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Review

The standard model of strong, weak, and electromagnetic interactions is a relativistic quantum field theory that describes all known interactions of quarks and leptons. This chapter provides a quick review of features of the standard model that are relevant for heavy quark systems, and of basic field theory techniques such as the operator product expansion. It will also serve the purpose of defining some of the normalization conventions and notation to be used in the rest of the book.

1.1 The standard model

The standard model is a gauge theory based on the gauge group $SU(3) \times SU(2) \times U(1)$. The $SU(3)$ gauge group describes the strong color interactions among quarks, and the $SU(2) \times U(1)$ gauge group describes the electroweak interactions. At the present time three generations of quarks and leptons have been observed. The measured width of the Z boson does not permit a fourth generation with a massless (or light) neutrino. Many extensions of the minimal standard model have been proposed, and there is evidence in the present data for neutrino masses, which requires new physics beyond that in the minimal standard model. Low-energy supersymmetry, dynamical weak symmetry breaking, or something totally unexpected may be discovered at the next generation of high-energy particle accelerators.

The focus of this book is on understanding the physics of hadrons containing a bottom or charm quark. The technically difficult problem is understanding the role strong interactions play in determining the properties of these hadrons. For example, weak decays can be computed by using a low-energy effective weak Hamiltonian. Any new physics beyond the standard model can also be treated by using a local low-energy effective interaction, and the theoretical difficulties associated with evaluating hadronic matrix elements of this interaction are virtually identical to those for the weak interactions. For this reason, most of the

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discussion in this book will focus on the properties of heavy quark hadrons as computed in the standard model.

The matter fields in the minimal standard model are three families of spin-1/2 quarks and leptons, and a spin-zero Higgs boson, shown in Table 1.1. The index i on the Fermion fields is a family or generation index $i = 1, 2, 3$, and the subscripts L and R denote left- and right-handed fields, respectively,

$$\psi_L = P_L\psi, \qquad \psi_R = P_R\psi, \tag{1.1}$$

where P_L and P_R are the projection operators

$$P_L = \frac{1}{2}(1 - \gamma_5), \qquad P_R = \frac{1}{2}(1 + \gamma_5). \tag{1.2}$$

Q_L^i, u_R^i, d_R^i are the quark fields and L_L^i, e_R^i are the lepton fields. All the particles associated with the fields in Table 1.1 have been observed experimentally, except for the Higgs boson. The $SU(2) \times U(1)$ symmetry of the electroweak sector is not manifest at low energies. In the standard model, the $SU(2) \times U(1)$ symmetry is spontaneously broken by the vacuum expectation value of the Higgs doublet

Table 1.1. *Matter fields in the standard model[a]*

Field	$SU(3)$	$SU(2)$	$U(1)$	Lorentz
$Q_L^i = \begin{pmatrix} u_L^i \\ d_L^i \end{pmatrix}$	**3**	**2**	1/6	(1/2, 0)
u_R^i	**3**	**1**	2/3	(0, 1/2)
d_R^i	**3**	**1**	−1/3	(0, 1/2)
$L_L^i = \begin{pmatrix} v_L^i \\ e_L^i \end{pmatrix}$	**1**	**2**	−1/2	(1/2, 0)
e_R^i	**1**	**1**	−1	(0, 1/2)
$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix}$	**1**	**2**	1/2	(0, 0)

[a] The index i labels the quark and lepton family. The dimensions of the $SU(3)$ and $SU(2)$ representations and their $U(1)$ charge are listed in the second, third, and fourth columns, respectively. The transformation properties of the fermion fields under the Lorentz group $SO(3, 1)$ are listed in the last column.

H. The spontaneous breakdown of $SU(2) \times U(1)$ gives mass to the $W^{\pm}$ and Z^0 gauge bosons. A single Higgs doublet is the simplest way to achieve the observed pattern of spontaneous symmetry breaking, but a more complicated scalar sector, such as two doublets, is possible.

The terms in the standard model Lagrangian density that involve only the Higgs doublet

$$H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \tag{1.3}$$

are

$$\mathcal{L}_{\text{Higgs}} = (D_\mu H)^\dagger (D^\mu H) - V(H), \tag{1.4}$$

where D_μ is the covariant derivative and $V(H)$ is the Higgs potential

$$V(H) = \frac{\lambda}{4}(H^\dagger H - v^2/2)^2. \tag{1.5}$$

The Higgs potential is minimized when $H^\dagger H = v^2/2$. The $SU(2) \times U(1)$ symmetry can be used to rotate a general vacuum expectation value into the standard form

$$\langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}, \tag{1.6}$$

where v is real and positive.

The generators of the $SU(2)$ gauge symmetry acting on the Higgs (i.e., fundamental) representation are

$$T^a = \sigma^a/2, \quad a = 1, 2, 3, \tag{1.7}$$

where the Pauli spin matrices are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.8}$$

and the generators are normalized to $\operatorname{Tr} T^a T^b = \delta^{ab}/2$. The $U(1)$ generator Y is called hypercharge and is equal to $1/2$ acting on the Higgs doublet (see Table 1.1). One linear combination of $SU(2) \times U(1)$ generators is left unbroken by the vacuum expectation value of the Higgs field H given in Eq. (1.6). This linear combination is the electric charge generator $Q = T^3 + Y$, where

$$Q = T^3 + Y = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{1.9}$$

when acting on the Higgs representation. It is obvious from Eqs. (1.6) and (1.9) that

$$Q\langle H \rangle = 0, \tag{1.10}$$

so that electric charge is left unbroken. The $SU(3) \times SU(2) \times U(1)$ symmetry of the standard model is broken to $SU(3) \times U(1)_Q$ by the vacuum expectation value of H, where the unbroken electromagnetic $U(1)_Q$ is the linear combination of the original $U(1)$ hypercharge generator, Y, and the $SU(2)$ generator, T^3, given in Eq. (1.9).

Expanding H about its expectation value

$$H(x) = \begin{pmatrix} h^+(x) \\ v/\sqrt{2} + h^0(x) \end{pmatrix} \tag{1.11}$$

and substituting in Eq. (1.5) gives the Higgs potential

$$V(H) = \frac{\lambda}{4}(|h^+|^2 + |h^0|^2 + \sqrt{2}v \operatorname{Re} h^0)^2. \tag{1.12}$$

The fields h^+ and $\operatorname{Im} h^0$ are massless. This is an example of Goldstone's theorem. The potential has a continuous three-parameter family of degenerate vacua that are obtained from the reference vacuum in Eq. (1.6) by global $SU(2) \times U(1)$ transformations. [Of the four $SU(2) \times U(1)$ generators, one linear combination Q leaves the vacuum expectation value invariant, and so does not give a massless mode.] Field excitations along these degenerate directions cost no potential energy and so the fields h^+ and $\operatorname{Im} h^0$ are massless. There is one massive scalar that is destroyed by the (normalized) real scalar field $\sqrt{2} \operatorname{Re} h^0$. At tree level, its mass is

$$m_{\operatorname{Re} h^0} = \sqrt{\frac{\lambda}{2}}\, v. \tag{1.13}$$

Global $SU(2) \times U(1)$ transformations allow the space–time independent vacuum expectation value of H to be put into the form given in Eq. (1.6). Local $SU(2) \times U(1)$ transformations can be used to eliminate $h^+(x)$ and $\operatorname{Im} h^0(x)$ completely from the theory, and to write

$$H(x) = \begin{pmatrix} 0 \\ v/\sqrt{2} + \operatorname{Re} h^0(x) \end{pmatrix}. \tag{1.14}$$

This is the standard model in unitary gauge, in which the $W^\pm$ and Z bosons have explicit mass terms in the Lagrangian, as is shown below. In this gauge, the massless fields h^+ and $\operatorname{Im} h^0$ are eliminated, and so do not correspond to states in the spectrum of the theory.

The gauge covariant derivative acting on any field ψ is

$$D_\mu = \partial_\mu + ig A_\mu^A T^A + ig_2 W_\mu^a T^a + ig_1 B_\mu Y, \tag{1.15}$$

where T^A, $A = 1, \ldots, 8$, are the eight color $SU(3)$ generators T^a, $a = 1, 2, 3$ are the weak $SU(2)$ generators, and Y is the $U(1)$ hypercharge generator. The generators are chosen to be in the representation of the field ψ on which the covariant derivative acts. The gauge bosons and coupling constants associated with

these gauge groups are denoted A_μ^A, W_μ^a, and B_μ and g, g_2, and g_1, respectively. The kinetic term for the Higgs field contains a piece quadratic in the gauge fields when expanded about the Higgs vacuum expectation value using Eq. (1.11). The quadratic terms that produce a gauge-boson mass are

$$\mathcal{L}_{\substack{\text{gauge-boson} \\ \text{mass}}} = \frac{g_2^2 v^2}{8}(W^1 W^1 + W^2 W^2) + \frac{v^2}{8}(g_2 W^3 - g_1 B)^2, \qquad (1.16)$$

where for simplicity of notation Lorentz indices are suppressed. The charged W-boson fields

$$W^\pm = \frac{W^1 \mp i W^2}{\sqrt{2}} \qquad (1.17)$$

have mass

$$M_W = \frac{g_2 v}{2}. \qquad (1.18)$$

It is convenient to introduce the weak mixing angle θ_W defined by

$$\sin\theta_W = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}, \qquad \cos\theta_W = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}. \qquad (1.19)$$

The Z-boson field and photon field $\mathcal{A}$ are defined as linear combinations of the neutral gauge-boson fields W^3 and B,

$$\begin{aligned} Z &= \cos\theta_W W^3 - \sin\theta_W B, \\ \mathcal{A} &= \sin\theta_W W^3 + \cos\theta_W B. \end{aligned} \qquad (1.20)$$

The Z boson has a mass at tree level

$$M_Z = \frac{\sqrt{g_1^2 + g_2^2}}{2} v = \frac{M_W}{\cos\theta_W}, \qquad (1.21)$$

and the photon is massless.

The covariant derivative in Eq. (1.15) can be reexpressed in terms of the mass-eigenstate fields as

$$D_\mu = \partial_\mu + ig A_\mu^A T^A + i\frac{g_2}{\sqrt{2}}\left(W_\mu^+ T^+ + W_\mu^- T^-\right)$$

$$+ i\sqrt{g_1^2 + g_2^2}(T_3 - \sin^2\theta_W Q)Z_\mu + ig_2 \sin\theta_W Q \mathcal{A}_\mu, \qquad (1.22)$$

where $T^\pm = T^1 \pm iT^2$. The photon coupling constant in Eq. (1.22) leads to the relation between the electric charge e and the couplings $g_{1,2}$,

$$e = g_2 \sin\theta_W = \frac{g_2 g_1}{\sqrt{g_1^2 + g_2^2}}, \qquad (1.23)$$

so the Z coupling constant $\sqrt{g_1^2 + g_2^2}$ in Eq. (1.22) is conventionally written as $e/(\sin\theta_W \cos\theta_W)$.

Outside of unitary gauge the H kinetic term also has a piece quadratic in the fields where the Goldstone bosons h^+, $\text{Im}\,h^0$ mix with the longitudinal parts of the massive gauge bosons. This mixing piece can be removed by adding to the Lagrange density the 't Hooft gauge fixing term

$$\mathcal{L}_{\substack{\text{gauge} \\ \text{fix}}} = -\frac{1}{2\xi}\sum_a \left[\partial^\mu W_\mu^a + ig_2\xi(\langle H\rangle^\dagger T^a H - H^\dagger T^a \langle H\rangle)\right]^2$$
$$-\frac{1}{2\xi}\left[\partial^\mu B_\mu + ig_1\xi(\langle H\rangle^\dagger Y H - H^\dagger Y \langle H\rangle)\right]^2, \qquad (1.24)$$

which gives the Lagrangian in R_ξ gauge, where ξ is an arbitrary parameter. The fields $h^\pm$ and $\text{Im}\,h^0$ have mass terms proportional to the gauge fixing constant ξ. In Feynman gauge $\xi = 1$ (the easiest for doing calculations), these masses are the same as those of the $W^\pm$ and Z. $\text{Im}\,h^0$ and $h^\pm$ are not physical degrees of freedom since in unitary gauge $\xi \to \infty$ their masses are infinite and they decouple from the theory.

$SU(3) \times SU(2) \times U(1)$ gauge invariance prevents bare mass terms for the quarks and leptons from appearing in the Lagrange density. The quarks and leptons get mass because of their Yukawa couplings to the Higgs doublet,

$$\mathcal{L}_{\text{Yukawa}} = g_u^{ij}\,\bar{u}_R^i H^T \epsilon Q_L^j - g_d^{ij}\,\bar{d}_R^i H^\dagger Q_L^j - g_e^{ij}\,\bar{e}_R^i H^\dagger L_L^j + \text{h.c.} \quad (1.25)$$

where h.c. denotes Hermitian conjugate. Here repeated indices i, j are summed and the antisymmetric matrix ϵ is given by

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \qquad (1.26)$$

Color indices and spinor indices are suppressed in Eq. (1.25). Since H has a vacuum expectation value, the Yukawa couplings in Eq. (1.25) give rise to the 3×3 quark and lepton mass matrices

$$\mathcal{M}_u = vg_u/\sqrt{2}, \;\; \mathcal{M}_d = vg_d/\sqrt{2}, \;\; \text{and } \mathcal{M}_e = vg_e/\sqrt{2}. \qquad (1.27)$$

Neutrinos do not get mass from the Yukawa interactions in Eq. (1.25), since there is no right-handed neutrino field.

Any matrix M can be brought into diagonal form by separate unitary transformations on the left and right, $M \to LDR^\dagger$, where L and R are unitary, and D is real, diagonal and nonnegative. One can make separate unitary transformations on the left- and right-handed quark and lepton fields, while leaving the kinetic energy terms for the quarks, $\bar{Q}_L^i i\!\!\not{\partial}\, Q_L^i$, $\bar{u}_R^i i\!\!\not{\partial}\, u_R^i$, and $\bar{d}_R^i i\!\!\not{\partial}\, d_R^i$, and also those for

the leptons, invariant. The unitary transformations are

$$
\begin{aligned}
u_L &= \mathcal{U}(u, L)\, u'_L, & u_R &= \mathcal{U}(u, R)\, u'_R, \\
d_L &= \mathcal{U}(d, L)\, d'_L, & d_R &= \mathcal{U}(d, R)\, d'_R, \\
e_L &= \mathcal{U}(e, L)\, e'_L, & e_R &= \mathcal{U}(e, R)\, e'_R.
\end{aligned}
\tag{1.28}
$$

Here u, d, and e are three-component column vectors (in flavor space) for the quarks and leptons, and the primed fields represent the corresponding mass eigenstates. The transformation matrices $\mathcal{U}$ are 3×3 unitary matrices, which are chosen to diagonalize the mass matrices

$$
\mathcal{U}(u, R)^\dagger \, \mathcal{M}_u \, \mathcal{U}(u, L) =
\begin{pmatrix}
m_u & 0 & 0 \\
0 & m_c & 0 \\
0 & 0 & m_t
\end{pmatrix},
\tag{1.29}
$$

$$
\mathcal{U}(d, R)^\dagger \, \mathcal{M}_d \, \mathcal{U}(d, L) =
\begin{pmatrix}
m_d & 0 & 0 \\
0 & m_s & 0 \\
0 & 0 & m_b
\end{pmatrix},
\tag{1.30}
$$

and

$$
\mathcal{U}(e, R)^\dagger \, \mathcal{M}_e \, \mathcal{U}(e, L) =
\begin{pmatrix}
m_e & 0 & 0 \\
0 & m_\mu & 0 \\
0 & 0 & m_\tau
\end{pmatrix}.
\tag{1.31}
$$

Diagonalizing the quark mass matrices in Eqs. (1.29) and (1.30) requires different transformations of the u_L and d_L fields, which are part of the same $SU(2)$ doublet Q_L. The original quark doublet can be rewritten as

$$
\begin{pmatrix} u_L \\ d_L \end{pmatrix} =
\begin{pmatrix} \mathcal{U}(u, L)\, u'_L \\ \mathcal{U}(d, L)\, d'_L \end{pmatrix} =
\mathcal{U}(u, L)
\begin{pmatrix} u'_L \\ V d'_L \end{pmatrix},
\tag{1.32}
$$

where the Cabibbo-Kobayashi-Maskawa (CKM) mixing matrix V is defined by

$$
V = \mathcal{U}(u, L)^\dagger \, \mathcal{U}(d, L).
\tag{1.33}
$$

It is convenient to reexpress the standard model Lagrangian in terms of the primed mass-eigenstate fields. The unitary matrices in Eq. (1.32) leave the quark kinetic terms unchanged. The Z and $\mathcal{A}$ couplings are also unaffected, so there are no flavor-changing neutral currents in the Lagrangian at tree level. The W couplings are left unchanged by $\mathcal{U}(u, L)$, but not by V, so that

$$
\frac{g_2}{\sqrt{2}}\, W^+\, \bar{u}_L \gamma^\mu d_L = \frac{g_2}{\sqrt{2}}\, W^+\, \bar{u}'_L \gamma^\mu V d'_L.
\tag{1.34}
$$

As a result there are flavor-changing charged currents at tree level.

The CKM matrix V is a 3×3 unitary matrix, and so is completely specified by nine real parameters. Some of these can be eliminated by making phase redefinitions of the quark fields. The u and d quark mass matrices are unchanged if one makes independent phase rotations on the six quarks, provided the same

phase is used for the left- and right-handed quarks of a given flavor. An overall equal phase rotation on all the quarks leaves the CKM matrix unchanged, but the remaining five rotations can be used to eliminate five parameters, so that V is written in terms of four parameters. The original Kobayashi-Maskawa parameterization of V is

$$
V = \begin{pmatrix}
c_1 & s_1 c_3 & s_1 s_3 \\
-s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\
-s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta}
\end{pmatrix}, \qquad (1.35)
$$

where $c_i \equiv \cos\theta_i$, and $s_i \equiv \sin\theta_i$ for $i = 1, 2, 3$. The angles θ_1, θ_2, and θ_3 can be chosen to lie in the first quadrant, where their sines and cosines are positive. Experimentally it is known that these angles are quite small. The CKM matrix is real if $\delta = 0$, so that $\delta \neq 0$ is a signal of CP violation in the weak interactions. It describes the unitary transformation between the mass-eigenstate basis $d^{i\prime}$, and the weak interaction eigenstate basis d^i. The standard notation for the mass-eigenstate fields is $u'^1 = u$, $u'^2 = c$, $u'^3 = t$, $d'^1 = d$, $d'^2 = s$, $d'^3 = b$.

So far we have only considered the left-handed quark couplings to the gauge bosons. For the right-handed quarks there are no W-boson interactions in the standard model, and in the primed mass-eigenstate basis the couplings of the Z, photon, and color gauge bosons are flavor diagonal. The analysis for leptons is similar to that for quarks, with one notable difference – because the neutrinos are massless, one can choose to make the same unitary transformation on the left-handed charged leptons and neutrinos. The analog of the CKM matrix in the lepton sector can be chosen to be the unit matrix, and the leptons can be chosen to be simultaneously mass and weak eigenstates. We adopt the notation $\nu'^1 = \nu_e$, $\nu'^2 = \nu_\mu$, $\nu'^3 = \nu_\tau$, $e'^1 = e$, $e'^2 = \mu$, $e'^3 = \tau$. From now on, we will use the mass-eigenstate basis for labeling the quark and lepton fields.

1.2 Loops

Loop diagrams in the standard model have divergences from the high-momentum (ultraviolet) region of the momentum integrals. These divergences are interpreted by a renormalization procedure; the theory is regulated in some way and terms that diverge as the regulator is removed are absorbed into the definitions of the couplings and masses. Theories in which all divergences in physical quantities (e.g., S-matrix elements) can be removed in this way using a finite number of counterterms are called renormalizable. In the unitary gauge, $\xi \to \infty$, the standard model is manifestly unitary (i.e., only physical degrees of freedom propagate because the "ghost" Higgs associated with $h^\pm$ and Im h^0 have infinite

mass). The vector-boson propagator

$$-i\,\frac{g_{\mu\nu} - k_\mu k_\nu / M_{W,Z}^2}{k^2 - M_{W,Z}^2} \tag{1.36}$$

is finite as $k \to \infty$, and naive power counting suggests that the standard model is not renormalizable. In the Feynman gauge, $\xi = 1$, the vector-boson propagator is

$$-i\,\frac{g_{\mu\nu}}{k^2 - M_{W,Z}^2}, \tag{1.37}$$

which falls off as $1/k^2$, and naive power counting shows that the standard model is renormalizable. The potentially disastrous divergences that occur in the unitary gauge must cancel. However, unitarity is not manifest in the Feynman gauge because the unphysical degrees of freedom associated with $h^\pm$ and $\mathrm{Im}\,h^0$ are included as intermediate states in Feynman diagrams. The standard model is manifestly unitary in one gauge and manifestly renormalizable in another. Gauge invariance assures us that the theory is both unitary and renormalizable.

In this book we will regularize Feynman diagrams by using dimensional regularization. Diagrams are calculated in $n = 4 - \epsilon$ dimensions, and the ultraviolet divergences that occur in four dimensions appear as factors of $1/\epsilon$, as $\epsilon \to 0$.

To review how dimensional regularization works, consider the quantum electrodynamics (QED) Lagrangian

$$\mathcal{L}_{\mathrm{QED}} = -\frac{1}{4}F_{\mu\nu}^{(0)}F^{(0)\mu\nu} + i\bar{\psi}^{(0)}\gamma^\mu\big(\partial_\mu - ie^{(0)}\mathcal{A}_\mu^{(0)}\big)\psi^{(0)} - m_e^{(0)}\bar{\psi}^{(0)}\psi^{(0)}, \tag{1.38}$$

which is part of the standard model Lagrangian. The superscript (0) is used to denote a bare quantity. Here

$$F_{\mu\nu}^{(0)} = \partial_\mu\mathcal{A}_\nu^{(0)} - \partial_\nu\mathcal{A}_\mu^{(0)} \tag{1.39}$$

is the bare electromagnetic field strength tensor. In n dimensions, the action

$$S_{\mathrm{QED}} = \int d^n x\,\mathcal{L}_{\mathrm{QED}} \tag{1.40}$$

is dimensionless, since $e^{iS_{\mathrm{QED}}}$ is the measure in the Feynman path integral (we use units where $\hbar = c = 1$). It follows that the dimensions of the fields, the coupling constant $e^{(0)}$, and the electron mass, $m_e^{(0)}$, are

$$\begin{aligned}
\big[\mathcal{A}^{(0)}\big] &= (n-2)/2 = 1 - \epsilon/2, \\
\big[\psi^{(0)}\big] &= (n-1)/2 = 3/2 - \epsilon/2, \\
\big[e^{(0)}\big] &= (4-n)/2 = \epsilon/2, \\
\big[m_e^{(0)}\big] &= 1.
\end{aligned} \tag{1.41}$$

The bare fields are related to the renormalized fields by

$$\mathcal{A}_\mu = \frac{1}{\sqrt{Z_A}}\mathcal{A}_\mu^{(0)},$$

$$\psi = \frac{1}{\sqrt{Z_\psi}}\psi^{(0)},$$

$$e = \frac{1}{Z_e}\mu^{-\epsilon/2}e^{(0)}, \tag{1.42}$$

$$m_e = \frac{1}{Z_m}m_e^{(0)}.$$

The factor of $\mu^{-\epsilon/2}$ is included in the relation between the bare and renormalized electric couplings so that the renormalized coupling is dimensionless. Here μ is a parameter with dimensions of mass and is called the subtraction point or renormalization scale of dimensional regularization. In terms of these renormalized quantities the Lagrange density is

$$\begin{aligned}
\mathcal{L}_{\text{QED}} &= -\frac{1}{4}Z_A F_{\mu\nu}F^{\mu\nu} + iZ_\psi\bar{\psi}\gamma^\mu\big(\partial_\mu - i\mu^{\epsilon/2}Z_e\sqrt{Z_A}e\mathcal{A}_\mu\big)\psi \\
&\quad - Z_m Z_\psi m_e\bar{\psi}\psi, \\
&= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^\mu\big(\partial_\mu - i\mu^{\epsilon/2}e\mathcal{A}_\mu\big)\psi - m_e\bar{\psi}\psi + \text{ counterterms.}
\end{aligned} \tag{1.43}$$

It is straightforward to compute the renormalization constants $Z_{A,\psi,e,m}$ by using the formula for one-loop integrals in dimensional regularization,

$$\begin{aligned}
\int &\frac{d^n q}{(2\pi)^n}\frac{(q^2)^\alpha}{(q^2 - M^2)^\beta} \\
&= \frac{i}{2^n\pi^{n/2}}(-1)^{\alpha+\beta}(M^2)^{\alpha-\beta+n/2}\frac{\Gamma(\alpha + n/2)\Gamma(\beta - \alpha - n/2)}{\Gamma(n/2)\Gamma(\beta)}, \tag{1.44}
\end{aligned}$$

and the Feynman trick for combining denominators,

$$\begin{aligned}
\frac{1}{a_1^{m_1}\cdots a_n^{m_n}} &= \frac{\Gamma(M)}{\Gamma(m_1)\cdots\Gamma(m_n)} \\
&\quad \times \int_0^1 dx_1 x_1^{m_1-1}\cdots\int_0^1 dx_n x_n^{m_n-1}\frac{\delta\big(1 - \sum_{i=1}^n x_i\big)}{(x_1 a_1 + \cdots + x_n a_n)^M}, \tag{1.45}
\end{aligned}$$

where

$$M = \sum_{i=1}^n m_i. $$

The Z's are determined by the condition that time-ordered products of renormalized fields (i.e., Green's functions) be finite when expressed in terms of the

renormalized coupling and mass. This condition still leaves considerable freedom in how the Z's are chosen. The precise way that the Z's are chosen is called the subtraction scheme. The Z's can be chosen to have the form

$$Z = 1 + \sum_{p=1}^{\infty} \frac{Z_p(e)}{\epsilon^p}, \tag{1.46}$$

where the $Z_p(e)$ are independent of ϵ. This choice is called minimal subtraction (MS) because only the poles in ϵ are subtracted and no additional finite pieces are put into the Z's. We will use the $\overline{\text{MS}}$ scheme, which is minimal subtraction followed by the rescaling $\mu^2 \to \mu^2 e^{\gamma}/4\pi$, where $\gamma = 0.577\ldots$ is Euler's constant.

The photon wavefunction renormalization Z_A to order e^2 can be determined by computing the photon–photon correlation function. There are two pieces to this order; the first is a tree-level contribution from the counterterm

$$-\frac{1}{4}(Z_A - 1)F_{\mu\nu}F^{\mu\nu}. \tag{1.47}$$

After truncating the external photon propagators, it gives

$$i(Z_A - 1)(p_\mu p_\nu - p^2 g_{\mu\nu}), \tag{1.48}$$

where p is the photon four momentum. The second contribution is from the one-loop diagram Fig. 1.1,

$$(-1)(ie)^2 \mu^\epsilon \int \frac{d^n q}{(2\pi)^n} \frac{\text{Tr}[\gamma_\mu i(\slashed{q} + \slashed{p} + m_e)\gamma_\nu i(\slashed{q} + m_e)]}{[(q+p)^2 - m_e^2][q^2 - m_e^2]}. \tag{1.49}$$

The factor of (-1) arises from the closed fermion loop. The renormalization constant only depends on the $1/\epsilon$ pole, so the γ matrix algebra can be performed in four dimensions. Expanding

$$\mu^\epsilon = 1 + \epsilon \ln \mu + \cdots, \tag{1.50}$$

one sees that μ^ϵ can be set to unity for the infinite part of the diagram, and finite
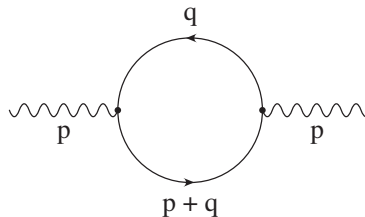

Fig. 1.1. One-loop vacuum polarization contribution to the photon propagator.

parts only depend logarithmically on μ. The denominators are combined using Eq. (1.45):

$$\frac{1}{[(q+p)^2 - m_e^2][q^2 - m_e^2]} = \int_0^1 dx \frac{1}{[q^2 + 2xq \cdot p + p^2 x - m_e^2]^2}. \quad (1.51)$$

Making the change of variables $k = q + px$ gives

$$-4e^2 \int_0^1 dx \int \frac{d^n k}{(2\pi)^n} \frac{1}{[k^2 + p^2 x(1-x) - m_e^2]^2}$$
$$\times \left[2k_\mu k_\nu - (k^2 - m_e^2) g_{\mu\nu} - 2x(1-x)p_\mu p_\nu + p^2 x(1-x)g_{\mu\nu}\right]. \quad (1.52)$$

Terms odd in k vanish upon integration and have been dropped. Evaluating the k integral using Eq. (1.44), keeping only the part proportional to $1/\epsilon$ (using $\Gamma(\epsilon/2) = 2/\epsilon + \cdots$), and doing the x integral gives the divergent part of the one-loop contribution:

$$\frac{i}{16\pi^2\epsilon} \left(\frac{8e^2}{3}\right) (p_\mu p_\nu - p^2 g_{\mu\nu}). \quad (1.53)$$

For the photon two-point correlation function to be finite as $\epsilon \to 0$, the sum of Eqs. (1.53) and (1.48) must be finite. One therefore chooses

$$Z_A = 1 - \frac{8}{3} \left(\frac{e^2}{16\pi^2\epsilon}\right). \quad (1.54)$$

The wave-function renormalization constant Z_ψ for the electron field ψ is obtained from the electron propagator. The counterterms

$$(Z_\psi - 1)\bar{\psi} i \slashed{\partial} \psi - (Z_m Z_\psi - 1)m_e \bar{\psi} \psi \quad (1.55)$$

contribute

$$i(Z_\psi - 1)\slashed{p} - i(Z_m Z_\psi - 1)m_e \quad (1.56)$$

to the propagator. In the Feynman gauge, the one-loop diagram Fig. 1.2 is

$$\mu^\epsilon (ie)^2 \int \frac{d^n q}{(2\pi)^n} \gamma_\nu i \frac{\slashed{p} + \slashed{q} + m_e}{(p+q)^2 - m_e^2} \gamma_\mu \frac{(-i)g^{\mu\nu}}{q^2}. \quad (1.57)$$
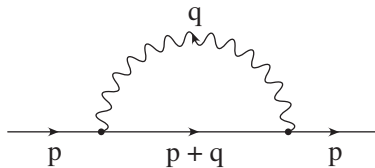

Fig. 1.2.   One-loop correction to the electron propagator.

Combining denominators and shifting the momentum integration as in the previous case gives

$$2e^2 \int \frac{d^n k}{(2\pi)^n} \int_0^1 dx \frac{-2m_e + \not{p}(1-x)}{\left[k^2 - m_e^2 x + p^2 x(1-x)\right]^2}. \tag{1.58}$$

Performing the k integration by using Eq. (1.44) and then the x integration gives

$$\frac{i}{16\pi^2\epsilon}(4e^2)\left(-2m_e + \frac{1}{2}\not{p}\right) \tag{1.59}$$

for the divergent contribution. The electron propagator is finite if

$$Z_\psi = 1 - 2\left(\frac{e^2}{16\pi^2\epsilon}\right) \tag{1.60}$$

and

$$Z_m = 1 - 6\left(\frac{e^2}{16\pi^2\epsilon}\right) \tag{1.61}$$

in the Feynman gauge.

The remaining renormalization factor Z_e can be determined by computing the $\psi\bar{\psi}A$ three-point function to order e^2. The Feynman graph that has to be computed is the vertex renormalization graph of Fig. 1.3. The counterterm is

$$Z_e = 1 + \frac{4}{3}\left(\frac{e^2}{16\pi^2\epsilon}\right). \tag{1.62}$$

Note that $Z_e = 1/\sqrt{Z_A}$ to order e^2.

The relation between the bare and renormalized couplings at order e^2 is

$$e^{(0)} = \mu^{\epsilon/2}eZ_e = \mu^{\epsilon/2}e\left[1 + \frac{4}{3}\left(\frac{e^2}{16\pi^2\epsilon}\right)\right], \tag{1.63}$$

using Eqs. (1.62) and (1.42). The bare fields, coupling, and mass are independent of the subtraction point μ, which is an arbitrary quantity with dimensions of mass introduced so that the renormalized coupling is dimensionless. Since the bare
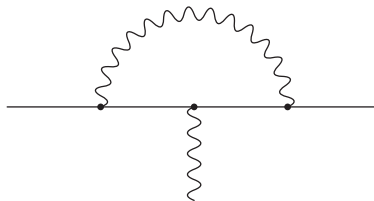

Fig. 1.3.   One-loop vertex correction.

coupling constant is independent of μ,

$$0 = \mu \frac{d}{d\mu} e^{(0)} = \mu \frac{d}{d\mu} \mu^{\epsilon/2} e Z_e = \mu^{\epsilon/2} e Z_e \left[\frac{\epsilon}{2} + \frac{1}{e} \beta(e) + \frac{\mu}{Z_e} \frac{dZ_e}{d\mu} \right], \quad (1.64)$$

where the β function is defined by

$$\beta(e) = \mu \frac{de}{d\mu}. \quad (1.65)$$

This gives

$$\beta(e) = -\frac{\epsilon}{2} e - e \frac{d \ln Z_e}{d \ln \mu}. \quad (1.66)$$

Using Eq. (1.62),

$$\frac{d \ln Z_e}{d \ln \mu} = \left(\frac{4}{3} \right) \frac{1}{16\pi^2 \epsilon} \mu \frac{d}{d\mu} e^2 + \cdots$$

$$= -\frac{e^2}{12\pi^2} + \cdots, \quad (1.67)$$

where the ellipses denote terms of higher order in e^2. The one-loop β function is

$$\beta(e) = -\frac{\epsilon}{2} e + \frac{e^3}{12\pi^2} + \cdots, \quad (1.68)$$

which is finite as $\epsilon \to 0$,

$$\beta(e) = \frac{e^3}{12\pi^2} + \cdots. \quad (1.69)$$

The β function gives the μ dependence of the renormalized coupling e. Here μ is an arbitrary scale parameter, so physical quantities do not depend on μ. However, some choices for μ are more convenient than others for computations. Consider the cross section for $\sigma(e^+ e^- \to \text{anything})$ at a center of mass energy squared, $s = (p_{e^+} + p_{e^-})^2 \gg m_e^2$. In QED this cross section is finite as $m_e \to 0$ and so for large s we neglect m_e. The cross section has a power series expansion in the coupling $e(\mu)$, and it is independent of the subtraction point μ. The implicit μ dependence in the coupling is canceled by an explicit μ dependence in the Feynman diagrams. (One can see this by computing, e.g., the finite parts of Figs. 1.1–1.3.) Typically one finds that terms in the perturbation series have the form $[\alpha(\mu)/4\pi]^n \ln^m s/\mu^2$, with $m \leq n$, where

$$\alpha(\mu) = \frac{e^2(\mu)}{4\pi} \quad (1.70)$$

is the (scale-dependent) fine structure constant. If s/μ^2 is not of the order of unity, the logarithms can get large and cause a breakdown of perturbation theory.

One usually chooses $\mu^2 \sim s$, which "minimizes" the higher-order terms in the perturbation expansion *that have not been computed*. With this choice of μ, one expects that perturbation theory is an expansion in $\alpha(\sqrt{s})/4\pi$.

When perturbation theory is valid we can use Eqs. (1.65) and (1.69) to solve explicitly for the dependence of the coupling on μ at one loop:

$$\frac{1}{e^2(\mu_2)} = \frac{1}{e^2(\mu_1)} - \frac{1}{12\pi^2} \ln\left(\frac{\mu_2^2}{\mu_1^2}\right). \tag{1.71}$$

The β function in Eq. (1.69) is positive, so e increases as μ increases, as can be seen explicitly from the solution in Eq. (1.71).

1.3   Composite operators

Composite operators involve products of fields at the same space–time point. Consider, for example, the bare mass operator

$$S^{(0)} = \bar{\psi}^{(0)} \psi^{(0)}(x). \tag{1.72}$$

Green's functions with an insertion of $S^{(0)}$ are usually divergent. An additional operator renormalization (beyond wave-function renormalization) is required to make the Green's functions finite. The renormalized operator S is

$$S = \frac{1}{Z_S} S^{(0)} = \frac{1}{Z_S} \bar{\psi}^{(0)} \psi^{(0)} = \frac{Z_\psi}{Z_S} \bar{\psi}\psi, \tag{1.73}$$

where Z_S is the additional operator renormalization. The operator $S = \bar{\psi}\psi +$ counterterms is conventionally denoted by just $\bar{\psi}\psi$, with the counterterms implicit. Green's functions with insertions of S are finite in perturbation theory.

The renormalization factor Z_S can be computed from the three-point function of the time-ordered product of ψ, $\bar{\psi}$, and S. It is simpler to use the one-particle irreducible Green's function Γ rather than the full Green's function G to compute Z_S. The counterterm contribution to the one-particle irreducible Green's function is

$$\frac{Z_\psi}{Z_S} - 1. \tag{1.74}$$

The one-loop contribution to Γ is shown in Fig. 1.4. The operator S contains no derivatives (and Z_S is mass independent in the $\overline{\text{MS}}$ scheme), so Z_S can be determined by evaluating Fig. 1.4 at zero external momentum (and neglecting the electron mass), giving

$$\mu^\epsilon (ie)^2 \int \frac{d^n q}{(2\pi)^n} \gamma^\alpha \frac{i\slashed{q}}{q^2} \frac{i\slashed{q}}{q^2} \gamma^\beta \frac{(-i)g_{\alpha\beta}}{q^2} = -4ie^2 \int \frac{d^n q}{(2\pi)^n} \frac{1}{(q^2)^2} + \cdots, \tag{1.75}$$
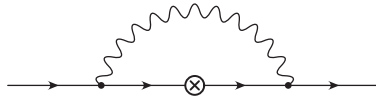
Fig. 1.4.   One-loop graph with an insertion of a fermion-bilinear composite operator (denoted by $\otimes$) such as $\bar{\psi}\psi$.

where the ellipsis denote terms finite as $\epsilon \to 0$. Note that neglecting external momenta and the electron mass has produced an infrared (i.e., low momentum) divergence. Regulating this with a mass m by replacing q^2 in the denominator with $(q^2 - m^2)$ gives

$$\frac{8e^2}{16\pi^2 \epsilon}, \tag{1.76}$$

for the ultraviolet divergent part of Eq. (1.75). Adding Eqs. (1.74) and (1.76) together and using Eq. (1.60), we find that the $1/\epsilon$ divergence cancels, provided

$$Z_S = 1 + 6\left(\frac{e^2}{16\pi^2\epsilon}\right). \tag{1.77}$$

The anomalous dimension of the composite operator S is defined by

$$\gamma_S = \mu \frac{d \ln Z_S}{d\mu} \tag{1.78}$$

so that

$$\gamma_S = -\frac{6e^2}{16\pi^2}. \tag{1.79}$$

Similar calculations can be performed for the vector and axial vector currents $\bar{\psi}\gamma_\mu\psi$ and $\bar{\psi}\gamma_\mu\gamma_5\psi$, and one finds $Z_V = Z_A = 1$, so that the currents are not renormalized and their anomalous dimensions vanish at one loop. Note that $Z = 1$ means that the infinite part of Fig. 1.4 is canceled by wave-function renormalization, not that Fig. 1.4 is finite. The result $Z = 1$ arises because for $m_e = 0$ both the axial and vector currents are conserved and the zero-component of these currents (integrated over all space) are charges $Q_{A,V}$ with commutation relations of the form

$$[Q_V, \psi] = -\psi, \tag{1.80}$$

for example. A conserved charge Q cannot be multiplicatively renormalized since that would spoil such commutation relations. In dimensional regularization with minimal subtraction, electron mass effects cannot induce a renormalization for the axial current because the renormalization factors are independent of particle masses. This is an example of a general result that "soft" symmetry

breaking effects, i.e., symmetry breaking terms with operator dimensions less than four, do not affect renormalization in the $\overline{\text{MS}}$ scheme.

The axial current is not conserved at one loop because of the axial anomaly. The divergence of the axial current is proportional to the dimension-four operator $F\tilde{F}$, so that symmetry breaking because of the anomaly is not soft. It produces an anomalous dimension for the axial current at two loops.

We have considered a particularly simple example in which the operator S was multiplicatively renormalized, since there are no other gauge invariant local operators with the same quantum numbers. In general, one can have many different operators O_i with the same quantum numbers, and one needs a renormalization matrix,

$$O_i^{(0)} = Z_{ij} O_j. \tag{1.81}$$

This is referred to as operator mixing. In the $\overline{\text{MS}}$ scheme, Z_{ij} is dimensionless, so operators can only mix with other operators of the same dimension. This greatly simplifies the analysis of operator mixing. In a general mass-dependent scheme, operators can also mix with operators of lower dimension.

1.4 Quantum chromodynamics and chiral symmetry

The portion of the standard model that describes the strong interactions of quarks and gluons is called quantum chromodynamics (QCD). The QCD Lagrange density including for the moment only the "light" u, d, and s quark flavors is

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G_{\mu\nu}^A G^{A\mu\nu} + \bar{q}(i\slashed{D} - m_q)q + \text{counterterms}, \tag{1.82}$$

where q is the triplet of light quarks

$$q = \begin{pmatrix} u \\ d \\ s \end{pmatrix}, \tag{1.83}$$

and m_q is the quark mass matrix

$$m_q = \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}. \tag{1.84}$$

Here $D_\mu = \partial_\mu + igA_\mu^A T^A$ is the $SU(3)$ color covariant derivative and $G_{\mu\nu}^A$ is the gluon field strength tensor,

$$G_{\mu\nu}^A = \partial_\mu A_\nu^A - \partial_\nu A_\mu^A - gf^{ABC} A_\mu^B A_\nu^C, \tag{1.85}$$

where the structure constants f^{ABC} are defined by $[T^A, T^B] = if^{ABC}T^C$. The QCD renormalization factors can be calculated at order g^2 in a manner similar
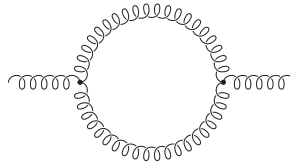
Fig. 1.5.   One-loop gluon contribution to the vacuum polarization.

to that for QED. For example, quark wave-function and mass renormalization Z_q and Z_m are given by Fig. 1.2 with the photon replaced by a gluon. They can be obtained from the QED result by replacing e^2 by $g^2 T^A T^A$, where $T^A T^A = (4/3)\mathbb{1}$ for quarks in QCD. In the Feynman gauge, the order g^2 wave-function and mass renormalization factors are

$$\sqrt{Z_q} = 1 - \frac{g^2}{12\pi^2\epsilon}, \qquad Z_m = 1 - \frac{g^2}{2\pi^2\epsilon}. \tag{1.86}$$

A major difference between QCD and QED occurs in the coupling constant renormalization. The β function for QCD is

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(11 - \frac{2}{3}N_q\right) + \mathcal{O}(g^5), \tag{1.87}$$

where N_q is the number of quark flavors. The quark contribution to the β function can be computed from Fig. 1.1 with the photon replaced by a gluon. It is obtained from the QED calculation by the replacement $e^2 \to N_q g^2/2$, since $\mathrm{Tr}\, T^A T^B = \delta^{AB}/2$ for each quark flavor in the loop. The other term in the β function is from gluon self-interactions, as in Fig. 1.5, and is not present in an Abelian gauge theory such as QED. The QCD β function is negative, as long as the number of quark flavors N_q is less than 16, so the QCD fine structure constant

$$\alpha_s(\mu) = \frac{g^2(\mu)}{4\pi} \tag{1.88}$$

becomes smaller at larger μ, a phenomenon known as asymptotic freedom. At high energies, the coupling constant is small, and QCD perturbation theory should be reliable. We can explicitly solve for the μ-dependence of α_s just as in QED:

$$\alpha_s(\mu_2) = \frac{1}{\left[1/\alpha_s(\mu_1) + \beta_0 \ln\left(\mu_2^2/\mu_1^2\right)\right]}, \tag{1.89}$$

where β_0 is proportional to the first term in the QCD β function,

$$\beta_0 = \left(\frac{33 - 2N_q}{12\pi}\right). \tag{1.90}$$

Equation (1.89) is valid as long as μ_1 and μ_2 are large enough that the order g^5 terms in Eq. (1.87) can be neglected, i.e., as long as $\alpha_s(\mu_1)$ and $\alpha_s(\mu_2)$ are

both small. It is convenient to introduce a subtraction-point independent constant Λ_{QCD} with dimensions of mass, defined by

$$\Lambda_{\text{QCD}} = \mu e^{-1/[2\beta_0\alpha_s(\mu)]}. \tag{1.91}$$

Then our expression for the strong interaction fine structure constant becomes

$$\alpha_s(\mu) = \frac{12\pi}{(33 - 2N_q)\ln\left(\mu^2/\Lambda_{\text{QCD}}^2\right)}. \tag{1.92}$$

Equation (1.92) suggests that the QCD coupling constant diverges as $\mu \to \Lambda_{\text{QCD}}$. Of course, this expression for α_s ceases to be valid when α_s gets large. Nevertheless, one can still view Λ_{QCD} as the scale at which QCD becomes strongly coupled so that perturbation theory breaks down and nonperturbative effects become important. Experimentally, Λ_{QCD} is ~200 MeV, and it sets the scale for nonperturbative strong interaction effects. One expects hadron masses such as the ρ meson mass to be dimensionless multiples of Λ_{QCD}. It is believed that QCD is a confining theory at long distances, i.e., the spectrum of physical states consists of color singlet states called hadrons; there are no colored hadrons. Bosonic hadrons are called mesons and fermionic hadrons are called baryons. The simplest ways to form color singlet combinations of the quark fields are $\bar{q}^\alpha q_\alpha$ and $\varepsilon^{\alpha\beta\gamma} q_\alpha q_\beta q_\gamma$.

The u, d, and s quark masses are small compared with the scale Λ_{QCD} of nonperturbative strong interaction physics, and so it is useful to consider an approximation to QCD in which the masses of these light quarks are set to zero, and to do perturbation theory in m_q about this limit. The limit $m_q \to 0$ is known as the chiral limit, because the light quark Lagrangian

$$\mathcal{L}_{\substack{\text{light} \\ \text{quarks}}} = \bar{q}\, i\slashed{D}\, q = \bar{q}_L\, i\slashed{D}\, q_L + \bar{q}_R\, i\slashed{D}\, q_R \tag{1.93}$$

has an $SU(3)_L \times SU(3)_R$ chiral symmetry

$$q_L \to L\, q_L \qquad q_R \to R\, q_R, \tag{1.94}$$

[$L \in SU(3)_L$, $R \in SU(3)_R$] under which the right- and left-handed quark fields transform differently. The Lagrange density in Eq. (1.93) also has a baryon number $U(1)$ symmetry where the left- and right-handed quarks transform by a common phase, and an axial $U(1)$ where all the left-handed quarks transform by a phase and all the right-handed quarks transform by the opposite phase. Although these axial $U(1)$ transformations leave the Lagrange density invariant, they change the measure in the path integral, an effect known as the axial anomaly. Hence, the axial $U(1)$ is not a symmetry of QCD.

The chiral $SU(3)_L \times SU(3)_R$ symmetry of massless three-flavor QCD is spontaneously broken by the vacuum expectation value of quark bilinears

$$\langle \bar{q}_R^j q_L^k \rangle = v\,\delta^{kj}, \tag{1.95}$$

where v is of order Λ_{QCD}^3. [Here v should not be confused with the Higgs vacuum

expectation value.] The indices j and k are flavor indices, $q^1 = u, q^2 = d, q^3 = s$, and color indices are suppressed. If we make a $SU(3)_L \times SU(3)_R$ transformation $q \to q'$,

$$\langle \bar{q}_R'^{\,j} q_L'^{\,k} \rangle = v(LR^\dagger)^{kj}. \tag{1.96}$$

Transformations with $L = R$ leave the vacuum expectation value unchanged. Thus the nonperturbative strong interaction dynamics spontaneously breaks the $SU(3)_L \times SU(3)_R$ chiral symmetry to its diagonal subgroup $SU(3)_V$. The eight broken $SU(3)_L \times SU(3)_R$ generators transform the composite field $\bar{q}_R^{\,j} q_L^{\,k}$ along symmetry directions, and so leave the potential energy unchanged. Fluctuations in field space along these eight directions are eight massless Goldstone bosons. We can describe the Goldstone boson fields by a 3×3 special unitary matrix $\Sigma(x)$, which represents the possible low-energy long-wavelength excitations of $\bar{q}_R q_L$. Here $v\Sigma_{kj}(x) \sim \bar{q}_R^{\,j}(x) q_L^{\,k}(x)$ gives the local orientation of the quark condensate. Σ has vacuum expectation value $\langle \Sigma \rangle = \mathbb{1}$. Under $SU(3)_L \times SU(3)_R$ transformations,

$$\Sigma \to L \, \Sigma \, R^\dagger. \tag{1.97}$$

The low-momentum strong interactions of the Goldstone bosons are described by an effective Lagrangian for $\Sigma(x)$ that is invariant under the chiral symmetry transformation in Eq. (1.97). The most general Lagrangian is

$$\mathcal{L}_{\text{eff}} = \frac{f^2}{8} \, \text{Tr} \, \partial^\mu \Sigma \, \partial_\mu \Sigma^\dagger + \text{ higher derivative terms}, \tag{1.98}$$

where f is a constant with dimensions of mass. There are no terms without any derivatives since $\text{Tr} \, \Sigma \Sigma^\dagger = 3$. At a low enough momentum the effects of the higher derivative terms can be neglected since they are suppressed by powers of $p_{\text{typ}}^2/\Lambda_{\text{CSB}}^2$, where p_{typ} is a typical momentum and Λ_{CSB} is the scale associated with chiral symmetry breaking, $\Lambda_{\text{CSB}} \sim 1$ GeV.

The field $\Sigma(x)$ is an $SU(3)$ matrix and it can be written as the exponential

$$\Sigma = \exp\left(\frac{2iM}{f}\right), \tag{1.99}$$

of M, a traceless 3×3 Hermitian matrix. Under the unbroken $SU(3)_V$ subgroup $(L = R = V)$, $\Sigma \to V \Sigma V^\dagger$, which implies that $M \to V M V^\dagger$, i.e., M transforms as the adjoint representation. M can be written out explicitly in terms of eight Goldstone boson fields:

$$M = \begin{pmatrix} \pi^0/\sqrt{2} + \eta/\sqrt{6} & \pi^+ & K^+ \\ \pi^- & -\pi^0/\sqrt{2} + \eta/\sqrt{6} & K^0 \\ K^- & \bar{K}^0 & -2\eta/\sqrt{6} \end{pmatrix}. \tag{1.100}$$

The factor of $2/f$ is inserted in Eq. (1.99) so that the Lagrangian in Eq. (1.98) gives kinetic-energy terms for the Goldstone bosons with the standard normalization.

In the QCD Lagrangian the light quark mass terms,

$$\mathcal{L}_{\text{mass}} = \bar{q}_L m_q q_R + \text{h.c.}, \tag{1.101}$$

transform under chiral $SU(3)_L \times SU(3)_R$ as $(\bar{\mathbf{3}}_L, \mathbf{3}_R) + (\mathbf{3}_L, \bar{\mathbf{3}}_R)$. We can include the effects of quark masses (to first order) on the strong interactions of the pseudo-Goldstone bosons, π, K, and η, by adding terms linear in m_q to Eq. (1.98) that transform in this way. Equivalently we can view the quark mass matrix itself as transforming like $m_q \to L m_q R^\dagger$ under $SU(3)_L \times SU(3)_R$. Then the Lagrange density in Eq. (1.101) is invariant under chiral $SU(3)_L \times SU(3)_R$. With this transformation rule for m_q, we include the effects of quark masses in the strong interactions of the π, K, and η by adding to Eq. (1.98) terms linear in m_q and $m_q^\dagger$ that are invariant under $SU(3)_L \times SU(3)_R$. This gives

$$\mathcal{L}_{\text{eff}} = \frac{f^2}{8} \text{Tr} \, \partial^\mu \Sigma \, \partial_\mu \Sigma^\dagger + v \, \text{Tr}(m_q^\dagger \Sigma + m_q \Sigma^\dagger) + \cdots. \tag{1.102}$$

The ellipses in Eq. (1.102) represent terms with more derivatives or more insertions of the light quark mass matrix. The quark mass terms in the Lagrange density in Eq. (1.102) give masses to the Goldstone bosons

$$m_{\pi^\pm}^2 = \frac{4v}{f^2}(m_u + m_d),$$

$$m_{K^\pm}^2 = \frac{4v}{f^2}(m_u + m_s), \tag{1.103}$$

$$m_{K^0}^2 = m_{\bar{K}^0}^2 = \frac{4v}{f^2}(m_d + m_s),$$

and hence the π, K, and η are referred to as pseudo-Goldstone bosons. The kaon masses are much larger than the pion masses, implying that $m_s \gg m_{u,d}$. For the $\eta - \pi^0$ system there is a mass-squared matrix with elements

$$m_{\pi^0 \pi^0}^2 = \frac{4v}{f^2}(m_u + m_d),$$

$$m_{\eta \pi^0}^2 = m_{\pi^0 \eta}^2 = \frac{4v}{\sqrt{3} f^2}(m_u - m_d), \tag{1.104}$$

$$m_{\eta \eta}^2 = \frac{4v}{3 f^2}(4m_s + m_u + m_d).$$

Because $m_s \gg m_{u,d}$, the off-diagonal terms are small compared with $m_{\eta \eta}^2$. Hence,

up to corrections suppressed by $(m_u - m_d)^2/m_s^2$,

$$m_{\pi^0}^2 \simeq \frac{4v}{f^2}(m_u + m_d) \tag{1.105}$$

and

$$m_\eta^2 \simeq \frac{4v}{3f^2}\left(4m_s + m_u + m_d\right). \tag{1.106}$$

It is interesting to note that the neutral pion mass is near the charged pion masses, not because m_u/m_d is near unity, but rather because $m_u - m_d$ is small compared with m_s. A more detailed study of mass relations, including electromagnetic corrections, leads to the expectation that $m_u/m_d \simeq 1/2$.

The chiral Lagrangian in Eq. (1.102) contains two parameters, v with dimensions of (mass)^3 and f with dimensions of mass. Since the quark masses always appear in conjunction with v it is not possible using the effective Lagrangian in Eq. (1.102) to determine the quark masses themselves. The effective theory describing the low-momentum interactions of the pseudo-Goldstone bosons only determines the ratios of quark masses, since v cancels out.

Equation (1.102) is an effective Lagrangian that describes the low-momentum interactions of the pseudo-Goldstone bosons. One can use the effective theory to compute scattering processes, such as $\pi - \pi$ scattering. Expanding out Σ in terms of the meson fields, one finds that the Tr $\partial_\mu \Sigma \, \partial^\mu \Sigma^\dagger$ part of the Lagrangian has the four-meson interaction term,

$$\frac{1}{6f^2}\mathrm{Tr}[M, \partial_\mu M][M, \partial^\mu M]. \tag{1.107}$$

Its tree-level matrix element (shown in Fig. 1.6) gives a contribution to the $\pi-\pi$ scattering amplitude of the form

$$\mathcal{M} \sim \frac{p_{\mathrm{typ}}^2}{f^2}, \tag{1.108}$$

where p_{typ} is a typical momentum. The amplitude is of the order of p_{typ}^2 since the vertex contains two derivatives. The mass terms also give a contribution of this form if we set $p_{\mathrm{typ}}^2 \sim m_\pi^2$. The contributions of higher derivative operators
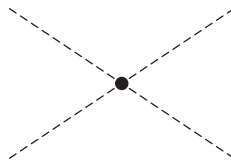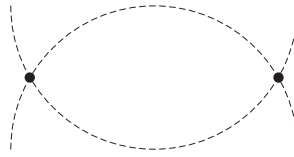

Fig. 1.6.   Tree-level contribution to $\pi-\pi$ scattering.

Fig. 1.7. One-loop contribution to π–π scattering.

in the chiral Lagrangian are suppressed by more factors of the small momentum p_{typ}.

What about loop diagrams? There are one-loop diagrams with two insertions of the $\pi\pi\pi\pi$ vertex, such as Fig. 1.7. Each vertex gives a factor of p^2/f^2, the two meson propagators give a factor of $1/p^4$, and the loop integration gives a factor of p^4. The resulting amplitude in the $\overline{\text{MS}}$ scheme is

$$\mathcal{M} \sim \frac{p_{\text{typ}}^4}{16\pi^2 f^4} \ln\big(p_{\text{typ}}^2/\mu^2\big). \qquad (1.109)$$

The factor of p_{typ}^4 in the numerator is required by dimensional analysis, since there is a factor of f^4 in the denominator and the subtraction point μ, which also has dimensions of mass, only occurs in the argument of logarithms. The $16\pi^2$ in the denominator typically occurs in the evaluation of one-loop diagrams. The one-loop diagram gives a contribution of the same order in the momentum expansion as operators in the chiral Lagrangian with four derivatives (or two insertions of the quark mass matrix). The total amplitude at order p^4 is the sum of one-loop diagrams containing order p^2 vertices and of tree graphs from the p^4 terms in the Lagrangian. The total p^4 amplitude is μ independent; the μ dependence in Eq. (1.109) is canceled by μ dependence in the coefficients of the p^4 terms in the Lagrangian.

The pattern we have just observed holds in general. More loops give a contribution of the same order as a term in the Lagrangian with more derivatives. One can show that a graph with L loops, and n_k insertions of vertices of order p^k, produces an amplitude of order p^D, where (see Problem 6)

$$D = 2 + 2L + \sum_k (k-2)\, n_k. \qquad (1.110)$$

Thus each loop increases D by two, and each insertion of a vertex of order p^k increases D by $k-2$. Note that $k-2 \geq 0$, since the Lagrangian starts at order p^2, so that each term in Eq. (1.110) is positive. Loop corrections and higher derivative operators are of comparable importance when the mass scale Λ_{CSB} that suppresses higher derivative operators is approximately equal to $4\pi f$. The computation of pseudo-Goldstone scattering amplitudes in a momentum expansion using an effective Lagrangian is known as chiral perturbation theory.

Although the u, d, and s quark masses are small, the spectrum of QCD suggests that the theory contains quasi-particles that transform like u, d, and s under the unbroken $SU(3)_V$ group but have a larger mass of approximately 350 MeV. These quasi-particles are called constituent quarks, and the hadronic spectrum is consistent at least qualitatively with spectra calculated from nonrelativistic potential models for the interactions of constituent quarks.

1.5 Integrating out heavy quarks

The top, bottom, and charm quark masses are $m_t \simeq 175$ GeV, $m_b \simeq 4.8$ GeV, and $m_c \simeq 1.4$ GeV. For processes that occur at energies well below the masses of these quarks, it is appropriate to go over to an effective theory of the strong interactions where these heavy quarks are integrated out of the theory and no longer occur as explicit degrees of freedom in the Lagrangian. The effects of Feynman diagrams with a virtual heavy quark Q are taken into account by non-renormalizable operators suppressed by factors of $1/m_Q$, and through shifts in the coupling constants of renormalizable terms in the effective Lagrangian. For definiteness, imagine integrating out the top quark and making a transition from the six-quark theory of the strong interactions to an effective five-quark theory. The strong coupling in the original theory with six flavors will be denoted by $g^{(6)}$, and in the effective five-quark theory by $g^{(5)}$. The relation between the two couplings is determined by ensuring that the scattering amplitudes computed in the five- and six-quark theories are the same. The general form of the relation is $g^{(5)}(\mu) = g^{(5)}[m_t/\mu, g^{(6)}(\mu)]$, since the g's are dimensionless. The power series expansion of $g^{(5)}$ in powers of $g^{(6)}$ has coefficients that, for μ very different from m_t, contain large logarithms of m_t^2/μ^2. If instead we pick $\mu = m_t$, $g^{(5)}$ has a power series expansion in $g^{(6)}(m_t)$ with coefficients that are not enhanced by any large logarithms. At tree level, $g^{(5)}(\mu) = g^{(6)}(\mu)$, so one expects

$$g^{(5)}(m_t) = g^{(6)}(m_t) \left\{ 1 + \mathcal{O}[\alpha_s^{(6)}(m_t)] \right\}. \tag{1.111}$$

An explicit calculation shows that the one-loop term in this equation vanishes, so the first nontrivial contribution is at two loops. The strong coupling in the effective theory with n quarks is written as in Eq. (1.92), where the value of Λ_{QCD} now depends on which particular effective theory is being used (i.e., $\Lambda_{\text{QCD}} \to \Lambda_{\text{QCD}}^{(n)}$). Equation (1.111) implies that, at leading order, the coupling constants are continuous at $\mu = m_t$. Combining this with Eq. (1.92), we find that

$$\Lambda_{\text{QCD}}^{(5)} = \Lambda_{\text{QCD}}^{(6)} \left(\frac{m_t}{\Lambda_{\text{QCD}}^{(6)}} \right)^{2/23}. \tag{1.112}$$

Integrating out the bottom and charm quarks to go over to effective four- and three-quark theories gives

$$\Lambda_{\text{QCD}}^{(4)} = \Lambda_{\text{QCD}}^{(5)} \left(\frac{m_b}{\Lambda_{\text{QCD}}^{(5)}} \right)^{2/25}, \tag{1.113}$$

$$\Lambda_{\text{QCD}}^{(3)} = \Lambda_{\text{QCD}}^{(4)} \left(\frac{m_c}{\Lambda_{\text{QCD}}^{(4)}} \right)^{2/27}. \tag{1.114}$$

Equations (1.112)–(1.114) determine the most important influence of virtual heavy quarks on low-energy physics. For example, the proton mass m_p is generated by nonperturbative dynamics in the effective three-quark theory so $m_p \propto \Lambda_{\text{QCD}}^{(3)}$, where the constant of proportionality is independent of the heavy quark masses. Imagining that the value of the strong coupling is fixed at some very-high-energy scale (e.g., the unification scale), Eqs. (1.112)–(1.114) give the dependence of the proton mass on the heavy quark masses. For example, doubling the charm quark mass increases the proton mass by the factor $2^{2/27} \simeq 1.05$.

1.6 Effective Hamiltonians for weak decays

The strong and electromagnetic interactions conserve quark and lepton flavor, so many particles can only decay by means of the weak interactions. The simplest example of such a decay is the weak decay of a muon, $\mu \to e \nu_\mu \bar{\nu}_e$. This decay is a purely leptonic process, since it does not involve any quark fields. The lowest-order graph for this decay has a single W boson exchanged, as shown in Fig. 1.8. The tree-level amplitude for the decay is

$$\mathcal{M}(\mu \to e \nu_\mu \bar{\nu}_e) = \left(\frac{g_2}{\sqrt{2}} \right)^2 \left[\bar{u}(p_{\nu_\mu}) \gamma_\alpha P_L u(p_\mu) \right] \left[\bar{u}(p_e) \gamma_\beta P_L v(p_{\nu_e}) \right]$$

$$\times \frac{1}{\left[(p_\mu - p_{\nu_\mu})^2 - M_W^2 \right]} \left[g^{\alpha\beta} - \frac{(p_\mu - p_{\nu_\mu})^\alpha (p_\mu - p_{\nu_\mu})^\beta}{M_W^2} \right], \tag{1.115}$$
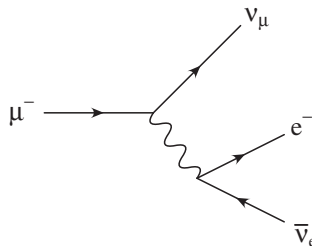

Fig. 1.8. Lowest-order diagram for μ decay.

where g_2 is the weak $SU(2)$ coupling constant, and the W propagator has been written in the unitary gauge. The muon mass is much smaller than the W-boson mass M_W, so the momenta of all the leptons involved in μ decay are much smaller than M_W. As a result, we can approximate the denominator of the W-boson propagator, $(p_\mu - p_{\nu_\mu})^2 - M_W^2$, by $-M_W^2$ and neglect the factor of $(p_\mu - p_{\nu_\mu})^\alpha (p_\mu - p_{\nu_\mu})^\beta / M_W^2$ in the numerator of the W-boson propagator. This approximation simplifies the decay amplitude to

$$\mathcal{M}(\mu \to e\nu_\mu \bar{\nu}_e) \simeq -\frac{4G_F}{\sqrt{2}} \big[\bar{u}\big(p_{\nu_\mu}\big)\gamma_\alpha P_L u\big(p_\mu\big)\big]\big[\bar{u}(p_e)\gamma^\alpha P_L v\big(p_{\nu_e}\big)\big], \quad (1.116)$$

where the Fermi constant G_F is defined by

$$\frac{G_F}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}. \tag{1.117}$$

The decay amplitude Eq. (1.116) is the same as that produced by the tree-level matrix element of the local effective Hamiltonian:

$$H_W = -\mathcal{L}_W = \frac{4G_F}{\sqrt{2}} [\bar{\nu}_\mu \gamma_\alpha P_L \mu][\bar{e}\gamma^\alpha P_L \nu_e]. \tag{1.118}$$

It is simpler to use an effective Hamiltonian description of the weak interactions in computing weak decay amplitudes at energies much smaller than M_W and M_Z, particularly if one wants to compute radiative corrections to decay amplitudes.

Electromagnetic loop corrections to the $\mu \to e\bar{\nu}_e \nu_\mu$ decay amplitude go partly into matrix elements of the Hamiltonian in Eq. (1.118) and partly into modifying the Hamiltonian itself. The corrections to the Hamiltonian are calculated by comparing amplitudes in the full theory with the W boson present as a dynamical field to amplitudes in the effective theory with the W boson removed. These corrections come from regions of loop momenta of order M_W, since the effective Hamiltonian has been chosen to correctly reproduce the full Hamiltonian for momenta much smaller than M_W. For this reason, the electron and muon masses occur as $m_{e,\mu}/M_W$ in the effective Hamiltonian, and they can be neglected at leading order. They are, of course, very important for the matrix elements of the effective Hamiltonian.

Neglecting the electron and muon masses, we know that the Hamiltonian must be of the form in Eq. (1.118). In this limit electromagnetic corrections do not change chirality and so $[\bar{\nu}_\mu \gamma_\alpha P_L \mu][\bar{e}\gamma^\alpha P_L \nu_e]$ and $[\bar{\nu}_\mu \gamma_\alpha P_L \nu_e][\bar{e}\gamma^\alpha P_L \mu]$ are the only possible dimension-six operators that can occur. Terms with three gamma matrices between the fermion fields can be reduced to single gamma matrices by using the identity

$$\gamma_\alpha \gamma_\beta \gamma_\nu = g_{\alpha\beta}\gamma_\nu + g_{\beta\nu}\gamma_\alpha - g_{\alpha\nu}\gamma_\beta - i\epsilon_{\alpha\beta\nu\eta}\gamma^\eta\gamma_5, \tag{1.119}$$

where the sign convention is $\epsilon_{0123} = 1$. Higher dimension operators are

negligible, being suppressed by powers of $1/M_W$. The Fierz operator identity,

$$[\bar{\psi}_1 \gamma_\alpha P_L \psi_2][\bar{\psi}_3 \gamma^\alpha P_L \psi_4] = [\bar{\psi}_1 \gamma_\alpha P_L \psi_4][\bar{\psi}_3 \gamma^\alpha P_L \psi_2], \qquad (1.120)$$

allows one to replace $[\bar{\nu}_\mu \gamma_\alpha P_L \nu_e][\bar{e} \gamma^\alpha P_L \mu]$ by $[\bar{\nu}_\mu \gamma_\alpha P_L \mu][\bar{e} \gamma^\alpha P_L \nu_e]$. So beyond tree level the effective Hamiltonian is modified to

$$H_W = \frac{4G_F}{\sqrt{2}} C\left[\frac{M_W}{\mu}, \alpha(\mu)\right][\bar{\nu}_\mu \gamma^\alpha P_L \mu][\bar{e} \gamma_\alpha P_L \nu_e], \qquad (1.121)$$

where μ is the subtraction point, and α is the electromagnetic fine structure constant. The only modification due to radiative corrections is the coefficient C, which is unity at tree level. Loop corrections at $\mu = M_W$ with virtual loop momenta of order M_W determine the deviation of the coefficient C from unity, so one expects

$$C[1, \alpha(M_W)] = 1 + \mathcal{O}[\alpha(M_W)]. \qquad (1.122)$$

Any dependence of the matrix elements of the four-fermion operator $[\bar{\nu}_\mu \gamma_\alpha P_L \mu][\bar{e} \gamma^\alpha P_L \nu_e]$ on the subtraction point is canceled by the μ dependence of C, so that physical quantities such as decay rates do not depend on μ. If the Hamiltonian above is used to calculate the muon decay rate, with $\mu = M_W$ naively one would think that there are large logarithms of (m_μ^2/M_W^2) in the perturbative expansion of the matrix elements of the Hamiltonian. In fact we know that C is μ-independent and hence such logarithms do not occur. A simple explanation of this fact follows using the Fierz identity in Eq. (1.120), which allows us to rewrite the effective Hamiltonian in the form $[\bar{\nu}_\mu \gamma_\alpha P_L \nu_e][\bar{e} \gamma^\alpha P_L \mu]$. The neutrino fields do not interact electromagnetically, so the only renormalization is that of $\bar{e} \gamma^\alpha P_L \mu$. In the limit $m_e = m_\mu = 0$, $\bar{e} \gamma^\alpha P_L \mu$ is a conserved current and does not get renormalized.

The electromagnetic coupling α is so small that even when it is multiplied by large logarithms, perturbation theory is usually adequate. However, this is not the case for the strong interactions. For an example in which such logarithms are important and must be summed, consider the effective Hamiltonian for nonleptonic $b \rightarrow c$ decays at tree level,

$$H_W^{(\Delta c = 1)} = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* [\bar{c}^\alpha \gamma_\mu P_L b_\alpha][\bar{d}^\beta \gamma^\mu P_L u_\beta]. \qquad (1.123)$$

In Eq. (1.123) α and β are color indices and repeated indices are summed. There is a contribution to the effective Hamiltonian for nonleptonic $b \rightarrow c$ decays where the d quark is replaced by a s quark. It has a coefficient that is suppressed by $|V_{us}/V_{ud}| \approx 0.2$ compared to Eq. (1.123). This "Cabibbo suppressed" contribution is neglected here. Also, we are focusing on $\Delta c = 1$ decays. There are nonleptonic decays where, at tree level, the final state has both a c and $\bar{c}$ quark. For these decays the coefficient in the effective Hamiltonian $H_W^{(\Delta c = 0)}$ is not smaller than Eq. (1.123).

Strong interaction loop corrections change the form of the Hamiltonian for $b \to c$ decays. An argument similar to that used for μ decay shows that there are two possible terms in the $\Delta c = 1$ effective Hamiltonian,

$$H_W = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \left\{ C_1 \left[\frac{M_W}{\mu}, \alpha_s(\mu) \right] O_1(\mu) + C_2 \left[\frac{M_W}{\mu}, \alpha_s(\mu) \right] O_2(\mu) \right\},$$

(1.124)

where

$$O_1(\mu) = [\bar{c}^\alpha \gamma_\mu P_L b_\alpha][\bar{d}^\beta \gamma^\mu P_L u_\beta],$$
$$O_2(\mu) = [\bar{c}^\beta \gamma_\mu P_L b_\alpha][\bar{d}^\alpha \gamma^\mu P_L u_\beta].$$

(1.125)

The coefficients $C_{1,2}$ are determined by comparing Feynman diagrams in the effective theory with the W-boson integrated out with analogous diagrams in the full theory. At $\mu = M_W$ we have from Eq. (1.123) that

$$C_1[1, \alpha_s(M_W)] = 1 + \mathcal{O}[\alpha_s(M_W)],$$
$$C_2[1, \alpha_s(M_W)] = 0 + \mathcal{O}[\alpha_s(M_W)].$$

(1.126)

Subtraction-point dependence of the operators $O_{1,2}$ cancels that in the coefficients $C_{1,2}$. Here $O_{1,2}$ are local four-quark operators, and they must be renormalized to render their matrix elements finite. The relationship between bare and renormalized operators has the form

$$O_i^{(0)} = Z_{ij} O_j,$$

(1.127)

where $i, j = \{1, 2\}$ and the repeated index j is summed over. Since the bare operator is μ independent,

$$0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} O_i^{(0)}(\mu) = \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_{ij} \right) O_j + Z_{ij} \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} O_j \right),$$

(1.128)

which implies that

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} O_j = -\gamma_{ji} O_i(\mu),$$

(1.129)

where

$$\gamma_{ji} = Z_{jk}^{-1} \left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} Z_{ki} \right).$$

(1.130)

Here $\gamma_{ij}(g)$ is called the anomalous dimension matrix. It can be calculated order by order in the coupling constant from the Z's. The subtraction-point independence of the weak Hamiltonian implies that

$$0 = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} H_W = \mu \frac{\mathrm{d}}{\mathrm{d}\mu} (C_j O_j),$$

(1.131)

yielding

$$\left(\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_j\right) O_j - C_j \gamma_{ji} O_i = 0. \tag{1.132}$$

Since the operators $O_{1,2}$ are independent we conclude that

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_i = \gamma_{ji} C_j. \tag{1.133}$$

The solution to this differential equation is

$$C_i\left[\frac{M_W}{\mu}, \alpha_s(\mu)\right] = P \exp\left[\int_{g(M_W)}^{g(\mu)} \frac{\gamma^T(g)}{\beta(g)} dg\right]_{ij} C_j[1, \alpha_s(M_W)]. \tag{1.134}$$

Here P denotes "coupling constant ordering" of the anomalous dimension matrices in the exponent, and γ^T is the transpose of γ.

It is straightforward to calculate the anomalous dimension matrix for $O_{1,2}$. At one loop, it is

$$\gamma(g) = \frac{g^2}{8\pi^2}\begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix}. \tag{1.135}$$

It is convenient to diagonalize this matrix by forming the linear combinations of operators

$$O_\pm = O_1 \pm O_2. \tag{1.136}$$

Using the Fierz identity in Eq. (1.120), it is evident that O_+ is symmetric under interchange of the d and c quark fields, whereas O_- is antisymmetric. Under an $SU(2)$ flavor group under which the d and c quark fields form a doublet, O_- is a singlet and O_+ is a triplet. The c, d mass difference breaks this flavor symmetry. Quark masses do not affect the renormalization constants Z_{ij}, so mixing between O_+ and O_- is forbidden by this symmetry. In terms of $O_\pm$ the effective weak Hamiltonian is

$$H_W = \frac{4G_F}{\sqrt{2}} V_{cb} V_{ud}^* \left\{ C_+\left[\frac{M_W}{\mu}, \alpha_s(\mu)\right] O_+(\mu) + C_-\left[\frac{M_W}{\mu}, \alpha_s(\mu)\right] O_-(\mu) \right\}, \tag{1.137}$$

where

$$C_\pm[1, \alpha_s(M_W)] = \frac{1}{2} + \mathcal{O}[\alpha_s(M_W)]. \tag{1.138}$$

At any other subtraction point

$$C_\pm\left[\frac{M_W}{\mu}, \alpha_s(\mu)\right] = \exp\left[\int_{g(M_W)}^{g(\mu)} \frac{\gamma_\pm(g)}{\beta(g)} dg\right] C_\pm[1, \alpha_s(M_W)], \tag{1.139}$$

where

$$\gamma_+(g) = \frac{g^2}{4\pi^2} + \mathcal{O}(g^4),$$

$$\gamma_-(g) = -\frac{g^2}{2\pi^2} + \mathcal{O}(g^4),$$

(1.140)

and $\beta(g)$ is given by Eq. (1.87). Provided $\mu \gg \Lambda_{\text{QCD}}$, the strong coupling $\alpha_s(\mu)$ is small over the range of integration in Eq. (1.139) and higher-order terms in g can be neglected in $\gamma_\pm$ and β. This gives

$$C_\pm\left[\frac{M_W}{\mu}, \alpha_s(\mu)\right] = \frac{1}{2}\left[\frac{\alpha_s(M_W)}{\alpha_s(\mu)}\right]^{a_\pm},$$

(1.141)

where

$$a_+ = \frac{6}{33 - 2N_q}, \qquad a_- = -\frac{12}{33 - 2N_q}.$$

(1.142)

Expressing $\alpha_s(M_W)$ in terms of $\alpha_s(\mu)$ using Eq. (1.89), the perturbative power series expansions of $C_\pm$ have the form

$$\frac{1}{2} + a_1\,\alpha_s(\mu)\ln(M_W/\mu) + a_2\,\alpha_s^2(\mu)\ln^2(M_W/\mu) + \cdots.$$

(1.143)

The expression for $C_\pm$ in Eq. (1.143) sums all leading logarithms of the form $\alpha_s^n(\mu)\ln^n(M_W/\mu)$, neglecting subleading logarithms of order $\alpha_s^n\ln^{n-1}(M_W/\mu)$. The series of subleading logarithms can be summed by using two-loop renormalization group equations, and so on. The subtraction-point dependence in the coefficients $C_\pm$ cancels that in the matrix elements of the operators $O_\pm$ so any value of μ can be used. However, if p_{typ} is the typical momentum in a nonleptonic decay, the matrix elements of $O_\pm$ will contain large logarithms of (μ^2/p_{typ}^2), for μ very different from p_{typ}. Roughly these logarithms come from integrations over momenta in the region between p_{typ} and μ. They are summed by scaling the coefficients down from the subtraction point M_W to one of order p_{typ}, which moves the logarithms from the matrix elements of $O_\pm$ to the coefficients $C_\pm$.

The exponents $a_\pm$ in Eq. (1.142) depend on the number of quark flavors N_q. It is convenient to integrate out the top quark at the same time as the W boson so that $N_q = 5$. For inclusive weak decay of a hadron containing a b quark, the typical momenta of the decay products are of the order of the b-quark mass, and the large logarithms of $(M_W/m_b)^2$ are summed by evaluating the coefficients $C_\pm$ at $\mu = m_b$. In this case,

$$C_+(m_b) = 0.42, \qquad C_-(m_b) = 0.70,$$

(1.144)

using $\alpha_s(M_W) = 0.12$ and $\alpha_s(m_b) = 0.22$.

1.7 The pion decay constant

Weak pion decay $\pi^- \to \mu \bar{\nu}_\mu$ determines the value for the parameter f that occurs in the chiral Lagrangian for pion strong interactions in Eq. (1.98). Neglecting electromagnetic corrections, the effective Hamiltonian for $\pi^- \to \mu \bar{\nu}_\mu$ decay is

$$H_{\text{eff}} = \frac{4G_F}{\sqrt{2}} V_{ud} [\bar{u} \gamma_\alpha P_L d][\bar{\mu} \gamma^\alpha P_L \nu_\mu]. \tag{1.145}$$

Here color indices on the quark fields are suppressed. The current $\bar{u} \gamma_\alpha P_L d$ is conserved in the limit $m_{u,d} \to 0$, and consequently its strong interaction matrix elements are subtraction-point independent. Taking the $\pi^- \to \mu \bar{\nu}_\mu$ matrix element of Eq. (1.145) gives the pion decay amplitude

$$\mathcal{M} = -i\sqrt{2} G_F V_{ud} f_\pi \, \bar{u}(p_\mu) \slashed{p}_\pi P_L \upsilon(p_{\nu_\mu}), \tag{1.146}$$

where the pion decay constant, f_π, is the value of the pion-to-vacuum matrix element of the axial current,

$$\langle 0 | \bar{u} \gamma^\alpha \gamma_5 d | \pi^-(p_\pi) \rangle = -i f_\pi p_\pi^\alpha. \tag{1.147}$$

The measured pion decay rate gives $f_\pi \simeq 131$ MeV. In Eq. (1.147) the pion field is normalized by using the standard covariant norm: $\langle \pi(p_\pi') | \pi(p_\pi) \rangle = 2E_\pi (2\pi)^3 \delta^3(\mathbf{p}_\pi' - \mathbf{p}_\pi)$. Parity invariance of the strong interactions implies that only the axial current part of the left-handed current contributes in Eq. (1.146).

In the limit $m_{u,d,s} = 0$, global $SU(3)_L$ transformations are a symmetry of QCD. The conserved currents associated with this symmetry can be derived by considering the change in the QCD Lagrangian under infinitesimal local $SU(3)_L$ transformations,

$$L = 1 + i\epsilon_L^A T^A, \tag{1.148}$$

with space–time dependent infinitesimal parameters $\epsilon_L^A(x)$. The change in the QCD Lagrange density, Eq. (1.93), under this transformation is

$$\delta \mathcal{L}_{\text{QCD}} = -J_{L\mu}^A \partial^\mu \epsilon_L^A, \tag{1.149}$$

where

$$J_{L\mu}^A = \bar{q}_L T^A \gamma_\mu q_L \tag{1.150}$$

are the conserved currents associated with $SU(3)_L$ transformations. We also know how left-handed transformations act on the meson fields in Σ. The change in the chiral Lagrange density under an infinitesimal left-handed transformation on the Σ in Eq. (1.98) is

$$\delta \mathcal{L}_{\text{eff}} = -J_{L\mu}^A \partial^\mu \epsilon_L^A, \tag{1.151}$$

where

$$J_{L\mu}^A = -\frac{if^2}{4}\operatorname{Tr} T^A \Sigma \, \partial_\mu \Sigma^\dagger. \tag{1.152}$$

Comparing Eqs. (1.150) and (1.152) gives

$$\bar{q}_L T^A \gamma_\mu q_L = -i\frac{f^2}{4}\operatorname{Tr} T^A \Sigma \partial_\mu \Sigma^\dagger + \cdots, \tag{1.153}$$

where the ellipses are contributions from higher derivative terms in the chiral Lagrangian. Matrix elements of the quark current involving the pseudo-Goldstone boson can be calculated by expanding Σ in terms of M on the right-hand side of Eq. (1.153). In particular, the part linear in M yields the tree-level relation $f = f_\pi$. Loops and higher derivatives operators in the chiral Lagrangian give corrections to the relation between f and f_π. The kaon decay constant is defined by

$$\langle 0|\bar{u}\gamma^\alpha\gamma_5 s|K^-(p_K)\rangle = -if_K p_K^\alpha. \tag{1.154}$$

The measured $K^- \to \mu\bar{\nu}_\mu$ decay rate determines f_K to be $\sim 25\%$ larger than the pion decay constant, $f_K \simeq 164\,\mathrm{MeV}$. At leading order in chiral $SU(3)_L \times SU(3)_R$, $f = f_K = f_\pi$, and the 25% difference between f_π and f_K is the typical size of $SU(3)_V$ breaking arising from the nonzero value of the strange quark mass.

At higher orders in chiral perturbation theory, the Noether procedure for finding the representation of $\bar{q}_L T^A \gamma_\mu q_L$ in terms of pseudo-Goldstone boson fields becomes ambiguous. Total derivative operators in the chiral Lagrangian can give a contribution to the current $J_{L\mu}^A$ (although not to the charges $Q_L^A = \int d^3x\, J_{L0}^A$), even though they are usually omitted from the chiral Lagrangian because they do not contribute to pseudo-Goldstone boson S-matrix elements. Note that at leading order in chiral perturbation theory there are no possible total derivative operators since $\partial^\mu(\operatorname{Tr}\Sigma^\dagger\partial_\mu\Sigma) = 0$.

1.8 The operator product expansion

The operator product expansion (OPE) is an important tool in particle physics and condensed matter physics, and it will be applied later in this book to describe inclusive B decay and to discuss sum rules. The use of the operator product expansion is best illustrated by an explicit example. In this section, the OPE will be applied to the study of deep inelastic lepton–proton scattering. The main purpose of the discussion is to explain the use of the OPE, so the presentation of the phenomenology of deep inelastic scattering will be kept to a minimum.

The basic deep inelastic scattering process is $\ell(k) + \mathrm{proton}(p) \to \ell(k') + X(p+q)$, in which an incoming lepton ℓ with momentum k scatters off a target proton, to produce an outgoing lepton ℓ with momentum k', plus anything X. The Feynman graph in Fig. 1.9 is the leading term in an expansion in the
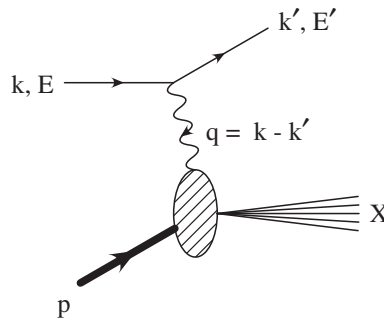
Fig. 1.9. The basic diagram for deep inelastic lepton–hadron scattering. The virtual photon momentum is q. The final hadronic state is not measured and is denoted by X.

electromagnetic fine structure constant α. The traditional kinematical variables used to describe the inclusive scattering process are the momentum transfer $Q^2 = -(k' - k)^2$, and the dimensionless variable x defined by

$$x = \frac{Q^2}{2p \cdot q}, \tag{1.155}$$

where $q = k - k'$. Note that for deep inelastic scattering, $Q^2 > 0$. It is also useful to define $\omega = 1/x$. The deep inelastic scattering cross section is the inclusive cross section in the limit that Q^2 is large with x fixed. The total cross section is obtained from squaring the amplitude represented by Fig. 1.9 and performing the appropriate phase space integrations. The lepton and photon parts of this amplitude as well as the phase space integrals can easily be computed. The nontrivial quantity is the square of the hadronic part of the diagram, which is

$$\sum_X (2\pi)^4 \delta^4(q + p - p_X) \langle p | J_{\text{em}}^\mu(0) | X \rangle \langle X | J_{\text{em}}^\nu(0) | p \rangle, \tag{1.156}$$

where the sum is over all possible final states X, and J_{em}^μ is the electromagnetic current. For convenience momentum and spin labels on the state vectors are suppressed. A spin average over the proton states $|p\rangle$ is also understood.

It is conventional to define the hadronic tensor

$$W^{\mu\nu}(p, q) = \frac{1}{4\pi} \int d^4x \, e^{iq \cdot x} \langle p | \big[J_{\text{em}}^\mu(x), J_{\text{em}}^\nu(0) \big] | p \rangle. \tag{1.157}$$

Inserting a complete set of states gives

$$W^{\mu\nu}(p, q) = \frac{1}{4\pi} \sum_X \int d^4x \, e^{iq \cdot x} \big[\langle p | J_{\text{em}}^\mu(x) | X \rangle \langle X | J_{\text{em}}^\nu(0) | p \rangle$$

$$- \langle p | J_{\text{em}}^\nu(0) | X \rangle \langle X | J_{\text{em}}^\mu(x) | p \rangle \big], \tag{1.158}$$

where the sum on X is a sum over all final states, as well as an integral over the

allowed final state phase space. Translation invariance implies that

$$\langle p|J_{\text{em}}^{\mu}(x)|X\rangle = \langle p|J_{\text{em}}^{\mu}(0)|X\rangle e^{i(p-p_X)\cdot x},$$
$$\langle X|J_{\text{em}}^{\mu}(x)|p\rangle = \langle X|J_{\text{em}}^{\mu}(0)|p\rangle e^{i(p_X-p)\cdot x}. \tag{1.159}$$

Inserting Eq. (1.159) into Eq. (1.158) gives

$$W^{\mu\nu}(p, q) = \frac{1}{4\pi} \sum_X \Big[(2\pi)^4 \delta^4(q + p - p_X)\langle p|J_{\text{em}}^{\mu}(0)|X\rangle\langle X|J_{\text{em}}^{\nu}(0)|p\rangle$$
$$- (2\pi)^4 \delta^4(q + p_X - p)\langle p|J_{\text{em}}^{\nu}(0)|X\rangle\langle X|J_{\text{em}}^{\mu}(0)|p\rangle \Big]. \tag{1.160}$$

The only allowed final states are those with $p_X^0 \geq p^0$, since the baryon number is conserved. For $q^0 > 0$, only the first delta function in Eq. (1.160) can be satisfied, and the sum in $W_{\mu\nu}$ reduces to the expression in Eq. (1.156) involving the hadronic currents, and the energy-momentum conserving delta function (up to a factor of $1/4\pi$). Since only the first term in Eq. (1.158) contributes, one could have defined $W_{\mu\nu}$ in Eq. (1.157) simply as the matrix element of $J_{\text{em}}^{\mu}(x)J_{\text{em}}^{\nu}(0)$ without the commutator. The reason for using the commutator is that then $W_{\mu\nu}$ has a nicer analytic structure when continued away from the physical region. The most general form of $W_{\mu\nu}$ consistent with current conservation, parity and time-reversal invariance is

$$W_{\mu\nu} = F_1 \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{F_2}{p \cdot q} \left(p_\mu - \frac{p \cdot q \, q_\mu}{q^2} \right)\left(p_\nu - \frac{p \cdot q \, q_\nu}{q^2} \right), \tag{1.161}$$

where $F_{1,2}$ can be written as functions of x and Q^2. Here $F_{1,2}$ are called structure functions.

The Q^2 dependence of the structure functions can be calculated in quantum chromodynamics. The starting point in the derivation is the time-ordered product of two currents:

$$t^{\mu\nu} \equiv i \int d^4x \, e^{iq \cdot x} \, T\left[J_{\text{em}}^{\mu}(x) \, J_{\text{em}}^{\nu}(0) \right]. \tag{1.162}$$

The proton matrix element of $t_{\mu\nu}$,

$$T_{\mu\nu} = \langle p|t_{\mu\nu}|p\rangle, \tag{1.163}$$

can also be written in terms of structure functions,

$$T_{\mu\nu} = T_1 \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) + \frac{T_2}{p \cdot q} \left(p_\mu - \frac{p \cdot q \, q_\mu}{q^2} \right)\left(p_\nu - \frac{p \cdot q \, q_\nu}{q^2} \right). \tag{1.164}$$

The analytic structure of $T_{1,2}$ as a function of ω for fixed Q^2 is shown in Fig. 1.10. There are cuts in the physical region $1 \leq |\omega|$. The discontinuity across the
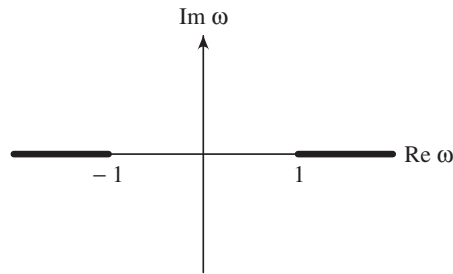
Fig. 1.10. The analytic structure of $T_{\mu\nu}$ in the complex ω plane. The discontinuity across the cuts $1 \leq |\omega| \leq \infty$ is related to $W_{\mu\nu}$.

right-hand cut for $T_{1,2}$ is $F_{1,2}$,

$$\mathrm{Im}\, T_{1,2}(\omega + i\epsilon, Q^2) = 2\pi F_{1,2}(\omega, Q^2). \qquad (1.165)$$

[The discontinuity across the left-hand cut gives the structure functions for deep inelastic scattering off antiprotons.]

The key idea that permits the computation of $T_{\mu\nu}$ in certain limiting cases is the operator product expansion. Consider the time-ordered product of two local operators separated in position by z:

$$T\left[O_a(z)\, O_b(0)\right]. \qquad (1.166)$$

For small z, the operators are at practically the same point. In this limit, the operator product can be written as an expansion in local operators,

$$T\left[O_a(z)\, O_b(0)\right] = \sum_k C_{abk}(z) O_k(0). \qquad (1.167)$$

The coefficient functions depend on the separation z. Low-momentum (compared with $1/z$) matrix elements of the left-hand side are completely equivalent to matrix elements of the right-hand side. Thus one can replace the product $T\left[O_a(z)O_b(0)\right]$ in the computation of matrix elements by the expansion in Eq. (1.167), where the coefficients $C_{abk}(z)$ are independent of the matrix elements, provided that the external states have momentum components that are small compared with the inverse separation $1/z$. In QCD, the coupling constant is small at short distances because of asymptotic freedom. Thus the coefficient functions can be computed in perturbation theory, since all nonperturbative effects occur at scales that are much larger than z, and do not affect the computation of the coefficient functions.

The momentum space version of the operator product expansion is for the product

$$\int d^4z\, e^{iq \cdot z} T\left[O_a(z)\, O_b(0)\right]. \qquad (1.168)$$

In the limit that $q \to \infty$, the Fourier transform in Eq. (1.168) forces $z \to 0$, and again the operator product can be expanded in terms of local operators with coefficient functions that depend on q. For large q,

$$\int d^4z\, e^{iq \cdot z} T[O_a(z)\, O_b(0)] = \sum_k C_{abk}(q) O_k(0). \qquad (1.169)$$

This expansion is valid for all matrix elements, provided q is much larger than the characteristic momentum in any of the external states.

We will use the Fourier transform version of the operator product expansion, Eq. (1.169). The product of two electromagnetic currents in Eq. (1.162) can be expanded in terms of a sum of local operators multiplied by coefficients that are functions of q. This expansion will be valid for proton matrix elements, Eq. (1.163), provided that q is much larger than the typical hadronic mass scale Λ_{QCD}. The local operators in the operator product expansion for QCD are quark and gluon operators with arbitrary dimension d and spin n. An operator with spin n and dimension d can be written as $O_{d,n}^{\mu_1 \cdots \mu_n}$, where $O_{d,n}$ is symmetric and traceless in $\mu_1 \cdots \mu_n$. The matrix element of $O_{d,n}$ in the spin-averaged proton target is proportional to $m_p^{d-n-2}\, \mathcal{S}[p^{\mu_1} \cdots p^{\mu_n}]$. $\mathcal{S}$ acts on a tensor to project out the completely symmetric traceless component. The power of m_p follows from dimensional analysis, since a proton state with the conventional relativistic normalization has dimension minus one. The coefficient functions in the operator product expansion are functions only of q. Thus the free indices on the operator O must be either μ, ν or be contracted with q^α. Every index on O contracted with q^α produces a factor of $p \cdot q$, which is of the order of Q^2 in the deep inelastic limit. An index μ or ν is contracted with the lepton momentum, and produces a factor of $p \cdot k$ or $p \cdot k'$, both of which are also of the order of Q^2 in the deep inelastic limit. In addition, since $t_{\mu\nu}$ has dimension two, the coefficient of O must have dimension $[\text{mass}]^{2-d}$ in the operator product expansion. Thus, the contribution of any operator O to the differential cross section is of the order of

$$\begin{aligned}
C_{\mu_1 \cdots \mu_n} O_{d,n}^{\mu_1 \cdots \mu_n} &\to \frac{q_{\mu_1}}{Q} \cdots \frac{q_{\mu_n}}{Q} Q^{2-d} \langle O_{d,n}^{\mu_1 \cdots \mu_n} \rangle, \\
&\to \frac{q_{\mu_1}}{Q} \cdots \frac{q_{\mu_n}}{Q} Q^{2-d} m_p^{d-n-2}\, p^{\mu_1} \cdots p^{\mu_n}, \\
&\to \frac{(p \cdot q)^n}{Q^n} Q^{2-d} m_p^{d-n-2}, \\
&\to \omega^n \left(\frac{Q}{m_p}\right)^{2+n-d} = \omega^n \left(\frac{Q}{m_p}\right)^{2-t}, \qquad (1.170)
\end{aligned}$$

where the twist t is defined as

$$t = d - n = \text{dimension} - \text{spin}. \qquad (1.171)$$

Table 1.2. *Dimension, spin, and twist*
for the basic objects
in the QCD Lagrangian

Parameter	q	$G_{\mu\nu}$	D^μ
Dimension	3/2	2	1
Spin	1/2	1	1
Twist	1	1	0

The most important operators in the operator product expansion are those with the lowest possible twist. Twist-two operators contribute a finite amount to the structure functions in the deep inelastic limit, twist three contributions are suppressed by m_p/Q, and so on. The fundamental fields in QCD are quark and gluon fields, so the gauge invariant operators in the operator product expansion can be written in terms of quark fields q, the gluon field strength $G_{\mu\nu}$, and the covariant derivative D^μ. Table 1.2 lists the basic objects, with their dimension and twist. Any gauge invariant operator must contain at least two quark fields, or two gluon field strength tensors. Thus the lowest possible twist is two. A twist-two operator has either two q's or two $G_{\mu\nu}$'s and an arbitrary number of covariant derivatives. The indices of the covariant derivatives are not contracted, because an operator such as D^2 has twist two, whereas the traceless symmetric part of $D^\alpha D^\beta$ has twist zero.

The first step in doing an operator product expansion is to determine all the linearly independent operators that can occur. We have just seen that the leading operators are twist-two quark and gluon operators. We will simplify the analysis by considering not the electromagnetic current but rather $J_\mu = \bar{q}\gamma_\mu q$ for a single quark flavor q. Results for the realistic case can be obtained by summing over flavors weighted by the square of quark charges. The Lorentz structure of the quark operators must be either $\bar{q}\gamma^\mu q$ or $\bar{q}\gamma^\mu \gamma_5 q$ in the limit that light quark masses can be neglected, because the operator product $J^\mu J^\nu$ does not change chirality. The conventional basis for twist-two quark operators is:

$$O_{q,V}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} \mathcal{S}\left\{\bar{q}\,\gamma^{\mu_1}\,\overset{\leftrightarrow}{D}{}^{\mu_2}\cdots\overset{\leftrightarrow}{D}{}^{\mu_n} q\right\}, \qquad (1.172)$$

$$O_{q,A}^{\mu_1\cdots\mu_n} = \frac{1}{2}\left(\frac{i}{2}\right)^{n-1} \mathcal{S}\left\{\bar{q}\,\gamma^{\mu_1}\,\overset{\leftrightarrow}{D}{}^{\mu_2}\cdots\overset{\leftrightarrow}{D}{}^{\mu_n}\gamma_5 q\right\}, \qquad (1.173)$$

where

$$\bar{A}\,\overset{\leftrightarrow}{D}{}^\mu\,B = \bar{A}\,\vec{D}{}^\mu B - \bar{A}\,\overset{\leftarrow}{D}{}^\mu B. \qquad (1.174)$$

The operators $O_{q,A}^{\mu_1 \cdots \mu_n}$ have matrix elements proportional to the proton spin, and so do not contribute to spin-averaged scattering. The tower of twist-two gluon operators needed for scattering from unpolarized protons is

$$O_{g,V}^{\mu_1 \cdots \mu_n} = -\frac{1}{2} \left(\frac{i}{2} \right)^{n-2} \mathcal{S} \left\{ G_A^{\mu_1 \alpha} \overset{\leftrightarrow}{D}^{\mu_2} \ldots \overset{\leftrightarrow}{D}^{\mu_{n-1}} G_{A\alpha}{}^{\mu_n} \right\}. \tag{1.175}$$

We will only compute the operator product expansion to lowest order in α_s, so the gluon operators do not occur.

The most general form for $t^{\mu\nu}$ consistent with current conservation and using only twist-two operators is

$$t_{\mu\nu} = \sum_{n=2,4,\ldots}^{\infty} \left(-g_{\mu\nu} + \frac{q_\mu q_\nu}{q^2} \right) \frac{2^n q_{\mu_1} \cdots q_{\mu_n}}{(-q^2)^n} \sum_{j=q,g} 2\, C_{j,n}^{(1)} \, O_{j,V}^{\mu_1 \cdots \mu_n}$$

$$+ \sum_{n=2,4,\ldots}^{\infty} \left(g_{\mu\mu_1} - \frac{q_\mu q_{\mu_1}}{q^2} \right) \left(g_{\nu\mu_2} - \frac{q_\nu q_{\mu_2}}{q^2} \right)$$

$$\times \frac{2^n q_{\mu_3} \cdots q_{\mu_n}}{(-q^2)^{n-1}} \sum_{j=q,g} 2 C_{j,n}^{(2)} O_{j,V}^{\mu_1 \cdots \mu_n}, \tag{1.176}$$

where the unknown coefficients are $C_{j,n}^{(1)}$ and $C_{j,n}^{(2)}$, and the factors of two and signs have been chosen for later convenience.

The second step in doing an operator product expansion is to determine the coefficients of the operators, $C_{j,n}^{(1)}$ and $C_{j,n}^{(2)}$. The best way to do this is to evaluate enough on-shell matrix elements to determine all the coefficients. Since we have argued that the coefficients can be computed using any matrix elements, we will evaluate the coefficients by taking matrix elements in on-shell quark and gluon states. We will only illustrate the computation of the coefficients to lowest nontrivial order, i.e., $(\alpha_s)^0$, in this chapter.

A generic term in the operator product expansion can be written as

$$J J \sim C_q O_q + C_g O_g, \tag{1.177}$$

where q and g refer to quark and gluon operators. Taking the matrix element of both sides in a free quark state gives

$$\langle q| J J |q \rangle \sim C_q \langle q| O_q |q \rangle + C_g \langle q| O_g |q \rangle. \tag{1.178}$$

The electromagnetic current is a quark operator. Thus the left-hand side is of the order of $(\alpha_s)^0$. The matrix element $\langle q| O_q |q \rangle$ is also of the order of $(\alpha_s)^0$, whereas the matrix element $\langle q| O_g |q \rangle$ is of the order of $(\alpha_s)^1$ since there are at least two gluons in O_g, each of which contributes a factor of the QCD coupling constant g to the matrix element. Thus, one can determine C_q to leading order by taking the quark matrix element of both sides of the operator product expansion, neglecting the gluon operators.
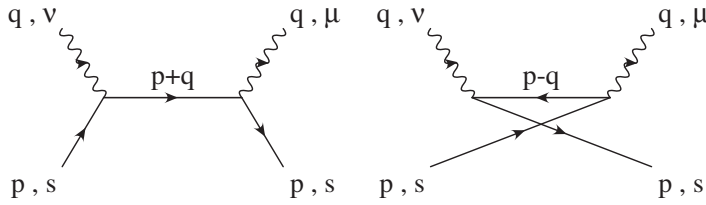
Fig. 1.11. The lowest order diagrams contributing to the quark matrix element of the product of two electromagnetic currents.

As mentioned previously, we work in a theory with a single quark flavor with charge one. The quark matrix element of the left-hand side of the operator product expansion, Eq. (1.169), is given by the Feynman graphs in Fig. 1.11,

$$\mathcal{M}^{\mu\nu} = i\bar{u}(p, s)\,\gamma^\mu i\,\frac{\not{p} + \not{q}}{(p + q)^2}\,\gamma^\nu\,u(p, s) + i\bar{u}(p, s)\,\gamma^\nu i\,\frac{\not{p} - \not{q}}{(p - q)^2}\,\gamma^\mu u(p, s).$$
(1.179)

Note that there is an overall factor of i because we are computing i times the time-ordered product in Eq. (1.162). The crossed diagram (second term) can be obtained by the replacement $\mu \leftrightarrow \nu, q \to -q$ from the direct diagram (first term), so we concentrate on simplifying the first term. Expanding the denominator gives

$$(p+q)^2 = 2p\cdot q + q^2 = q^2\left(1 + \frac{2p\cdot q}{q^2}\right) = q^2\left(1 - \omega\right), \quad (1.180)$$

since $p^2 = 0$ for an on-shell massless quark. The numerator can be simplified using the γ matrix identity in Eq. (1.119):

$$\bar{u}(p, s)\gamma^\mu(\not{p} + \not{q})\gamma^\nu u(p, s) = \bar{u}(p, s)[(p + q)^\mu\gamma^\nu + (p + q)^\nu\gamma^\mu$$
$$-g^{\mu\nu}(\not{p} + \not{q}) + i\epsilon^{\mu\nu\alpha\lambda}(p + q)_\alpha\gamma_\lambda\gamma_5]u(p, s).$$
(1.181)

For an on-shell massless quark,

$$\not{p}\,u(p, s) = 0, \quad \bar{u}(p, s)\,\gamma_\lambda\,u(p, s) = 2p_\lambda, \quad \bar{u}(p, s)\,\gamma_\lambda\gamma_5\,u(p, s) = 2h\,p_\lambda,$$
(1.182)

where h is the quark helicity. Thus the $\not{p}$ and $\epsilon^{\mu\nu\alpha\lambda}p_\alpha\gamma_\lambda\gamma_5$ terms both give zero. For spin-averaged matrix elements the sum over helicities gives zero and so we neglect the part of $\mathcal{M}^{\mu\nu}$ proportional to h. Combining the various terms and using

$$(1 - \omega)^{-1} = \sum_{n=0}^{\infty} \omega^n$$
(1.183)

gives

$$\mathcal{M}^{\mu\nu} = -\frac{2}{q^2} \sum_{n=0}^{\infty} \omega^n [(p+q)^\mu p^\nu + (p+q)^\nu p^\mu - g^{\mu\nu} p \cdot q]. \quad (1.184)$$

To complete the operator product expansion, we need the free quark matrix element of the right-hand side of the operator product. The matrix element of the quark operators of Eq. (1.172) in a free quark state with momentum p is

$$\langle q(p)| O_{q,V}^{\mu_1 \cdots \mu_n} |q(p)\rangle = \mathcal{S}[p^{\mu_1} \cdots p^{\mu_n}] = p^{\mu_1} \cdots p^{\mu_n}, \quad (1.185)$$

since $p^2 = 0$. The factors of i and 2 in Eqs. (1.172) and (1.173) were chosen so that no such factors appear in the matrix elements.

We determine the coefficient functions for the spin-independent terms in the operator product expansion. Including the crossed diagram, the spin-independent terms on the left-hand side of the operator product are

$$\mathcal{M}^{\mu\nu} = -\frac{2}{q^2} \sum_{n=0}^{\infty} \omega^n [(p+q)^\mu p^\nu + (p+q)^\nu p^\mu - g^{\mu\nu} p \cdot q]$$
$$+ (\mu \leftrightarrow \nu, \ q \to -q, \ \omega \to -\omega), \quad (1.186)$$

since ω is odd in q. The crossed diagram causes half the terms to cancel, so that the matrix element is

$$\mathcal{M}^{\mu\nu} = -\frac{4}{q^2} \sum_{n=0,2,4}^{\infty} \omega^n 2p^\mu p^\nu - \frac{4}{q^2} \sum_{n=1,3,5}^{\infty} \omega^n (q^\mu p^\nu + q^\nu p^\mu - g^{\mu\nu} p \cdot q)$$
$$= -\frac{8}{q^2} \sum_{n=0,2,4}^{\infty} \frac{2^n (p \cdot q)^n}{(-q^2)^n} \left(p^\mu - \frac{p \cdot q q^\mu}{q^2}\right) \left(p^\nu - \frac{p \cdot q q^\nu}{q^2}\right)$$
$$- \frac{4}{q^2} \sum_{n=1,3,5}^{\infty} \frac{2^n (p \cdot q)^{n+1}}{(-q^2)^n} \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right). \quad (1.187)$$

Equation (1.187) can be rewritten in the form

$$\mathcal{M}^{\mu\nu} = -\frac{8}{q^2} \sum_{n=0,2,4}^{\infty} \frac{2^n q^{\mu_3} \cdots q^{\mu_{n+2}}}{(-q^2)^n}$$
$$\times \left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2}\right) \left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2}\right) p_{\mu_1} \cdots p_{\mu_{n+2}}$$
$$- \frac{4}{q^2} \sum_{n=1,3,5}^{\infty} \frac{2^n q^{\mu_1} \cdots q^{\mu_{n+1}}}{(-q^2)^n} \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) p_{\mu_1} \cdots p_{\mu_{n+1}}, \quad (1.188)$$

which separates the q and p dependence.

The coefficient functions in the operator product depend only on q, and the matrix elements depend only on p. We have separated the operator product into

pieces which depend only on q and only on p. By comparing with Eq. (1.185), we can write Eq. (1.188) as

$$\mathcal{M}^{\mu\nu} = -\frac{8}{q^2} \sum_{n=0,2,4}^{\infty} \frac{2^n q^{\mu_3} \cdots q^{\mu_{n+2}}}{(-q^2)^n}$$

$$\times \left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2} \right) \left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2} \right) \langle p|O_{q,V\mu_1\cdots\mu_{n+2}}|p\rangle$$

$$-\frac{4}{q^2} \sum_{n=1,3,5}^{\infty} \frac{2^n q^{\mu_1} \cdots q^{\mu_{n+1}}}{(-q^2)^n} \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) \langle p|O_{q,V\mu_1\cdots\mu_{n+1}}|p\rangle$$

$$\tag{1.189}$$

so that

$$t^{\mu\nu} = 2 \sum_{n=2,4,6}^{\infty} \frac{2^n q^{\mu_3} \cdots q^{\mu_n}}{(-q^2)^{n-1}} \left(g^{\mu\mu_1} - \frac{q^\mu q^{\mu_1}}{q^2} \right) \left(g^{\nu\mu_2} - \frac{q^\nu q^{\mu_2}}{q^2} \right) O_{q,V\mu_1\cdots\mu_n}$$

$$+ 2 \sum_{n=2,4,6}^{\infty} \frac{2^n q^{\mu_1} \cdots q^{\mu_n}}{(-q^2)^n} \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) O_{q,V\mu_1\cdots\mu_n}. \tag{1.190}$$

This is the operator product expansion for the spin-independent part of $t_{\mu\nu}$, i.e., the part involving only vector operators. Only vector operators with n even occur in the operator product expansion, because $t^{\mu\nu}$ is even under charge conjugation.

Comparing with the most general form for the operator product in Eq. (1.176), we see that at lowest order in α_s the coefficients $C_{q,n}^{(1,2)} = 1$. Considering a gluon matrix element gives $C_{g,n} = 0$ at lowest order in α_s. At higher orders in α_s the coefficient functions and the operator matrix elements depend on a subtraction point μ. Since the physical quantity $t_{\mu\nu}$ is independent of the arbitrary choice of subtraction point, a renormalization group equation similar to that for coefficients in the weak nonleptonic decay Hamiltonian in Eq. (1.132) can be derived for the coefficients $C_{j,n}^{(1,2)}$. At $\mu = Q$ there are no large logarithms in the coefficients $C_{j,n}^{(1,2)}$. Therefore, we have that

$$C_{q,n}^{(1,2)}[1, \alpha_s(Q)] = 1 + \mathcal{O}[\alpha_s(Q)],$$
$$C_{g,n}^{(1,2)}[1, \alpha_s(Q)] = 0 + \mathcal{O}[\alpha_s(Q)]. \tag{1.191}$$

However, at $\mu = Q$ there are large logarithms of Q/Λ_{QCD} in the nucleon matrix element of the twist-two operators. It is convenient to use the renormalization group equations that the $C_{j,n}^{(1,2)}$ satisfy and the initial conditions in Eqs. (1.191) to move the Q dependence from the matrix elements into the coefficients by scaling the subtraction point down to a value $\mu \ll Q$. It is this calculable Q dependence that results in the dependence of the structure functions $T_{1,2}$ and hence $F_{1,2}$ on Q, without which they would just be functions of x. So quantum chromodynamics predicts a calculable logarithmic dependence of the structure functions $F_{1,2}$ on Q,
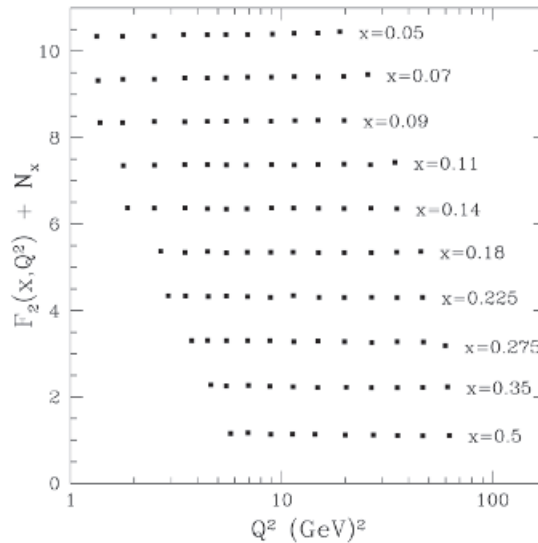
Fig. 1.12. The proton structure function $F_2(x, Q^2)$, measured in deep inelastic muon scattering by the NMC Collaboration [M. Arneodo et al., Phys. Lett. 364B (1995) 107]. The data is shown as a function of Q^2 for different values of x. For clarity, the plots for different x values are offset by one unit vertically, so that what is plotted is $F_2 + N_x$, where N_x is an integer equal to 1 for $x = 0.5$, 2 for $x = 0.35$, etc.

which has been verified experimentally. The fact that this dependence is weak at large Q is a consequence of asymptotic freedom. In free field theory the structure functions $F_{1,2}$ are independent of Q, which is called scaling. The logarithmic Q dependence is usually called a scaling violation. Some experimental data showing the approximate scaling of F_2 are shown in Fig. 1.12.

1.9 Problems

1. Consider an $SU(5)$ gauge theory with a scalar field Φ that transforms in the adjoint representation

$$\Phi \to U \Phi U^\dagger, \ \ U \in SU(5).$$

Suppose Φ gets the vacuum expectation value

$$\langle \Phi \rangle = v \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{bmatrix}.$$

(a) What is the unbroken subgroup H of $SU(5)$?

(b) What are the H quantum numbers of the massive $SU(5)$ gauge bosons?

2. If there are N generations of quarks and leptons, show that the CKM matrix contains $(N-1)^2$ real parameters.

3. Calculate the vertex renormalization constant Z_e given in Eq. (1.62).

4. Calculate to order g^2 the renormalization matrix Z_{ij} defined in Eq. (1.127) for the operators O_1 and O_2 defined in Eq. (1.125). Use it to deduce the anomalous dimension matrix in Eq. (1.135).

5. Calculate the cross section $\sigma(\pi^+\pi^- \to \pi^+\pi^-)$ at center of mass energy E to leading order in the chiral perturbation theory expansion.

6. In chiral perturbation theory, any Feynman diagram contributing to π–π scattering has L loops, n_k insertions of vertices of order p^k, and N_π internal pion lines. The resulting amplitude is of order p^D, where

$$D = (\text{powers of } p \text{ in numerator}) - (\text{powers of } p \text{ in denominator}).$$

    Using the identity $L = N_\pi - \sum_k n_k + 1$, derive Eq. (1.110) for D.

7. Calculate the decay amplitude for $K^- \to \pi^0 e \bar{\nu}_e$ at leading order in chiral perturbation theory.

8. (a) Calculate the semileptonic free quark decay rate $\Gamma(b \to c e \bar{\nu}_e)$.
    (b) Using the renormalization group improved effective Hamiltonian in Eq. (1.124), calculate the nonleptonic free quark decay rate $\Gamma(b \to c d \bar{u})$.

    Neglect all masses except those of the b and c quarks.

1.10   References

The material in this chapter can be found in many textbooks. Some suggested references are:

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