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Sign Changes of the Liouville Function on Quadratics

Peter Borwein, Stephen K. K. Choi, and Himadri Ganguli

Abstract.

Let $\lambda(n)$ denote the Liouville function. Complementary to the prime number theorem, Chowla conjectured that

$$(*) \qquad \sum_{n\le x} \lambda(f(n)) = o(x)$$

for any polynomial $f(x)$ with integer coefficients which is not of form $bg(x)^2$.

When $f(x) = x$, $(*)$ is equivalent to the prime number theorem. Chowla's conjecture has been proved for linear functions, but for degree greater than 1, the conjecture seems to be extremely hard and remains wide open. One can consider a weaker form of Chowla's conjecture.

Conjecture 1 (Cassaigne et al.) *If $f(x) \in \mathbb{Z}[x]$ and is not in the form of $bg^2(x)$ for some $g(x) \in \mathbb{Z}[x]$, then $\lambda(f(n))$ changes sign infinitely often.*

Clearly, Chowla's conjecture implies Conjecture 1. Although weaker, Conjecture 1 is still wide open for polynomials of degree > 1. In this article, we study Conjecture 1 for quadratic polynomials. One of our main theorems is the following.

Theorem 1 *Let $f(x) = ax^2 + bx + c$ with $a > 0$ and l be a positive integer such that al is not a perfect square. If the equation $f(n) = lm^2$ has one solution $(n_0, m_0) \in \mathbb{Z}^2$, then it has infinitely many positive solutions $(n, m) \in \mathbb{N}^2$.*

As a direct consequence of Theorem 1, we prove the following.

Theorem 2 *Let $f(x) = ax^2 + bx + c$ with $a \in \mathbb{N}$ and $b, c \in \mathbb{Z}$. Let*

$$A_0 = \Big[\frac{|b| + (|D| + 1)/2}{2a}\Big] + 1.$$

Then either the binary sequence $\{\lambda(f(n))\}_{n=A_0}^{\infty}$ is a constant sequence or it changes sign infinitely often.

Some partial results of Conjecture 1 for quadratic polynomials are also proved using Theorem 1.

1 Introduction

Let $\lambda(n)$ denote the Liouville function, *i.e.*, $\lambda(n) = (-1)^{\Omega(n)}$, where $\Omega(n)$ denotes the number of prime factors of n counted with multiplicity. Alternatively, $\lambda(n)$ is the completely multiplicative function defined by $\lambda(p) = -1$ for each prime. Let $\zeta(s)$

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denote the Riemann zeta function, defined for complex s with $\Re(s) > 1$ by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \Big(1 - \frac{1}{p^s}\Big)^{-1},$$

where the product is over all prime numbers p. Thus

$$\frac{\zeta(2s)}{\zeta(s)} = \prod_p \Big(1 + \frac{1}{p^s}\Big)^{-1} = \prod_p \Big(1 - \frac{\lambda(p)}{p^s}\Big)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}. \tag{1.1}$$

Let $L(x)$ denote the average of the values of $\lambda(n)$ up to x,

$$L(x) := \sum_{1 \le n \le x} \lambda(n)$$

so that $L(x)$ records the difference of the number of positive integers up to x with an even number of prime factors (counted with multiplicity) and those with an odd number. In 1919 Pólya [10] showed that the Riemann Hypothesis, *i.e.*, that all nontrivial zeros of $\zeta(s)$ are on the critical line $\Re(s) = 1/2$, will follow if $L(x)$ does not change sign for sufficiently large n. There is a vast literature about the study of sign changes of $L(x)$. In 1958, Haselgrove proved that $L(x)$ changes sign infinitely often in [4]. For more discussion about this problem, we refer the reader to [1].

It is well known that the prime number theorem is equivalent to

$$L(x) = \sum_{n \le x} \lambda(n) = o(x). \tag{1.2}$$

In fact, the prime number theorem is equivalent to that fact that $\zeta(s) \neq 0$ on the vertical line $\Re(s) = 1$; and this is equivalent to (1.2) in view of (1.1). Complementary to the prime number theorem, Chowla [3] made the following conjecture.

Conjecture 1 (Chowla) *Let $f(x) \in \mathbb{Z}[x]$ be any polynomial which is not of form $bg^2(x)$ for some $b \neq 0, g(x) \in \mathbb{Z}[x]$. Then*

$$\sum_{n \le x} \lambda(f(n)) = o(x). \tag{1.3}$$

Clearly, Chowla's conjecture is equivalent to the prime number theorem when $f(x) = x$. For polynomials of degree > 1, Chowla's conjecture seems to be extremely hard and remains wide open. One can consider a weaker form of Chowla's conjecture.

Conjecture 2 (Cassaigne et al.) *If $f(x) \in \mathbb{Z}[x]$ is not of form $bg^2(x)$, then $\lambda(f(n))$ changes sign infinitely often.*

Clearly, Chowla's conjecture implies Conjecture 2. In fact, suppose it is not true, *i.e.*, there is n_0 such that $\lambda(f(n)) = \epsilon$ for all $n \geq n_0$, where ϵ is either -1 or $+1$. Then it follows that

$$\sum_{n \leq x} \lambda(f(n)) = \epsilon x + O(1),$$

which contradicts (1.3).

Although it is weaker, Conjecture 2 is still wide open for polynomials of degree greater than 1. In [2], Conjecture 2 was studied for special polynomials, and some partial results were proved.

Theorem 1.1 (Cassaigne et al.) *Let $f(n) = (an + b_1)(an + b_2) \cdots (an + b_k)$, where $a, k \in \mathbb{N}$, $b_1, \dots, b_k$ are distinct integers with $b_1 \equiv \cdots \equiv b_k \pmod{a}$. Then $\lambda(f(n))$ changes sign infinitely often.*

Proof This is Corollary 2 in [2]. ■

For certain quadratic polynomials, they proved the following theorem.

Theorem 1.2 (Cassaigne et al.) *If $f(n) = (n + a)(bn + c)$, where $a, c \in \mathbb{Z}$, $b \in \mathbb{N}$, $ab \neq c$, then $\lambda(f(n))$ changes sign infinitely often.*

Proof This is Theorem 4 in [2]. ■

Theorem 1.3 (Cassaigne et al.) *Let $a \in \mathbb{N}$, $b, c \in \mathbb{Z}$, and write $f(n) = an^2 + bn + c$, $D = b^2 - 4ac$. Assume that a, b, and c satisfy the following conditions:*

(i) $2a \mid b$,

(ii) $D < 0$,

(iii) *there is a positive integer k with*

$$\lambda\Big(-\frac{D}{4}k^2 + 1\Big) = -1.$$

Then $\lambda(f(n))$ changes sign infinitely often.

Proof This is Theorem 3 in [2]. ■

In this article, we continue to study Conjecture 2 for the quadratic case. One of our main results is Theorem 2.2 below. By Theorem 2.2, in order to show that the sequence $\{\lambda(f(n))\}_{n=1}^{\infty}$ changes sign infinitely often, we only need find one pair of large integers n_1 and n_2 such that $\lambda(f(n_1)) \neq \lambda(f(n_2))$. This will make the conjecture much easier to handle. Some partial results from Theorem 2.2 are also deduced in the next section.

2 Main Results

Conjecture 2 for linear polynomials is easily settled by the following result.

Theorem 2.1 *Let $P := \{n \in \mathbb{N} : \lambda(n) = +1\}$ and $N := \{n \in \mathbb{N} : \lambda(n) = -1\}$. Then neither P nor N can contain infinite arithmetic progression. In particular, $\lambda(an+b)$ changes sign infinitely often in n.*

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Proof We claim that neither P nor N can contain any infinite arithmetic progression. Otherwise, there are an n_0 and l such that

$$\lambda(n_0 + lk) = \lambda(n_0) \tag{2.1}$$

for $k = 0, 1, 2, \dots$. Pick a prime p which is of the form $lm + 1$. Now put $k = mn_0$ and consider

$$\lambda(n_0 + lk) = \lambda(n_0 + lmn_0) = \lambda(n_0)\lambda(lm + 1) = \lambda(n_0)\lambda(p) = -\lambda(n_0).$$

This contradicts (2.1). Hence our claim is attained. ■

One of the main results in this paper is the following theorem.

Theorem 2.2 *Let $f(x) = ax^2 + bx + c$ with $a > 0$ and let l be a positive integer such that al is not a perfect square. If the equation $f(n) = lm^2$ has a solution $(n_0, m_0) \in \mathbb{Z}^2$, then it has infinitely many positive solutions $(n, m) \in \mathbb{N}^2$.*

Proof Let $D = b^2 - 4ac$ be the discriminant of $f(x)$. Solving the quadratic equation

$$an^2 + bn + c = lm^2, \tag{2.2}$$

for n, we get

$$n_0 = \frac{-b \pm \sqrt{b^2 - 4a(c - lm_0^2)}}{2a} = \frac{-b \pm \sqrt{D + 4alm_0^2}}{2a}.$$

It follows that $D + 4alm_0^2 = t_0^2$ for some integer t_0. By choosing a suitable sign of t_0, we may assume

$$t_0 \equiv b \pmod{2a}, \quad \text{and} \quad n_0 = \frac{-b + t_0}{2a} \in \mathbb{Z}. \tag{2.3}$$

This leads us to consider the diophantine equation

$$t^2 = 4alm^2 + D. \tag{2.4}$$

Suppose that (t_0, m_0) and (t, m) are solutions of (2.4). Then we have

$$t^2 = 4alm^2 + D \quad \text{and} \quad t_0^2 = 4alm_0^2 + D.$$

Subtracting the above two equations, we get

$$(t - t_0)(t + t_0) = l(m - m_0)(4am + 4am_0).$$

We now let s and r be

$$r(m - m_0) = 2as(t + t_0) \quad \text{and} \quad 2as(4alm + 4alm_0) = r(t - t_0). \tag{2.5}$$

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By eliminating the terms t and m respectively in (2.5), we get

$$(r^2 - 16a^3ls^2)m = r^2m_0 + 16a^3ls^2m_0 + 4arst_0, \tag{2.6}$$

$$(r^2 - 16a^3ls^2)t = r^2t_0 + 16a^2ls^2m_0 + 16a^3s^2lt_0. \tag{2.7}$$

Note that by our assumption, $16a^3l$ is not a perfect square. So the Pell equation,

$$r^2 - 16a^3ls^2 = 1, \tag{2.8}$$

always has infinitely many solutions $(r, s) \in \mathbb{Z}^2$. Furthermore, through (2.6) and (2.7), each solution (r, s) of the Pell equation (2.8) gives integers m and t such that

$$m = r^2m_0 + 16a^3ls^2m_0 + 4arst_0,$$
$$t = r^2t_0 + 16a^2lsrm_0 + 16a^3s^2lt_0.$$

One can easily verify that if $(r, s) \neq (\pm 1, 0)$, these m and t satisfy equations (2.5) and hence satisfy equation (2.4). Note that $r^2 \equiv 1 \pmod{2a}$ and $r(m - m_0) \equiv 0 \pmod{2a}$. Hence we have $m \equiv m_0 \pmod{2a}$ and $t \equiv t_0 \pmod{2a}$ by (2.5). Since there are infinitely many solutions $(r, s) \in \mathbb{Z}^2$ of the Pell equation (2.8) and these will give infinitely many solutions $(m, t) \in \mathbb{Z}^2$ of the equation (2.5). In particular, there are infinite many positive integers t such that $t \equiv t_0 \pmod{2a}$ and that

$$n = \frac{-b + \sqrt{D + 4alm^2}}{2a} = \frac{-b + t}{2a}$$

is a positive integer by (2.3). Therefore, there are infinitely many positive solutions $(n, m) \in \mathbb{N}^2$ of (2.2). ∎

It is worth mentioning that one should not expect Theorem 2.2 to be true for polynomials of higher degree, because they may only have finitely many integer solutions by Siegel's theorem on integral points in [12].

In view of Theorem 2.2, to determine that the conjecture is true for a given quadratic polynomial $f(x)$, we only need to find one pair of positive integers n_1 and n_2 such that $\lambda(f(n_1)) \neq \lambda(f(n_2))$. This gives us the following theorem.

Theorem 2.3 *Let $f(x) = ax^2 + bx + c$ with $a \in \mathbb{N}$ and $b, c \in \mathbb{Z}$. Let*

$$A_0 = \Big[\frac{|b| + (|D| + 1)/2}{2a}\Big] + 1.$$

Then the binary sequence $\{\lambda(f(n))\}_{n=A_0}^{\infty}$ is either a constant sequence or it changes sign infinitely often.

Proof Suppose $\{\lambda(f(n))\}_{n=A_0}^{\infty}$ is not a constant sequence. Then there are positive integers $n_1 \neq n_2 \geq A_0$ such that $\lambda(f(n_1)) \neq \lambda(f(n_2))$. Hence there are positive integers l_1, l_2 and m_1, m_2 such that

$$\lambda(l_1) = +1, \quad \text{and} \quad \lambda(l_2) = -1,$$

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with
$$f(n_1) = l_1 m_1^2, \quad \text{and} \quad f(n_2) = l_2 m_2^2.$$

We claim that al_1 and al_2 are not perfect squares. If $al_j = N^2$ is a perfect square, then the diophantine equation $t^2 = D + 4al_j m^2$ has only finitely many solutions (t, m). In fact, since $(t_j - 2Nm_j)(t_j + 2Nm_j) = D$, there is a $d \neq 0$ such that $t_j + 2Nm_j = D/d$ and $t_j - 2Nm_j = d$. It follows that $2t_j = D/d + d$. Thus,

$$|t_j| \leq \frac{1}{2}\Big(\frac{|D|}{|d|} + |d|\Big) \leq \frac{|D| + 1}{2}.$$

Since $f(n_j) = l_j m_j^2$,

$$n_j = \Big| \frac{-b \pm \sqrt{D + 4al_j m_j}}{2a} \Big| \leq \frac{|b| + |t_j|}{2a} \leq \frac{|b| + (|D| + 1)/2}{2a} < A_0.$$

This contradicts $n_j \geq A_0$. Therefore from Theorem 2.2, there are infinitely many n_1 and n_2 such that $\lambda(f(n_1)) \neq \lambda(f(n_2))$, and hence $\lambda(f(n))$ changes sign infinitely often. ∎

As we remarked above, one should not expect Theorem 2.3 to be true for polynomials of higher degree.

We prove some partial results of special quadratic polynomials.

Theorem 2.4 *Let $f(n) = n^2 + bn + c$ with $b, c \in \mathbb{Z}$. Suppose there exists a positive integer $n_0 \geq A_0$ (with $a = 1$) such that $\lambda(f(n_0)) = -1$. Then $\lambda(f(n))$ changes sign infinitely often.*

Proof We observe the identity $f(n)f(n+1) = f(f(n) + n)$, which can be verified directly. Hence we have

$$\lambda(f(n))\lambda(f(n+1)) = \lambda(f(f(n) + n)). \tag{2.9}$$

If $\{\lambda(f(n))\}_{n=1}^{\infty}$ is a constant sequence, then it follows from (2.9) that

$$\lambda(f(n)) = +1, \qquad \text{for all } n = 1, 2, \dots .$$

Therefore if there is $n_0 \geq A_0$ such that $\lambda(f(n_0)) = -1$, then by Theorem 2.3, $\{\lambda(f(n))\}_{n=1}^{\infty}$ changes sign infinitely often. ∎

The proof of Theorem 2.2 shows that the solvability of the diophantine equation

$$X^2 - 4alY^2 = D \tag{2.10}$$

is critical in solving the problem. In general, there is no simple criterion to determine the solvability of equation (2.10) except if we know the central norm of the continued fraction of the irrational number $\sqrt{al}$. For more discussion in this area, we refer the reader to [6–9]. The following theorem deals with a special case of D for which we can solve equation (2.10).

Theorem 2.5 *Let $f(x) = px^2+bx+c$ with prime number p. Suppose the discriminant $D = b^2 - 4pc$ is a non-zero perfect square. Then $\lambda(f(n))$ changes sign infinitely often.*

Proof We first choose positive integers l_1 and l_2 such that pl_1 and pl_2 are not perfect squares and $\lambda(l_1) \neq \lambda(l_2)$. So the Pell equations

$$X^2 - 4p\, l_j Y^2 = 1, \quad j = 1, 2 \tag{2.11}$$

have infinitely many positive solutions (X, Y). Let $D = q^2$ with $q \geq 1$. Then any positive solution (X, Y) of (2.11) gives a positive solution (qX, qY) of

$$X^2 - 4p\, l_j Y^2 = D.$$

We can choose X large enough so that $-b + qX > 0$. On the other hand, we have $X^2 \equiv 1 \pmod{p}$ by (2.11) and $q^2 \equiv b^2 \pmod{p}$ because $D = b^2 - 4pc$. Therefore $(qX)^2 \equiv b^2 \pmod{p}$. Since p is a prime, either (i) $qX \equiv b \pmod{p}$ or (ii) $qX \equiv -b \pmod{p}$. We define

$$n = \frac{-b \pm qX}{2p}$$

where the sign $\pm$ is determined according to cases (i) or (ii) so that n is a positive integer. Therefore (n, qX) is a positive solution of the equations $f(n) = l_j m^2$. Then our theorem follows readily from Theorem 2.2. ■

References

[1] P. Borwein, R. Ferguson, and M. Mossinghoff, *Sign changes in sums of the Liouville function.* Math. Comp. **77**(2008), no. 263, 1681–1694. http://dx.doi.org/10.1090/S0025-5718-08-02036-X

[2] J. Cassaigne, S. Ferenczi, C. Mauduit, J. Rivat, and A. Sarkozy *On finite pseudorandom binary sequences. IV. The Liouville function. II.* Acta Arith. **95**(2000), no. 4, 343–359.

[3] S. Chowla, *The Riemann Hypothesis and Hilbert's Tenth Problem.* Gordon and Breach, New York, 1965.

[4] C. B. Haselgrove *A disproof of a conjecture of Pólya.* Mathematika **5**(1958), 141–145. http://dx.doi.org/10.1112/S0025579300001480

[5] I. Kátai, *Research Problems in Number Theory. II*, Ann. Univ. Sci. Budapest. Sect. Comput. **16**(1996), 223–251.

[6] R. Mollin, *Central norms and continued fractions*. Int. J. Pure Appl. Math. **55**(2009), no. 1, 1–8.

[7] ———, *Norm form equations and continued fractions*. Acta Math. Univ. Comenian. **74**(2005), no. 2, 273–278.

[8] ———, *Criteria for simultaneous solutions of $X^2 - DY^2 = c$ and $x^2 - Dy^2 = -c$*. Canad. Math. Bull. **45**(2002), no. 3, 428–435. http://dx.doi.org/10.4153/CMB-2002-045-2

[9] ———, *The Diophantine equation $ax^2 - by^2 = c$ and simple continued fractions*. Int. Math. J. **2**(2002), no. 1, 1–6.

[10] G. Pólya, *Verschiedene Bemerkungen zur Zahlentheorie.* Jahresber. Deutsch. Math.-Verein. **28**(1919), 31–40.

[11] A. Sarkozy, *Unsolved Problems in Number Theory.* Period. Math. Hungar. **42** (2001), no. 1-2, 17–35. http://dx.doi.org/10.1023/A:1015236305093

[12] C. L. Siegel, *Über einige Anwendungen Diophantischer Approximationen.* Abh. Preussischen Akademie der Wissenshaften, Phys. Math. Klasse (1929) pp. 41–69.

Department of Mathematics, Simon Fraser University, Burnaby, BC, V5C 1S6.
e-mail: pborwein@sfu.ca kkchoi@math.sfu.ca hganguli@sfu.ca