BLOWUP OF CYLINDRICALLY SYMMETRIC SOLUTIONS FOR BIHARMONIC NLS

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Abstract In this paper, we consider blowup of solutions to the Cauchy problem for the following biharmonic nonlinear Schrödinger equation (NLS),

$$\mathrm{i}\,\partial_t u = \Delta^2 u - \mu \Delta u - |u|^{2\sigma} u \quad \text{in } \mathbb{R} \times \mathbb{R}^d,$$

where $d \geq 1$, $\mu \in \mathbb{R}$ and $0 < \sigma < \infty$ if $1 \leq d \leq 4$ and $0 < \sigma < 4/(d-4)$ if $d \geq 5$. In the mass critical and supercritical cases, we establish the existence of blowup solutions to the problem for cylindrically symmetric data. The result extends the known ones with respect to blowup of solutions to the problem for radially symmetric data.

Keywords: blowup; biharmonic NLS; cylindrical symmetry solutions

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1. Introduction

In this paper, we are concerned with blowup of cylindrically symmetric solutions to the Cauchy problem for the following biharmonic nonlinear Schrödinger equation NLS,

$$\mathrm{i}\,\partial_t u = \Delta^2 u - \mu \Delta u - |u|^{2\sigma} u \quad \text{in } \mathbb{R} \times \mathbb{R}^d, \tag{1.1}$$

where $d \geq 1$, $\mu \in \mathbb{R}$ and $0 < \sigma < \infty$ if $1 \leq d \leq 4$ and $0 < \sigma < 4/(d-4)$ if $d \geq 5$. The first study of biharmonic NLS traces back to Karpman [14] and Karpman–Shagalov [15], where the authors investigated the regularization and stabilization effect of the fourth-order dispersion. Later, Fibich *et al.* [11] carried out a rigorous survey to biharmonic NLS from mathematical point of views and proved global existence in time of solutions to the Cauchy problem for (1.1). During recent years, there is a large number of literature mainly devoted to the study of well-posedness and scattering of solutions to the Cauchy problem for (1.1), see for example [9, 13, 17–21] and references therein. In [8], Boulenger and Lenzmann rigorously and completely discussed the existence of blowup solutions

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to the Cauchy problem for (1.1) with radially symmetric initial data, which in turn confirms a series of numerical studies conducted in [1–4]. We also refer to the readers to the papers due to *Bonheure et al.* [6, 7] with respect to orbital instability of radially symmetric standing waves to (1.1). Inspired by the aforementioned works, the aim of the present paper is to investigate blowup of solutions to the Cauchy problem for (1.1) with cylindrically symmetric initial data, i.e. initial data belong to Σ_d defined by

$$\Sigma_d := \left\{ u \in H^2(\mathbb{R}^d) : u(y, x_d) = u(|y|, x_d), x_d u \in L^2(\mathbb{R}^d) \right\},$$

where $x = (y, x_d) \in \mathbb{R}^d$ and $y = (x_1, \cdots, x_{d-1}) \in \mathbb{R}^{d-1}$.

For further clarifications, we shall fix some notation. Let us define

$$s_c := \frac{d}{2} - \frac{2}{\sigma}.$$

We refer to the cases $s_c < 0$, $s_c = 0$ and $s_c > 0$ as mass subcritical, critical and supercritical, respectively. The end case $s_c = 2$ is energy critical. Note that the cases $s_c = 0$ and $s_c = 2$ correspond to the exponents $\sigma = 4/d$ and $\sigma = 4/(d-4)$, respectively. For $1 \le p < \infty$, we denote by $L^q(\mathbb{R}^d)$ the usual Lebesgue space with the norm

$$\|u\|_p := \left(\int_{\mathbb{R}^d} |u|^p \, \mathrm{d}x \right)^{\frac{1}{p}}.$$

The Sobolev space $H^2(\mathbb{R}^d)$ is equipped with the standard norm

$$\|u\| := \|\Delta u\|_2 + \|\nabla u\|_2 + \|u\|_2.$$

In addition, we denote by $Q \in H^2(\mathbb{R}^d)$ a ground state to the following nonlinear elliptic equation,

$$\Delta^2 Q + Q - |Q|^{2\sigma} Q = 0 \quad \text{in } \mathbb{R}^d.$$

The main results of the present paper read as follows, which gives blowup criteria for solutions to the Cauchy problem for (1.1) with cylindrically symmetric data.

Theorem 1.1. *(Blowup for Mass-Supercritical Case) Let $d \ge 5$, $\mu \in \mathbb{R}$ and $0 < s_c < 2$ with $0 < \sigma \le 1$. Suppose that $u_0 \in \Sigma_d$ satisfies one of the following conditions.*

(i) *If $\mu \neq 0$, we assume that*

$$E[u_0] < \begin{cases} 0, & \text{for } \mu > 0, \\ -\chi \mu^2 M[u_0], & \text{for } \mu < 0, \end{cases}$$

with some constant $\chi = \chi(d, \sigma) > 0$.

(ii) *If $\mu = 0$, we assume that either $E[u_0] < 0$ or if $E[u_0] \geq 0$, we suppose that*

$$E[u_0]^{s_c} M[u_0]^{2-s_c} < E[Q]^{s_c} M[Q]^{2-s_c}$$

and

$$\|\Delta u_0\|_2^2 \|u_0\|_2^{2-s_c} > \|\Delta Q\|_2^2 \|Q\|_2^{2-s_c}.$$

Then the solution $u \in C([0, T), H^2(\mathbb{R}^d))$ to the Cauchy problem for (1.1) with initial datum u_0 blows up in finite time, i.e. $0 < T < +\infty$ and $\lim_{t \to T^-} \|\Delta u\|_2 = +\infty$.

Remark 1.1. The extra restriction on σ comes from the use of the well-known radial Sobolev inequality in $\mathbb{R}^{d-1}$. Note that if $4/d < \sigma \leq 1$, then $d \geq 5$. This is the reason that we need to assume that $d \geq 5$.

Theorem 1.2. *(Blowup for Mass-Critical Case) Let $d \geq 4$, $\mu \geq 0$ and $s_c = 0$. Let $u_0 \in \Sigma_d$ be such that $E(u_0) < 0$. Then the solution $u \in C([0, T), H^2(\mathbb{R}^d))$ to the Cauchy problem for (1.1) satisfies the following.*

  (i) *If $\mu > 0$, then $u(t)$ blows up in finite time.*
  (ii) *If $\mu = 0$, then $u(t)$ either blows up in finite time or $u(t)$ blows up in infinite time.*

To prove Theorems 1.1 and 1.2, the essential argument is to deduce the evolution of the localized virial quantity $M_{\varphi_R}[u(t)]$ defined by (2.2) along time, see Lemma 2.2. To this end, we shall make use of ideas from [8, 16]. It is worth mentioning [4, 5, 10, 12], where blowup of solutions to NLS for cylindrically symmetric data has been investigated. Comparing with the existing works, we deal with the evolution of the localized virial quantity to biharmonic NLS for cylindrically symmetric data and extra treatments are needed in the cylindrically symmetric context, because of the presence of the biharmonic term. For cylindrically symmetric solutions, the radial Sobolev inequality is only applicable in $\mathbb{R}^{d-1}$, which is different from the radially symmetric case handled in [8], we shall take advantage of ingredients in [16] to estimate error terms due to the nonlinearity in the process of discussion of the evolution of the localized virial quantity.

Remark 1.2. It seems possible to remove the condition that $x_d u_0 \in L^2(\mathbb{R}^d)$ to study blowup of solutions to (1.1) for cylindrically symmetric data in the spirit of work due to Martel [16]. In this case, more restrictive conditions should be imposed on σ. This shall be discussed in forthcoming publications.

2. Proofs of main results

In this section, we are going to prove Theorems 1.1 and 1.2. To do this, we first need to introduce a localized virial quantity, which is inspired by [8] and [16]. For $d \geq 2$, let $\psi : \mathbb{R}^{d-1} \to \mathbb{R}$ be a radially symmetric and smooth function such that $|\nabla \psi^j| \in L^\infty(\mathbb{R}^{d-1})$ for $1 \leq j \leq 6$ and

$$\psi(r) := \begin{cases} \dfrac{r^2}{2} & \text{for } r \leq 1, \\ \text{const.} & \text{for } r \geq 10, \end{cases} \quad \psi''(r) \leq 1 \text{ for } r \geq 0.$$

For $R > 0$ given, we define a radial function $\psi_R : \mathbb{R}^{d-1} \to \mathbb{R}$ by

$$\psi_R(r) := R^2 \varphi \left(\frac{r}{R} \right).$$

It follows from (3.3) in [8] that

$$1 - \psi_R''(r) \geq 0, \quad 1 - \frac{\psi_R'(r)}{r} \geq 0, \quad d - 1 - \Delta \psi_R(r) \geq 0, \quad r \geq 0. \tag{2.1}$$

Let

$$\varphi_R(x) := \psi_R(r) + \frac{x_d^2}{2}, \quad x = (y, x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}, \ r = |y|.$$

Define

$$\mathcal{M}_{\varphi_R}[u] := 2\text{Im} \int_{\mathbb{R}^d} \overline{u} \left(\nabla \varphi_R \cdot \nabla u \right) \, \mathrm{d}x. \tag{2.2}$$

It is simple see that $\mathcal{M}_{\varphi_R}[u]$ is well-defined for any $u \in \Sigma_d$. For later use, we shall give the well-known radial Sobolev's inequality in [22]. For every radial function $f \in H^1(\mathbb{R}^{d-1})$ with $d \geq 3$, then

$$|y|^{\frac{d-2}{2}} |f(y)| \leq 2\|f\|_2^{\frac{1}{2}} \|\nabla f\|_2^{\frac{1}{2}}, \quad y \neq 0. \tag{2.3}$$

We also present the well-known Gagliardo–Nirenberg's inequality in one dimension. For any $f \in H^1(\mathbb{R})$ and $p > 2$, then

$$\|f\|_p \leq C_p \|f'\|_2^{\alpha} \|f\|_2^{1-\alpha}, \quad \alpha = \frac{p-2}{2p}. \tag{2.4}$$

Let $f : \mathbb{R}^{d-1} \to \mathbb{C}$ be a radial and smooth function, then

$$\partial_{kl}^2 f = \left(\delta_{kl} - \frac{x_k x_l}{r^2} \right) \frac{\partial_r f}{r} + \frac{x_k x_l}{r^2} \partial_r^2 f. \tag{2.5}$$

Next we present the well-posedness of solutions to the Cauchy problem for (1.1) in $H^2(\mathbb{R}^d)$, which was established by Pausader [18].

Lemma 2.1. [*18, Proposition* 4.1] *Let $d \geq 1$, $\mu \in \mathbb{R}$ and $s_c < 2$. Then, for any $u_0 \in H^2(\mathbb{R}^d)$, there exist a constant $T > 0$ and a unique solution $u \in C([0, T), H^2(\mathbb{R}^d))$*

to the Cauchy problem for (*1.1*) with initial datum u_0. The solution has conserved mass and energy in the sense that

$$M[u(t)] = M[u_0], \quad E[u(t)] = E[u_0], \quad t \in [0, T),$$

where

$$M[u] := \int_{\mathbb{R}^d} |u|^2 \, \mathrm{d}x$$

and

$$E[u] = \frac{1}{2} \int_{\mathbb{R}^d} |\Delta u|^2 \, \mathrm{d}x + \frac{\mu}{2} \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, \mathrm{d}x.$$

Moreover, blowup alternative holds, i.e. either $T = +\infty$ or $\|u(t)\|_{H^2} = +\infty$ as $t \to T^-$. The solution map

$$u_0 \in H^2(\mathbb{R}^d) \mapsto u \in C([0, T), H^2(\mathbb{R}^d))$$

is continuous.

In the following, we give the evolution of $\mathcal{M}[u(t)]$ along time, which is the key argument to prove Theorems 1.1 and 1.2.

Lemma 2.2. *Let $d \geq 3$, $R > 0$ and $0 < \sigma \leq 1$. Suppose that $u \in C([0, T); H^2(\mathbb{R}^d))$ is the solution to the Cauchy problem for (1.1) with initial datum $u_0 \in \Sigma_d$. Then, for any $t \in [0, T)$, there holds that*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M}_R[u(t)] &\leq 8 \int_{\mathbb{R}^d} |\Delta u|^2 \, \mathrm{d}x + 4\mu \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x - \frac{2\sigma d}{\sigma + 1} \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, \mathrm{d}x + X_\mu[u] \\
&\quad + \mathcal{O}\left(R^{-4} + R^{-2}\|\nabla u\|_2^2 + R^{-\sigma(d-2)}\|\nabla u\|_2^{2\sigma} + |\mu|R^{-2} \right) \\
&= 4d\sigma E(u_0) - (2d\sigma - 8)\|\Delta u\|_2^2 - \mu(2d\sigma - 4)\|\nabla u\|_2^2 + X_\mu[u] \\
&\quad + \mathcal{O}\left(R^{-4} + R^{-2}\|\nabla u\|_2^2 + R^{-\sigma(d-2)}\|\nabla u\|_2^{2\sigma} + |\mu|R^{-2} \right),
\end{aligned}
$$

where

$$
X_\mu[u] \lesssim
\begin{cases}
0, & \text{for } \mu \geq 0, \\
|\mu|\|\nabla u\|_2^2, & \text{for } \mu < 0.
\end{cases}
$$

Proof. To achieve this, we shall adapt some elements from [8] and [16]. In view of Step 1 of the proof of [8, Lemma 3.1], we first have that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_R[u(t)] = 8 \sum_{k,l,m=1}^{d} \left\langle u, \partial_{kl}^2 \left(\partial_{lm}^2 \varphi_R\right) \partial_{mk}^2 u \right\rangle + 4 \sum_{k,l=1}^{d} \left\langle u, \partial_k \left(\partial_{kl}\Delta\varphi_R\right) \partial_l u \right\rangle$$

$$+ 2\sum_{k=1}^{d} \left\langle u, \partial_k \left(\Delta^2 \varphi_R\right) \partial_l u \right\rangle + \left\langle u, \left(\Delta^3 \varphi_R\right) u \right\rangle$$

$$- 4\mu \sum_{k,l=1}^{d} \left\langle u, \partial_k \left(\partial_{kl}^2 \varphi_R\right) \partial_l u \right\rangle - \mu \left\langle u, \left(\Delta^2 \varphi_R\right) u \right\rangle -$$

$$\left\langle u, [|u|^{2\sigma}, \nabla\varphi_R \cdot \nabla + \nabla \cdot \nabla\varphi_R]u \right\rangle,$$

$$=: \mathcal{A}_R^{(1)}[u] + \mathcal{A}_R^{(2)}[u] + \mathcal{B}_R[u].$$

In what follows, we are going to estimate the terms $\mathcal{A}_R^{(1)}[u]$, $\mathcal{A}_R^{(2)}[u]$ and $\mathcal{B}_R[u]$. The estimates of dispersive terms $\mathcal{A}_R^{(1)}[u]$ and $\mathcal{A}_R^{(2)}[u]$ are inspired by the proof of [8, Lemma 3.1]. Let us begin with treating the term $\mathcal{A}_R^{(1)}[u]$. Using integration by parts and the definition of φ_R, we are able to derive that

$$\sum_{k,l,m=1}^{d} \left\langle u, \partial_{kl}^2 \left(\partial_{lm}^2 \varphi_R\right) \partial_{mk}^2 u \right\rangle = \sum_{k,l,m=1}^{d} \int_{\mathbb{R}^d} \left(\partial_{kl}^2 \overline{u}\right) \left(\partial_{lm}^2 \varphi_R\right) \left(\partial_{mk}^2 u\right) \, \mathrm{d}x$$

$$= \sum_{k,l,m=1}^{d-1} \int_{\mathbb{R}^d} \left(\partial_{kl}^2 \overline{u}\right) \left(\partial_{lm}^2 \psi_R\right) \left(\partial_{mk}^2 u\right) \, \mathrm{d}x$$

$$+ \sum_{l,m=1}^{d-1} \int_{\mathbb{R}^d} \left(\partial_{dl}^2 \overline{u}\right) \left(\partial_{lm}^2 \psi_R\right) \left(\partial_{md}^2 u\right) \, \mathrm{d}x \tag{2.6}$$

$$+ \sum_{k=1}^{d-1} \int_{\mathbb{R}^d} \left(\partial_{kd}^2 \overline{u}\right) \left(\partial_{dk}^2 u\right) \, \mathrm{d}x + \int_{\mathbb{R}^d} \left|\partial_{dd}^2 u\right|^2 \, \mathrm{d}x.$$

We now compute each term in the right hand side of (2.6). Utilizing (2.5), we can derive that

$$\sum_{k,l,m=1}^{d-1} \int_{\mathbb{R}^d} \left(\partial_{kl}^2 \overline{u}\right) \left(\partial_{lm}^2 \psi_R\right) \left(\partial_{mk}^2 u\right) \, \mathrm{d}x$$

$$= \sum_{k,l,m=1}^{d-1} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \left(\partial_{kl}^2 \overline{u}\right) \left(\partial_{lm}^2 \psi_R\right) \left(\partial_{mk}^2 u\right) \, \mathrm{d}y\mathrm{d}x_d \tag{2.7}$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \left(\partial_r^2 \psi_R |\partial_r^2 u|^2 + \frac{d-2}{r^2}\frac{\partial_r \psi_R}{r}|\partial_r u|^2\right) \, \mathrm{d}y\mathrm{d}x_d$$

$$= \int_{\mathbb{R}^d} |\Delta_y u|^2 - \left(1 - \partial_r^2 \psi_R\right) |\partial_r^2 u|^2 - \left(1 - \frac{\partial_r \psi_R}{r}\right)\frac{d-2}{r^2}|\partial_r u|^2 \, \mathrm{d}x$$

and

$$
\sum_{l,m=1}^{d-1} \int_{\mathbb{R}^d} \left(\partial_{dl}^2 \overline{u} \right) \left(\partial_{lm}^2 \psi_R \right) \left(\partial_{md}^2 u \right) \, \mathrm{d}x = \sum_{k=1}^{d-1} \int_{\mathbb{R}^d} \left(\partial_r^2 \psi_R \right) \left| \partial_{kd}^2 u \right|^2 \, \mathrm{d}x
$$

$$
= \sum_{k=1}^{d-1} \int_{\mathbb{R}^d} \left| \partial_{kd}^2 u \right|^2 - \left(1 - \partial_r^2 \psi_R \right) \left| \partial_{kd}^2 u \right|^2 \, \mathrm{d}x
$$

$$
= \sum_{k=1}^{d-1} \int_{\mathbb{R}^d} \left(\partial_{kd}^2 \overline{u} \right) \left(\partial_{dk}^2 u \right) - \left(1 - \partial_r^2 \psi_R \right) \left| \partial_{kd}^2 u \right|^2 \, \mathrm{d}x. \tag{2.8}
$$

In addition, applying integration by parts, we have that

$$
\int_{\mathbb{R}^d} \left(\partial_{kd}^2 \overline{u} \right) \left(\partial_{dk}^2 u \right) \, \mathrm{d}x = \int_{\mathbb{R}^d} \left(\partial_{kk}^2 \overline{u} \right) \left(\partial_{dd}^2 u \right) \, \mathrm{d}x. \tag{2.9}
$$

As a consequence, coming back to (2.6) and using (2.1), (2.6), (2.7), (2.8) and (2.9), we now conclude that

$$
\sum_{k,l,m=1}^{d} \left\langle u, \partial_{kl}^2 \left(\partial_{lm}^2 \varphi_R \right) \partial_{mk}^2 u \right\rangle \leq \int_{\mathbb{R}^d} |\Delta u|^2 \, \mathrm{d}x. \tag{2.10}
$$

Furthermore, by the definitions of φ_R and ψ_R, there holds that

$$
\sum_{k,l=1}^{d} |\langle u, \partial_k \left(\partial_{kl} \Delta \varphi_R \right) \partial_l u \rangle| = \sum_{k,l=1}^{d-1} |\langle u, \partial_k \left(\partial_{kl} \Delta \psi_R \right) \partial_l u \rangle| \lesssim R^{-2} \|\nabla u\|_2^2,
$$

$$
\sum_{k,l=1}^{d} \left| \langle u, \partial_k \left(\Delta^2 \varphi_R \right) \partial_l u \rangle \right| = \sum_{k,l=1}^{d-1} \left| \langle u, \partial_k \left(\Delta^2 \psi_R \right) \partial_l u \rangle \right| \lesssim R^{-2} \|\nabla u\|_2^2,
$$

and

$$
\left| \langle u, \left(\Delta^3 \varphi_R \right) u \rangle \right| = \left| \langle u, \left(\Delta^3 \psi_R \right) u \rangle \right| \lesssim R^{-4} \|u\|_2^2.
$$

This along with (2.10) and the conservation of mass implies that

$$
\mathcal{A}_R^{(1)}[u] \leq 8 \int_{\mathbb{R}^d} |\Delta u|^2 \, \mathrm{d}x + \mathcal{O} \left(R^{-4} + R^{-2} \|\nabla u\|_2^2 \right).
$$

We next deal with the term $\mathcal{A}_R^{(2)}[u]$. In virtue of integration by parts, the definition of φ_R and (2.5), we can show that

$$
\mathcal{A}_R^{(2)}[u(t)] = 4\mu \sum_{k,l=1}^{d} \int_{\mathbb{R}^d} \left(\partial_k \overline{u} \right) \left(\partial_{kl}^2 \varphi_R \right) \left(\partial_l u \right) \, \mathrm{d}x - \mu \int_{\mathbb{R}^d} \left(\Delta^2 \varphi_R \right) |u|^2 \, \mathrm{d}x
$$

$$= 4\mu \sum_{k,l=1}^{d-1} \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \left(\partial_k \overline{u}\right) \left(\partial_{kl}^2 \psi_R\right) \left(\partial_l u\right) \, \mathrm{d}y \mathrm{d}x_d + 4\mu \int_{\mathbb{R}^d} \left|\partial_d u\right|^2 \, \mathrm{d}x$$

$$- \mu \int_{\mathbb{R}^d} \left(\Delta^2 \psi_R\right) |u|^2 \, \mathrm{d}x$$

$$= 4\mu \int_{\mathbb{R}} \int_{\mathbb{R}^{d-1}} \left(\partial_r^2 \psi_R\right) |\partial_r u|^2 \, \mathrm{d}y \mathrm{d}x_d + 4\mu \int_{\mathbb{R}^d} \left|\partial_d u\right|^2 \, \mathrm{d}x - \mu \int_{\mathbb{R}^d} \left(\Delta^2 \psi_R\right) |u|^2 \, \mathrm{d}x$$

$$= 4\mu \int_{\mathbb{R}^d} \left|\nabla_y u\right|^2 \, \mathrm{d}x + X_\mu[u] + 4\mu \int_{\mathbb{R}^d} \left|\partial_d u\right|^2 \, \mathrm{d}x - \mu \int_{\mathbb{R}^d} \left(\Delta^2 \psi_R\right) |u|^2 \, \mathrm{d}x$$

$$= 4\mu \int_{\mathbb{R}^d} \left|\nabla u\right|^2 \, \mathrm{d}x + X_\mu[u] - \mu \int_{\mathbb{R}^d} \left(\Delta^2 \psi_R\right) |u|^2 \, \mathrm{d}x$$

where

$$X_\mu[u] := -4\mu \int_{\mathbb{R}^d} \left(1 - \partial_r^2 \psi_R\right) |\partial_r u|^2 \, \mathrm{d}x.$$

Due to $\|\Delta^2 \psi_R\|_\infty \lesssim R^{-2}$, then

$$\mathcal{A}_R^{(2)}[u(t)] = 4\mu \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x + X_\mu[u] + \mathcal{O}(|\mu| R^{-2}).$$

We now turn to handle the term $\mathcal{B}_R[u]$. Here we need some special treatments. Applying integration by parts and the definition of φ_R, we first derive that

$$\mathcal{B}_R[u] = 2 \int_{\mathbb{R}^d} |u|^2 \nabla \varphi_R \cdot \nabla \left(|u|^{2\sigma}\right) = -\frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^d} \left(\Delta \varphi_R\right) |u|^{2\sigma+2} \, \mathrm{d}x$$

$$= -\frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^d} \left(\Delta \psi_R\right) |u|^{2\sigma+2} \, \mathrm{d}x$$

$$- \frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, \mathrm{d}x$$

$$= -\frac{2\sigma d}{\sigma+1} \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, \mathrm{d}x$$

$$- \frac{2\sigma}{\sigma+1} \int_{\mathbb{R}^d} \left(\Delta \psi_R - d + 1\right) |u|^{2\sigma+2} \, \mathrm{d}x.$$

In virtue of the definition of ψ_R and (2.5), then there holds that $\Delta \psi_R(r) - d + 1 = 0$ for $0 \leq r \leq R$. This further implies that

$$\mathcal{B}_R[u] = -\frac{2\sigma d}{\sigma+1} \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, \mathrm{d}x - \frac{2\sigma}{\sigma+1} \int_{\mathbb{R}} \int_{|y| \geq R} \left(\Delta \psi_R - d + 1\right) |u|^{2\sigma+2} \, \mathrm{d}y \mathrm{d}x_d. \quad (2.11)$$

In the following, we shall estimate the second term in the right hand side of (2.11). Observe first that

$$\int_{\mathbb{R}} \int_{|y| \geq R} |u|^{2\sigma+2} \, \mathrm{d}y \mathrm{d}x_d \leq \int_{\mathbb{R}} \|u\|_{L^\infty(|y| \geq R)}^{2\sigma} \|u\|_{L_y^2}^2 \, \mathrm{d}x_d. \tag{2.12}$$

To proceed the proof, we first consider the case that $\sigma = 1$. In this case, by (2.3), Hölder's inequality and the conservation of mass, then

$$\begin{aligned}
\int_{\mathbb{R}} \|u\|_{L^\infty(|y| \geq R)}^2 \, \mathrm{d}x_d &\lesssim R^{-(d-2)} \int_{\mathbb{R}} \|u\|_{L_y^2} \|\nabla_y u\|_{L_y^2} \, \mathrm{d}x_d \\
&\leq R^{-(d-2)} \|u\|_2 \|\nabla_y u\|_2 \lesssim R^{-(d-2)} \|\nabla_y u\|_2.
\end{aligned} \tag{2.13}$$

On the other hand, by Hölder's inequality and the conservation of mass, we know that

$$\begin{aligned}
\|u\|_{L^\infty(\mathbb{R}, L_y^2(\mathbb{R}^{d-1}))}^2 &= \sup_{x_d \in \mathbb{R}} \int_{\mathbb{R}^{d-1}} |u|^2 \, \mathrm{d}y = \sup_{x_d \in \mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{x_d} \partial_d \left(|u|^2 \right) \, \mathrm{d}y \mathrm{d}x_d \\
&= 2\mathrm{Re} \sup_{x_d \in \mathbb{R}} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{x_d} \overline{u} \left(\partial_d u \right) \, \mathrm{d}y \mathrm{d}x_d \leq 2\|u\|_2 \|\partial_d u\|_2 \lesssim \|\partial_d u\|_2.
\end{aligned} \tag{2.14}$$

Consequently, going back to (2.12) and using (2.13) and (2.14), we derive that

$$\int_{\mathbb{R}} \int_{|y| \geq R} |u|^4 \, \mathrm{d}y \mathrm{d}x_d \lesssim R^{-(d-2)} \|\nabla_y u\|_2 \|\partial_d u\|_2 \lesssim R^{-(d-2)} \|\nabla u\|_2^2. \tag{2.15}$$

We next consider the case that $0 < \sigma < 1$. In this case, from (2.12) and Hölder's inequality, it follows that

$$\int_{\mathbb{R}} \int_{|y| \geq R} |u|^{2\sigma+2} \, \mathrm{d}x \leq \left(\int_{\mathbb{R}} \|u\|_{L^\infty(|y| \geq R)}^2 \, \mathrm{d}x_d \right)^\sigma \left(\int_{\mathbb{R}} \|u\|_{L_y^2}^{\frac{2}{1-\sigma}} \, \mathrm{d}x_d \right)^{1-\sigma}. \tag{2.16}$$

In view of (2.4) and the conservation of mass, we get that

$$\begin{aligned}
\int_{\mathbb{R}} \|u\|_{L_y^2}^{\frac{2}{1-\sigma}} \, \mathrm{d}x_d &\leq \left(\int_{\mathbb{R}} \left| \partial_d \left(\|u\|_{L_y^2} \right) \right|^2 \, \mathrm{d}x_d \right)^{\frac{\sigma}{2(1-\sigma)}} \left(\int_{\mathbb{R}} \|u\|_{L_y^2}^2 \, \mathrm{d}x \right)^{\frac{2-\sigma}{2(1-\sigma)}} \\
&\lesssim \left(\int_{\mathbb{R}} \left| \partial_d \left(\|u\|_{L_y^2} \right) \right|^2 \, \mathrm{d}x_d \right)^{\frac{\sigma}{2(1-\sigma)}}.
\end{aligned} \tag{2.17}$$

Furthermore, notice that

$$\left| \partial_d \left(\|u\|_{L_y^2} \right) \right| \|u\|_{L_y^2} = \frac{1}{2} \left| \partial_d \left(\|u\|_{L_y^2}^2 \right) \right| = \frac{1}{2} \left| \mathrm{Re} \int_{\mathbb{R}^{d-1}} \overline{u} \left(\partial_d u \right) \, \mathrm{d}y \right| \leq \frac{1}{2} \|\partial_d u\|_{L_y^2} \|u\|_{L_y^2}.$$

This means that

$$\left| \partial_d \left(\|u\|_{L_y^2} \right) \right| \leq \frac{1}{2} \|\partial_d u\|_{L_y^2}. \tag{2.18}$$

As a result, via (2.17), we obtain that

$$\int_{\mathbb{R}} \|u\|_{L_y^2}^{\frac{2}{1-\sigma}} \, \mathrm{d}x_d \lesssim \left(\int_{\mathbb{R}} \|\partial_d u\|_{L_y^2}^2 \, \mathrm{d}x_d \right)^{\frac{\sigma}{2(1-\sigma)}} = \|\partial_d u\|_2^{\frac{\sigma}{1-\sigma}}. \tag{2.19}$$

Using (2.13) and (2.19), we then obtain from (2.16) that

$$\int_{\mathbb{R}} \int_{|y| \geq R} |u|^{2\sigma+2} \, \mathrm{d}x \lesssim R^{-\sigma(d-2)} \|\nabla_y u\|_2^\sigma \|\partial_d u\|_2^\sigma \lesssim R^{-\sigma(d-2)} \|\nabla u\|_2^{2\sigma}. \tag{2.20}$$

To sum up, it then follows from (2.11), (2.15) and (2.20) that

$$\mathcal{B}_R[u] \lesssim -\frac{2\sigma d}{\sigma+1} \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, \mathrm{d}x + R^{-\sigma(d-2)} \|\nabla u\|_2^{2\sigma}.$$

Accordingly, applying the estimates to $\mathcal{A}_R^{(1)}[u]$, $\mathcal{A}_R^{(2)}[u]$ and $\mathcal{B}_R[u]$ and the conservation of energy, we finally derive that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M}_R[u(t)] &\leq 8 \int_{\mathbb{R}^d} |\Delta u|^2 \, \mathrm{d}x + 4\mu \int_{\mathbb{R}^d} |\nabla u|^2 \, \mathrm{d}x - \frac{2\sigma d}{\sigma+1} \int_{\mathbb{R}^d} |u|^{2\sigma+2} \, \mathrm{d}x + X_\mu[u] \\
&\quad + \mathcal{O}\left(R^{-4} + R^{-2}\|\nabla u\|_2^2 + R^{-\sigma(d-2)}\|\nabla u\|_2^{2\sigma} + |\mu|R^{-2} \right) \\
&= 4d\sigma E(u_0) - (2d\sigma - 8)\|\Delta u\|_2^2 - \mu(2d\sigma - 4)\|\nabla u\|_2^2 + X_\mu[u] \\
&\quad + \mathcal{O}\left(R^{-4} + R^{-2}\|\nabla u\|_2^2 + R^{-\sigma(d-2)}\|\nabla u\|_2^{2\sigma} + |\mu|R^{-2} \right).
\end{aligned}$$

This completes the proof.   □

Proof of Theorem 1.1. Noting that $0 < \sigma \leq 1$ and applying Lemma 2.2, then the proof can be completed by closely following the one of [8, Theorem 1].   □

Proof of Theorem 1.2. If $\mu > 0$, using Lemma 2.2 and arguing as the proof of [8, Theorem 3], we can get the desired result. To complete the proof, we only need to consider the case that $\mu = 0$. In this case, we need to conduct a more refined analysis to the evolution of $\mathcal{M}[u(t)]$ along time. From the proof of Lemma 2.2, (2.1) and (2.5), we first have that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \mathcal{M}_R[u(t)] &\leq 16E(u_0) - 8 \int_{\mathbb{R}^d} \left(1 - \partial_r^2 \psi_R\right) |\partial_r^2 u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^d} \left(\Delta^3 \psi_R\right) |u|^2 \, \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} A_R |\partial_r u|^2 \, \mathrm{d}x + \int_{\mathbb{R}^N} B_R |u|^{2+\frac{8}{d}} \, \mathrm{d}x,
\end{aligned} \tag{2.21}$$

where

$$A_R := 4\partial_r^2 \Delta \psi_R + 2\Delta^2 \psi_R(r), \quad B_R := \frac{8}{d+4}\left(d-1-\Delta\psi_R\right).$$

Thereinafter, we shall estimate the last two terms in the right hand side of (2.21). Using (7.7) in [8] and the conservation of mass, we can derive that

$$\left|\int_{\mathbb{R}^d} A_R |\partial_r u|^2 \,\mathrm{d}x\right| = \left|\int_{\mathbb{R}}\int_{\mathbb{R}^{d-1}} A_R |\partial_r u|^2 \,\mathrm{d}y\mathrm{d}x_d\right| \lesssim 8\eta R^4 \int_{\mathbb{R}} \|A_R \partial_r^2 u\|_{L_y^2}^2 \,\mathrm{d}x_d$$
$$+ \eta^{-1} R^{-4} \int_{\mathbb{R}} \|u\|_{L_y^2}^2 \,\mathrm{d}x_d$$
$$= 8\eta R^4 \|A_R \partial_r^2 u\|_2^2 + \eta^{-1} R^{-4},$$

where $\eta > 0$ is an arbitrary constant. We now treat the other term. To do this, we first consider the case that $d = 4$. In this case, we have that

$$\left|\int_{\mathbb{R}^d} B_R |u|^4 \,\mathrm{d}x\right| = \left|\int_{\mathbb{R}}\int_{|y|\geq R} B_R |u|^4 \,\mathrm{d}y\mathrm{d}x_d\right| \leq \int_{\mathbb{R}} \|u\|_{L_y^2}^2 \|B_R^{\frac{1}{2}} u\|_{L^\infty(|y|\geq R)}^2 \,\mathrm{d}x_d$$
$$\leq \|u\|_{L^\infty(R, L_y^2(\mathbb{R}^{N-1}))}^2$$
$$\int_{\mathbb{R}} \|B_R^{\frac{1}{2}} u\|_{L^\infty(|y|\geq R)}^2 \,\mathrm{d}x_d, \qquad (2.22)$$

because of $B_R = 0$ for $0 \leq r \leq R$. In view of (2.3) with $d = 4$ and the definition of B_R, we are able to infer that

$$\int_{\mathbb{R}} \|B_R^{\frac{1}{2}} u\|_{L^\infty(|y|\geq R)}^2 \,\mathrm{d}x_d \lesssim R^{-2} \int_{\mathbb{R}} \|B_R^{\frac{1}{2}} u\|_{L_y^2} \|\nabla_y (B_R^{\frac{1}{2}} u)\|_{L_y^2} \,\mathrm{d}x_d$$
$$\lesssim R^{-2} \int_{\mathbb{R}} \|u\|_{L_y^2} \|\nabla_y (B_R^{\frac{1}{2}} u)\|_{L_y^2} \,\mathrm{d}x_d. \qquad (2.23)$$

It follows from (7.9) with $d = 4$ in [8] that

$$\|\nabla_y (B_R^{\frac{1}{2}} u)\|_{L_y^2}^2 \lesssim \left(\eta^{-\frac{1}{4}} + R^{-2}\right) \|u\|_{L_y^2}^2 + 8\eta^{\frac{1}{4}} \|B_R \partial_r^2 u\|_{L_y^2}^2.$$

By means of (2.23), the conservation of mass and Hölder's inequality, we then get that

$$\int_{\mathbb{R}} \|B_R^{\frac{1}{2}} u\|_{L^\infty(|y|\geq R)}^2 \,\mathrm{d}x_d \lesssim R^{-2}\left(\eta^{-\frac{1}{4}} + R^{-2}\right) + 8R^{-2}\eta^{\frac{1}{4}} \int_{\mathbb{R}^d} \|u\|_{L_y^2} \|B_R \partial_r^2 u\|_{L_y^2} \,\mathrm{d}x_d$$
$$\lesssim R^{-2}\left(\eta^{-\frac{1}{4}} + R^{-2}\right) + 8R^{-2}\eta^{\frac{1}{4}} \|B_R \partial_r^2 u\|_2. \qquad (2.24)$$

Going back to (2.22) and using (2.14) and (2.24), we then get that

$$\left|\int_{\mathbb{R}^N} B_R |u|^4 \,\mathrm{d}x\right| \lesssim R^{-2}\left(\eta^{-\frac{1}{4}} + R^{-2}\right) \|\partial_d u\|_2 + 8R^{-2}\eta^{\frac{1}{4}} \|B_R \partial_r^2 u\|_2 \|\partial_d u\|_2$$

$$\leq R^{-2}\left(\eta^{-\frac{1}{4}}+R^{-2}\right)\|\partial_d u\|_2 + R^{-4}\eta^{-\frac{1}{2}}\|\partial_d u\|_2^2 + 8\eta\|B_R \partial_r^2 u\|_2^2.$$

Taking into account (2.21) and noting that $\|\Delta^2 \varphi_R\|_\infty \lesssim R^{-2}$, we then obtain that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_R[u(t)] &\leq 16E(u_0) - 8\int_{\mathbb{R}^d}\left(1 - \partial_r^2\psi_R - \eta\left(R^4 A_R^2 + B_R^2\right)\right)|\partial_r^2 u|^2\,\mathrm{d}x \\
&\quad + R^{-2}\left(\eta^{-\frac{1}{4}} + R^{-2}\right)\|\partial_d u\|_2 + R^{-4}\eta^{-\frac{1}{2}}\|\partial_d u\|_2^2 + \eta^{-1}R^{-4} + R^{-2}.
\end{aligned}
\tag{2.25}
$$

Next we consider the case that $d \geq 5$. In this case, by Hölder's inequality and the definition of B_R, we can obtain that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N} B_R|u|^{2+\frac{8}{d}}\,\mathrm{d}x\right| &= \left|\int_{\mathbb{R}}\int_{|y|\geq R} B_R|u|^{2+\frac{8}{d}}\,\mathrm{d}y\mathrm{d}x_d\right| \leq \int_{\mathbb{R}}\|B_R^{\frac{1}{2}}u\|_{L_y^\infty(|y|\geq R)}^{\frac{8}{d}}\|B_R^{\frac{1}{2}-\frac{2}{d}}u\|_{L_y^2}^2\,\mathrm{d}x_d \\
&\leq \left(\int_{\mathbb{R}}\|B_R^{\frac{1}{2}}u\|_{L_y^\infty(|y|\geq R)}^2\,\mathrm{d}x_d\right)^{\frac{4}{d}}\left(\int_{\mathbb{R}}\|u\|_{L_y^2}^{\frac{2d}{d-4}}\,\mathrm{d}x_d\right)^{\frac{d-4}{d}}.
\end{aligned}
\tag{2.26}
$$

As an application of (2.4) leads to

$$
\int_{\mathbb{R}}\|u\|_{L_y^2}^{\frac{2d}{d-4}}\,\mathrm{d}x_d \lesssim \left(\int_{\mathbb{R}^d}\left|\partial_d\left(\|u\|_{L_y^2}\right)\right|^2\,\mathrm{d}x_d\right)^{\frac{2}{d-4}}\left(\int_{\mathbb{R}}\|u\|_{L_y^2}^2\,\mathrm{d}x_d\right)^{\frac{d-2}{d-4}}.
\tag{2.27}
$$

It then follows from (2.18), (2.27) and the conservation of mass that

$$
\int_{\mathbb{R}}\|u\|_{L_y^2}^{\frac{2d}{d-4}}\,\mathrm{d}x_d \lesssim \left(\int_{\mathbb{R}^d}\|\partial_d u\|_{L_y^2}^2\,\mathrm{d}x_d\right)^{\frac{2}{d-4}} = \|\partial_d u\|_2^{\frac{4}{d-4}}.
\tag{2.28}
$$

Therefore, coming back to (2.26) and using (2.24) and (2.28), we derive that

$$
\begin{aligned}
\left|\int_{\mathbb{R}^N} B_R|u|^{2+\frac{8}{d}}\,\mathrm{d}x\right| &\lesssim \left(\int_{\mathbb{R}}\|B_R^{\frac{1}{2}}u\|_{L_y^\infty(|y|\geq R)}^2\,\mathrm{d}x_d\right)^{\frac{4}{d}}\|\partial_d u\|_2^{\frac{4}{d}} \\
&\lesssim R\int_{\mathbb{R}}\|B_R^{\frac{1}{2}}u\|_{L_y^\infty(|y|\geq R)}^2\,\mathrm{d}x_d + R^{-\frac{4}{d-4}}\|\partial_d u\|_2^{\frac{4}{d-4}} \\
&\leq R^{-1}\left(\eta^{-\frac{1}{4}}+R^{-2}\right)\|\partial_d u\|_2 + 8R^{-1}\eta^{\frac{1}{4}}\|B_R\partial_r^2 u\|_2 \\
&\quad + R^{-\frac{4}{d-4}}\|\partial_d u\|_2^{\frac{4}{d-4}} \\
&\lesssim R^{-1}\left(\eta^{-\frac{1}{4}}+R^{-2}\right)\|\partial_d u\|_2 + 8\eta\|B_R\partial_r^2 u\|_2^2 + \\
&\quad R^{-\frac{4}{d-4}}\|\partial_d u\|_2^{\frac{4}{d-4}} + R^{-2}\eta^{-\frac{1}{2}}.
\end{aligned}
$$

As a consequence, invoking (2.21), we finally derive that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_R[u(t)] \leq{}& 16E(u_0) - 8\int_{\mathbb{R}^d}\left(1 - \partial_r^2\psi_R - \eta\left(R^4 A_R^2 + B_R^2\right)\right)|\partial_r^2 u|^2\,\mathrm{d}x \\
&+ R^{-1}\left(\eta^{-\frac{1}{4}} + R^{-2}\right)\|\partial_d u\|_2 + R^{-4}\eta^{-\frac{1}{2}}\|\partial_d u\|_2^2 \\
&+ R^{-\frac{4}{d-4}}\|\partial_d u\|_2^{\frac{4}{d-4}} + R^{-2}\eta^{-\frac{1}{2}} + \eta^{-1}R^{-4} + R^{-2}.
\end{aligned} \tag{2.29}
$$

At this point, using (2.25) and (2.29) and reasoning as the proof of [8, Theorem 3], we are able to finish the proof. This completes the proof. $\qquad\square$

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