives presumed to be of analytic rank 1. We compare their values numerically to the first derivative of the L -functions of the respective motives at $s = 2$.

Keywords: Birch–Swinnerton–Dyer-type conjecture; Calabi–Yau motives; hypergeometric differential equations

Introduction

The goal of this note is to explain, favoring expedience over detail, how one can systematically obtain explicit numerical evidence in support of a B-SD-type conjecture for hypergeometric Calabi–Yau motives. For a Calabi–Yau threefold X/\mathbb{Q} with Hodge numbers $h^{3,0} = h^{0,3} = 1, h^{2,1} = h^{1,2} = a$, Poincaré duality defines a non-degenerate alternating form on the third cohomology $H^3(X)$ for any Weil cohomology theory. We view the collection of such cohomological realizations as arising from a so-called *symplectic motive* of rank $2 + 2a$. We will focus on the case $a = 1$ here; these motives, now colloquially called “(1,1,1)-motives,” are expected to exist in 1-parameter families (Golyshev & van Straten, 2023). Their typical Euler factors can be obtained as characteristic polynomials of the geometric p -Frobenius acting in the l -adic cohomology of X over the algebraic closure. They take the form

$$\det \left(1 - T \cdot \text{Frob}_p \big|_{H_{\text{ét}}^3(\bar{X}, \mathbb{Q}_l)} \right) = 1 + \alpha_p T + \beta_p p T^2 + p^3 \alpha_p T^3 + p^6 T^4$$

with $\alpha_p, \beta_p \in \mathbb{Z}$.

It is believed that the completed L -function

$$\Lambda(s) = \left(\frac{N}{\pi^4} \right)^{s/2} \Gamma \left(\frac{s-1}{2} \right) \Gamma \left(\frac{s}{2} \right) \Gamma \left(\frac{s}{2} \right) \Gamma \left(\frac{s+1}{2} \right) L(H^3(X), s),$$

is entire and satisfies $\Lambda(s) = \pm \Lambda(4-s)$, where N is the conductor. The known meromorphicity and existence of a functional equation (Patrikis & Taylor, 2015) enable one in principle to study the leading coefficient of the Taylor series of $L(s)$ along the lines suggested by the conjectures of Deligne (1977) and Birch–Swinnerton–Dyer and Beilinson (1987), Bloch (1980; 1983), Gillet and Soulé (1984). More broadly, if one is to think of these (1,1,1)-motives as analogues of elliptic curves over \mathbb{Q} two dimensions

higher, a question arises of what standard motivic conjectures known to be true or confirmed numerically for elliptic curves survive in this new setup. The key and probably indispensable ingredient here will be a suitable automorphy theorem. Many believe, for instance, that a weight 3-paramodular newform (a Hecke-eigen (3,0)-regular form on the Siegel threefold parametrizing $(1,N)$ -polarized abelian surfaces) f_M could be associated with such a motive M of conductor N so that $L(f_M, s) = L(M, s)$. With automorphy proven—in general, or for any given motive M —one could try to proceed by relating the central L -value (or the leading coefficient) at $s=2$ obtained from an integral representation for the L -function to a certain Hodge-theoretic volume arising in a biextension of M , an idea that can be traced back to Bloch’s early work (Bloch, 1980); see also (Bloch & Kato, 1990; Kontsevich & Zagier, 2001; Scholl, 1991).

In analytic rank 1, one would seek a $\mathrm{GSp}(4)$ -analog of the Gross–Zagier formula (Gross & Zagier, 1986) that might express $L'(M, 2)$ in terms of the height pairing between certain curves on the Siegel threefold parametrizing special abelian surfaces. Its proof, however, is expected to be very difficult and not to be found soon, so a numerical study is desirable as a second-best. Once the Dirichlet series of M and the shape of the functional equation are known, the technology described in Dokchitser(2004) and implemented in Magma (Bosma et al., 1997), enables one (in principle) to compute the Taylor expansion of $L(M, s)$ to an arbitrary precision.

The paper (Roberts & Villegas, 2022) is an excellent introduction to hypergeometric motives and explains how to compute hypergeometric L -functions. The present note can be viewed by the reader as a companion paper. We show how a combination of two ideas specific to hypergeometric pencils enables one to write down closed formulas for the (archimedean) extension volumes and obtain evidence in support of B-SD. One is the principle that gamma structures (Golyshev & Mellit, 2014) give rise to Betti structures. The other says that the motive of the *total space* of a hypergeometric pencil can be used to provide every fiber with a biextension Hodge structure (Bloch et al., 2023). The relevant biextension can be viewed as joining together two Katz’s extensions [20, 8.4.7, 8.4.9] going the opposite directions.

Hypergeometric (1,1,1)-families

The arithmetic of some of these hypergeometric families was studied by, for example, Dwork (1969) and Schoen (1986). The interest in families of \mathbb{Q} -Calabi–Yau motives with points of maximally unipotent monodromy surged in the wake of the discovery of mirror symmetry (Candelas et al., 1991). The simplest are the 14 hypergeometric families, which directly generalize the famous Dwork pencil (Doran & Morgan, 2006; Hofmann & van Straten, 2015). These (and certain “quadratic twists” of these, as we will see) are probably the most amenable to direct computation with the l -adic and Betti-de Rham realizations.

N. Katz introduced implicitly the concept of a hypergeometric motivic sheaf in 1990 by analyzing in detail hypergeometric differential equations, that is, scalar differential equations of the form

$$L_{\alpha,\beta}S(z) = 0 \tag{*}$$

with

$$L_{\alpha,\beta} = \prod_{i=1}^n (D - \alpha_i) - \lambda z \prod_{j=1}^n (D - \beta_j), \quad D = z \frac{d}{dz},$$

and proving a theorem that states that an irreducible regular singular hypergeometric differential equation with rational indices (and $\lambda \in \mathbb{Q}$) is motivic, i.e., arises in a piece of relative cohomology in a pencil of algebraic varieties defined over a number field. An analog of Katz’s theorem holds for tame hypergeometric l -adic sheaves over $\mathbf{G}_m/\overline{\mathbb{Q}}$ whose local inertiae act quasiunipotently. If one furthermore requires that the sets $\exp(2\pi i\alpha_i)$ ’s and $\exp(2\pi i\beta_j)$ ’s are each $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -stable and $\lambda \in \mathbb{Q}$, a motivic construction can be defined over \mathbb{Q} , cf. (Beukers et al., 2015).

Gamma structures give rise to Betti structures

In order to refine hypergeometric D -modules to Hodge modules one needs to identify the \mathbb{Q} -bases of the spaces of local solutions that represent the periods of relative \mathbb{Q} -de Rham forms along \mathbb{Q} -Betti cycles. Following Dwork, one can think of hypergeometric families as deformations of Fermat hypersurfaces (with their relatively simple motivic structures) obtained by introducing an extra monomial to the defining equation. From this perspective, it is clear that the leading expansion coefficients of \mathbb{Q} -Betti solutions of hypergeometric Hodge modules should be proportional to products of the values of the gamma function at rational arguments corresponding to the hypergeometric indices. A theorem on hypergeometric monodromy in Golyshev and Mellit (2014) says, in particular, the following. Assume that **(A1)**:

- the sets $\exp(2\pi i\alpha_i)$'s and $\exp(2\pi i\beta_j)$'s are each $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ -stable and $\lambda \in \mathbb{Q}$
 - $\alpha_i \neq \beta_{i'} \pmod{\mathbb{Z}}$ for all i, i'
- and, merely to make our statement simpler, that
- $\alpha_i \neq \alpha_{i'} \pmod{\mathbb{Z}}$ for all $i \neq i'$ either.

Put

$$\Gamma(s) = \Gamma_{\alpha, \beta}(s) = \prod_{i=1}^n \Gamma(s - \alpha_i + 1)^{-1} \prod_{i=1}^n \Gamma(-s + \beta_i + 1)^{-1} (s \in \mathbb{C}),$$

and $A_i = e^{2\pi i\alpha_i}, B_j = e^{2\pi i\beta_j}$. To simplify notation, assume until the end of this paragraph that $\lambda = (-1)^n$. In general, $(*)$ comes with a gamma structure that is defined to be the set $\gamma = \left\{ \sum_{s \in s_0 + \mathbb{Z}} \Gamma(s) z^s \mid s_0 \in \mathbb{C} \right\}$ of formal solutions to $(*)$ and is meant to specialize to a Betti structure when the hypergeometric indices are rational. In particular, consider the basis of local solutions of $(*)$ at 0 given by

$$S_{A_j}(z) = \sum_{l=0}^{\infty} \Gamma(l + \alpha_j) z^{l + \alpha_j} \in \gamma.$$

Then the monodromy of $(*)$ around 0 is given by

$$M_0(S_{A_1}(z), \dots, S_{A_n}(z))^t = (A_1 S_{A_1}(z), \dots, A_n S_{A_n}(z))^t.$$

Denote by V_A the respective Vandermonde matrix

$$V_A = \begin{pmatrix} 1 & A_1 & \dots & A_1^{n-1} \\ 1 & A_2 & \dots & A_2^{n-1} \\ \vdots & \vdots & & \vdots \end{pmatrix}.$$

The global monodromy of $(*)$ in the basis $V_A^t(S_{A_1}(z), \dots, S_{A_n}(z))^t$ is shown in Golyshev and Mellit (2014) to be in $\text{GL}_n(\mathbb{Q})$, and in fact defines a \mathbb{Q} -local system that underlies a Hodge module.

To identify the Hodge filtration, we proceed as follows. For simplicity, let us further assume, as is the case with our hypergeometric $(1,1,1,1)$ -motives, that **(A2)**:

- n is divisible by 4;
- the sets A 's and B 's are *maximally non-interlaced* on the unit circle in the sense that it can be broken into two complementary sectors containing all A 's resp. B 's;
- $\{\alpha_i\} \subset [0, 1), \{\beta_j\} \subset (-1, 0]$.

To fix a scaling, set $\lambda = \exp \sum_i (\psi(\bar{\alpha}_i) - \psi(\bar{\beta}_i))$, where $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ and \bar{y} denotes the unique representative of the class $y \pmod{\mathbb{Z}}$ in $(0, 1]: \bar{y} = 1 - \{-y\}$. It follows from the multiplication formula for the gamma function that $\lambda \in \mathbb{Q}$. Let $\text{univ} : U \rightarrow (\mathbf{G}_m \setminus \{\lambda^{-1}\})^{\text{an}}$ denote the universal cover. Let \mathcal{U} be the

weight $1 - 2n$ VHS whose underlying local system is constant with the fiber \mathbb{Q}^n , and the Hodge filtration is given as follows: consider the matrix $\Pi_A(z)$ whose j th column is $(z \frac{d}{dz})^j V_A^t(S_{A_1}(\lambda z), \dots, S_{A_n}(\lambda z))^t$, and let $\text{Fil}^{-n/2-j}\mathcal{U}$ be the span of rows $0, \dots, j$ in $\mathbb{Q}^n \otimes \mathbb{C}$.

It is convenient to follow Deligne’s and Bloch’s convention and twist \mathcal{U} by $\mathbb{Q}(1 - n)$: there exists a unique weight -1 hypergeometric VHS \mathcal{V} on $(\mathbf{G}_m \setminus \{\lambda^{-1}\})^{\text{an}}$ such that $\mathcal{U} \otimes \mathbb{Q}(1 - n) = \text{univ}^* \mathcal{V}$. Katz’s weight convention is the opposite of ours for \mathcal{U} . For each $z_0 \neq \lambda^{-1}$ in $\mathbf{G}_m(\mathbb{Q})$, his theory of l -adic hypergeometric sheaves enables one to construct naturally a weight $(2n - 1)$ hypergeometric Galois representation R_{z_0} . Finally, Magma’s convention on hypergeometric motives is yet something different: there should exist a hypergeometric motive M_{z_0} of weight $n - 1$ such that $R_{z_0} = H_{\text{ét}}(M_{z_0} \otimes \mathbb{Q}(-n/2), \mathbb{Q}_l)$ and $\mathcal{V}_{z_0} = H_{\text{dR}}(M_{z_0} \otimes \mathbb{Q}(n/2))$. Conceptually, these are all minor details that affect the computations in a trivial way.

Deligne’s conjecture

Evidence for Deligne’s conjecture for Calabi–Yau motives has been obtained in the last decade ((Roberts, n.d.; Yang, 2021), and unpublished computations by Candelas–de la Ossa–van Straten). With the assumptions made in the previous paragraph, it says that the value $L(M_{z_0}, n/2)$ is proportional with a rational factor to a certain minor arising from the Betti to de Rham identification for \mathcal{V}_{z_0} , or equivalently, from the period matrix for the Hodge structure $\mathcal{V}_{z_0} \otimes \mathbb{Q}(-1)$. Concretely, one expects

$$\frac{L(M_{z_0}, n/2)}{\det (2\pi i)^n \text{Re } \Pi_A(z_0)_{\{0, \dots, n/2-1\}, \{0, \dots, n/2-1\}}} \in \mathbb{Q},$$

where the subscript indicates the top-left quarter of the period matrix. Experimenting with the L -functions (as implemented in Magma) for the case $n = 4$ corresponding to weight 3 Calabi–Yau motives, one checks the identity numerically for various different values of z_0 for the 7 out of the 14 MUM families that are non-resonant at $z = 0$ (a hypergeometric differential equation is non-resonant at 0 resp. ∞ if the eigenvalues A ’s resp. B ’s of the local monodromy operator are distinct). Concretely, the α ’s and β ’s in the seven families are as in the left table below.

The quadratic twist and the Birch–Swinnerton–Dyer period

Following a suggestion by Fernando Rodriguez Villegas, we twist the α ’s and β ’s by shifting all the indices by $-\frac{1}{2}$: $\tilde{\alpha}, \tilde{\beta}$ are, respectively, in the right table; $\tilde{\lambda}$ is now obtained from $\tilde{\alpha}, \tilde{\beta}$ by the same rule as above.

1	$\left[\frac{1}{12}, \frac{5}{12}, \frac{7}{12}, \frac{11}{12} \right]$	$[0, 0, 0, 0]$	$\tilde{1}$	$\left[-\frac{5}{12}, -\frac{1}{12}, \frac{1}{12}, \frac{5}{12} \right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right]$
2	$\left[\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10} \right]$	$[0, 0, 0, 0]$	$\tilde{2}$	$\left[-\frac{2}{5}, -\frac{1}{5}, \frac{1}{5}, \frac{2}{5} \right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right]$
3	$\left[\frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right]$	$[0, 0, 0, 0]$	$\tilde{3}$	$\left[-\frac{3}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8} \right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right]$
4	$\left[\frac{1}{6}, \frac{1}{4}, \frac{3}{4}, \frac{5}{6} \right]$	$[0, 0, 0, 0]$	$\tilde{4}$	$\left[-\frac{1}{3}, -\frac{1}{4}, \frac{1}{4}, \frac{1}{3} \right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right]$
5	$\left[\frac{1}{6}, \frac{1}{3}, \frac{2}{3}, \frac{5}{6} \right]$	$[0, 0, 0, 0]$	$\tilde{5}$	$\left[-\frac{1}{3}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{3} \right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right]$
6	$\left[\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5} \right]$	$[0, 0, 0, 0]$	$\tilde{6}$	$\left[-\frac{3}{10}, -\frac{1}{10}, \frac{1}{10}, \frac{3}{10} \right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right]$
7	$\left[\frac{1}{4}, \frac{1}{3}, \frac{2}{3}, \frac{3}{4} \right]$	$[0, 0, 0, 0]$	$\tilde{7}$	$\left[-\frac{1}{4}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{4} \right]$	$\left[-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right]$

Put $\tilde{\Gamma}_{\tilde{\alpha},\tilde{\beta}}(s) = \tilde{\lambda}^{1/2} \Gamma_{\tilde{\alpha},\tilde{\beta}}(s)$. One has $\tilde{\Gamma}_{\tilde{\alpha},\tilde{\beta}}(s) = \tilde{\lambda}^{1/2} \Gamma_{\alpha,\beta}(s + 1/2)$. All that has been said up to now about hypergeometric Hodge structures works identically for the seven families and the seven twists. However, we expect the twist to raise the “average” analytic rank in the family. Starting with an $L_{\tilde{\alpha},\tilde{\beta}}$ as above, we will construct a “bextension” variation of mixed Hodge structure formally in hypergeometric terms. Although it is not true in general that the product of two differential operators of motivic origin is again motivic, there are situations when one can construct mixed motivic variations formally.

Theorem

With the assumptions (A1) and (A2) made in 2 and 3, the differential equation $DL_{\tilde{\alpha},\tilde{\beta}} DS(z) = 0$ is motivic, that is, underlies a VMHS of geometric origin.

Proof

The idea is that under certain conditions that hold in our case we can pass from the D -module corresponding to a differential operator L to the one corresponding to DL by successively convoluting it with the star resp. the shriek extension of the “constant object” \mathcal{O} on $\mathbf{G}_m - \{1\}$ to \mathbf{G}_m . The background is Katz (1990); all references in the proof are to this book. Denote $\partial = \frac{d}{dz}, D = z\partial$ as above, $\mathcal{D} = \mathcal{D}_{\mathbf{G}_m} = \mathbb{C}[z, z^{-1}, \partial], \mathcal{D}_{\mathbb{A}^1} = \mathbb{C}[z, \partial]$. Let j be the open immersion $\mathbf{G}_m \hookrightarrow \mathbb{A}^1$, and let inv denote the inversion map on \mathbf{G}_m . We denote the Fourier transform functor by FT.

1. *Katz’s lemma on indicial polynomials.* [2.9.5] Write L as a polynomial in z whose coefficients are in turn polynomials in D : $L = \sum_{k=0}^d z^k P_k(D)$. Then: $P_0(y)$ has no zeroes in $\mathbb{Z}_{<0}$ iff

$$\mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1} L \cong j_!(\mathcal{D} / \mathcal{D}L);$$

$P_0(y)$ has no zeroes in $\mathbb{Z}_{\geq 0}$ iff

$$\mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1} L \cong j_*(\mathcal{D} / \mathcal{D}L).$$

2. *The D -modules* $F_k = \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1} (D - k)$ *with* $k \in \mathbb{Z}$. The lemma says that for $k \geq 0$, the D -module F_k is isomorphic to $j_! \mathcal{O}$; for $k < 0$, the D -module F_k is isomorphic to $j_* \mathcal{O}$.

We will need a version of this: put $E_k = \mathcal{D} / \mathcal{D}(D - z(D - k))$ with $k \in \mathbb{Z}$. Denote by j' the open immersion $\mathbf{G}_m - \{1\} \hookrightarrow \mathbf{G}_m$. We claim that for $k \geq 0$, the D -module E_k is $j'_! \mathcal{O}_{\mathbf{G}_m - \{1\}}$. For $k < 0$, the D -module E_k is $j'_* \mathcal{O}_{\mathbf{G}_m - \{1\}}$. Indeed, put $z = 1 + u$, then $D - z(D - k) = (1 + u)\partial - (1 + u)(1 + u)\partial + (1 + u)k = -(1 + u)(u\partial - k)$.

3. *Katz’s “key lemma.”* [5.2.3] Let the convolution sign stand for convolution with no supports on \mathbf{G}_m . For any holonomic module M on \mathbf{G}_m we have

$$j^* \text{FT}(j_* \text{inv}_*(M)) \cong M * (\mathcal{D} / \mathcal{D}(D - z)) \tag{1}$$

and [5.2.3.1]

$$\text{inv}_* j^* \text{FT}(j_* M) \cong M * (\mathcal{D} / \mathcal{D}(1 + zD)). \tag{2}$$

4. We define the star (resp. the shriek) Ur-object to be

$$(\mathcal{D} / \mathcal{D}(1 - zD)) * (\mathcal{D} / \mathcal{D}(D - z))$$

resp.

$$(\mathcal{D} / \mathcal{D}(1 - zD)) * !(\mathcal{D} / \mathcal{D}(D - z)).$$

Claim. The star Ur-object is E_0 . Proof (cf. [6.3.5]): use the key lemma with $M = \mathcal{D} / \mathcal{D}(1 - zD)$. The LHS becomes

$$j^* \text{FT}(\mathcal{D}_{\mathbb{A}^1_1}/\mathcal{D}_{\mathbb{A}^1_1}((D+1)+z)) \cong j^*(\mathcal{D}_{\mathbb{A}^1_1}/\mathcal{D}_{\mathbb{A}^1_1} \text{FT}((D+1)+z)) \cong j^*(\mathcal{D}_{\mathbb{A}^1_1}/\mathcal{D}_{\mathbb{A}^1_1}(-D+\partial)) \cong \mathcal{D}/\mathcal{D}(D-zD).$$

5. Let $H = P_0(D) - zP_1(D)$ be an irreducible hypergeometric operator, so that the sets of roots $\text{mod } \mathbb{Z}$ of P_0 and P_1 are disjoint. Assume further that P_1 has no integer roots and P_0 has no integer roots in $\mathbb{Z}_{\geq 0}$. We claim that

$$\mathcal{D}/\mathcal{D}H * E_0 \cong \mathcal{D}/\mathcal{D}(DP_0(D-1) - zDP_1(D-1)).$$

Indeed, in order to convolute with E_0 one convolutes first with $\mathcal{D}/\mathcal{D}(D-z)$ then with $\mathcal{D}/\mathcal{D}(1-zD) \cong [z \mapsto -z]_*(\mathcal{D}/\mathcal{D}(1+zD))$. The result of the first convolution is simply $\mathcal{D}/\mathcal{D}((D+1)P_0(D) - zP_1(D))$ as P_1 has no integer roots, [5.3.1]. In order to convolute with $\mathcal{D}/\mathcal{D}(1+zD)$ one now uses the second statement of the key lemma, obtaining

$$\text{inv}^*(\mathcal{D}/\mathcal{D}(-DP_0(-D-1) - \partial P_1(-D-1))) \cong \text{inv}^*(\mathcal{D}/\mathcal{D}(-zDP_0(-D-1) - DP_1(-D-1))) \cong \mathcal{D}/\mathcal{D}(DP_0(D-1) + zDP_1(D-1)).$$

Finally, the effect of $[z \mapsto -z]_*$ is in simply changing the sign of z .

6. Let \vee denote the “passing to adjoints” anti-automorphism sending t to t and ∂ to $-\partial$, so that the formal adjoint of $(P_0(D-1) - zP_1(D-1))D$ is $(-D-1)(P_0(-D-2) - P_1(-D-2)z)$. Assume now that P_0 has no integer roots. The previous consideration applies so convoluting with E_0 we get the \mathcal{D} -module corresponding to the operator

$$((-D-1)-1)(P_0(-D-1)-2) - P_1(-D-1)-2z)D = -D(P_0(-D-1) - P_1(-D-1)z)D.$$

Passing to adjoints again,

$$[-D(P_0(-D-1) - P_1(-D-1)z)D]^\vee = -(-D-1)(P_0(D) - zP_1(D))(-D-1)$$

we arrive at the \mathcal{D} -module

$$\mathcal{D}/\mathcal{D}((D+1)H(D+1)) \cong \mathcal{D}/\mathcal{D}(D(P_0(D-1) - zP_1(D-1))D).$$

7. To finish the proof, take H to be the hypergeometric operator whose indices are $\tilde{\alpha}$'s and $\tilde{\beta}$'s shifted by -1 , and the position of the singularities are the same. By Katz, H is motivic. By the argument above, one can pass from the \mathcal{D} -module $\mathcal{D}/\mathcal{D}H$ to the \mathcal{D} -module $\mathcal{D}/\mathcal{D}(DL_{\tilde{\alpha}, \tilde{\beta}}D)$ by successively applying the motivic operations of convolution with the motivic object E_{-1} and passage to duals. Hence, $\mathcal{D}/\mathcal{D}(DL_{\tilde{\alpha}, \tilde{\beta}}D)$ is itself motivic, namely $\mathcal{D}/\mathcal{D}(DL_{\tilde{\alpha}, \tilde{\beta}}D) \simeq \left((\mathcal{D}/\mathcal{D}L_{\tilde{\alpha}, \tilde{\beta}}) * j_! \mathcal{O} \right) * j_* \mathcal{O}$. ■

We remark that all these considerations translate immediately into the l -adic setting. We stick with Hodge modules, but what we need here is a concrete description suitable for computation. The significance of the twist is that the variation of mixed Hodge structure in question is a biextension VHS (Hain, 1990), that is, sits in a $\mathbb{Q}(1) \hookrightarrow \mathcal{V} \twoheadrightarrow \mathbb{Q}$; this would not be the case without the twist. Think of the fiber \mathcal{V} at $z_0 \in \mathbb{Q}$ as realized in $H^3(X_{z_0}, \mathbb{Q}(2))$ for a threefold X_{z_0} . By specializing this VMHS we construct a non-trivial (in general) biextension of $H^3(X_{z_0}, \mathbb{Q}(2))$, and by relaxing the structure to a once-extension, a class in absolute Hodge cohomology $H^4_{\text{Hodge}}(X_{z_0}, \mathbb{R}(2))$. According to the Beilinson rank conjecture, this class signals the presence of a non-trivial class c_{z_0} in $\text{CH}^{(2)}_0(X_{z_0}) \otimes \mathbb{Q}$.

In the language of period matrices, in addition to the 4 pure periods

$$(\Phi_1(z), \Phi_2(z), \Phi_3(z), \Phi_4(z)) = \left(S_{\tilde{A}_1}(\tilde{\lambda}z), \dots, S_{\tilde{A}_4}(\tilde{\lambda}z) \right) V_{\tilde{A}}$$

one introduces an extension solution $S_1(z) = \sum_{n=0}^{\infty} \tilde{\Gamma}_{\tilde{\alpha}, \tilde{\beta}}(n) z^n$ so that $DL_{\tilde{\alpha}, \tilde{\beta}} S_1(\tilde{\lambda}z) = 0$ and the (transposed) biextension period matrix

$$\Pi_A^{\text{biext}}(z) = \left(1, \frac{d}{dz}, \dots, \left(\frac{d}{dz} \right)^5 \right)^t \left(\int S_1(\lambda z) \frac{dz}{z}, \int \Phi_1(z) \frac{dz}{z}, \int \Phi_2(z) \frac{dz}{z}, \int \Phi_3(z) \frac{dz}{z}, \int \Phi_4(z) \frac{dz}{z}, \text{const} \right)$$

with the choice of the constant terms in the 0th row being

$$\left((1/\tilde{\alpha}_1 + 1/\tilde{\alpha}_2) \tilde{\Gamma}_{\tilde{\alpha}, \tilde{\beta}}(0), 0, 0, 0, 0, (2\pi i) \tilde{\Gamma}_{\tilde{\alpha}, \tilde{\beta}}(0) \right).$$

A version of the Birch–Swinnerton–Dyer-type conjecture (Bloch, 1980; Kontsevich & Zagier, 2001; Scholl, 1991) translates into the following statement. By analogy with the elliptic curve cases two dimensions lower, one expects that the archimedean component of the height of c_{z_0} is essentially the ratio of two minors of $\text{Re} \Pi_A^{\text{biext}}(z_0)$:

$$h_{\text{arch}}(c_{z_0}) = \tilde{\Gamma}_{\tilde{\alpha}, \tilde{\beta}}(0)^{-1} \cdot \frac{\det \text{Re} \Pi_A^{\text{biext}}(z_0)_{\{0,1,2\}, \{0,1,2\}}}{\det \text{Re} \Pi_A^{\text{biext}}(z_0)_{\{1,2\}, \{1,2\}}}.$$

Assume, in addition, that the modulus $z_0 \in \mathbb{Q}$ is chosen so that there are no non-archimedean components of the height. Since the minor $\det \text{Re} \Pi_A^{\text{biext}}(z_0)_{\{1,2\}, \{1,2\}}$ occurring in the denominator is nothing else but the $\tilde{\Gamma}_{\tilde{\alpha}, \tilde{\beta}}(0)^{-2}$ -scaled Deligne period of M_{z_0} , a version of B-SD for an analytic rank 1 motive M_{z_0} in a hypergeometric family as above would predict that

$$r(z_0) := \frac{L'(M_{z_0}, 2)}{\tilde{\Gamma}_{\tilde{\alpha}, \tilde{\beta}}(0)^{-3} \det \text{Re} \Pi_A^{\text{biext}}(z_0)_{\{0,1,2\}, \{0,1,2\}}} \in \mathbb{Q}^*.$$

Examples

Consider the hypergeometric family $\tilde{2}$ in the second table in 5. (so that $\tilde{\Gamma}_{\tilde{\alpha}, \tilde{\beta}}(0) = 32(2\pi i)^{-4}$). One finds numerically

$$r(1/2) \stackrel{?}{=} 5^{-2} \text{ and } r(1) \stackrel{?}{=} 2^3 \cdot 5^{-2}.$$

More:

α 's	t	$r(t)$	Conj. Value of $t^{-6} r(t)$
$[-5/12, -1/12, 1/12, 5/12]$	1/8	0.0000065104167	128/75
$[-2/5, -1/5, 1/5, 2/5]$	1/8	0.00031250000	2048/25
$[-2/5, -1/5, 1/5, 2/5]$	1/3	0.0070233196	128/25
$[-3/8, -1/8, 1/8, 3/8]$	1/8	0.000027126736	64/9
$[-3/8, -1/8, 1/8, 3/8]$	1/6	0.00060966316	256/9
$[-3/8, -1/8, 1/8, 3/8]$	1/2	0.0069444444	4/9
$[-1/3, -1/4, 1/4, 1/3]$	1/6	0.014631916	2048/3
$[-1/3, -1/4, 1/4, 1/3]$	1/3	0.058527664	128/3
$[-1/3, -1/4, 1/4, 1/3]$	1/3	0.058527664	128/3
$[-1/3, -1/4, 1/4, 1/3]$	1/2	0.33333333	64/3

α 's	t	$r(t)$	Conj. Value of $t^{-6}r(t)$
$[-1/3, -1/6, 1/6, 1/3]$	1/8	0.00021701389	512/9
$[-1/3, -1/6, 1/6, 1/3]$	1/7	0.00012088689	128/9
$[-1/3, -1/6, 1/6, 1/3]$	1/2	0.055555556	32/9
$[-3/10, -1/10, 1/10, 3/10]$	1/2	0.0022222222	32/225
$[-1/4, -1/6, 1/6, 1/4]$	1/2	0.16666667	32/3

Much of this can be generalized to higher-rank hypergeometrics or extended to cases involving certain higher regulators. The method can be extended to cases involving certain higher regulators as will be shown in a forthcoming paper with Matt Kerr. I thank Kilian Boenisch for checking the computations.

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Review 1: On \mathbb{Z} -derivatives and biextensions of Calabi–Yau motives

Reviewer: Dr. Wadim Zudilin 

Radboud University Nijmegen, Netherlands

Date of review: 19 April 2023

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Review 2: On \mathbb{L} -derivatives and biextensions of Calabi-Yau motives

Reviewer: Francois Brunault 

Date of review: 06 May 2023

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Comment


This article is an interesting contribution to the study of periods of Calabi-Yau and hypergeometric motives, and their relations with central values of the associated L-functions. It uses the language of Hodge modules and contains, in particular, experimental results in the analytic rank 1 case. This paves the way to understanding higher rank hypergeometric motives and higher regulators.

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