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COVERINGS AND EMBEDDINGS OF INVERSE SEMIGROUPS

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A correspondence is established between a class of coverings of an inverse semigroup S and a class of embeddings of S, generalising results of McAlister and Reilly on E-unitary covers of inverse semigroups.

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Introduction

The aim of this paper is to generalise two correspondences discovered by McAlister and Reilly [9]. To state their results precisely, we shall need a few definitions. An inverse monoid F with group of units $U(F)$ is said to be *factorisable* if for each $x \in F$ there exists $g \in U(F)$ such that $x \leqq g$. If G is any group then $K(G)$, the *coset semigroup* of G, has elements the cosets of subgroups of G and multiplication $(Ha) \otimes (Kb) = \langle H \cup aKa^{-1} \rangle ab$, where H and K are subgroups of G. It is easy to show that $K(G)$ is a factorisable inverse monoid with group of units isomorphic to G.

Now let S be a fixed inverse semigroup. Consider the following three classes of morphisms:

(I) *E-unitary covers* $\theta: P \to S$ of the inverse semigroup S through the group $G = P/\sigma$, where σ is the minimum group congruence on P.

(II) *Idempotent pure prehomomorphisms* $\psi: S \to K(G)$, where $G = \bigcup\{\psi(s): s \in S\}$.

(III) *Embeddings* $\iota: S \to F$, where F is a factorisable inverse monoid with group of units $U(F) = G$, such that for each $g \in U(F)$ there exists an element $s \in S$ such that $\iota(s) \leqq g$.

In [9], it is shown that there is a correspondence between coverings of S of type (I) and morphisms with domain S of type (II), and between morphisms of type (II) and embeddings of S of type (III). These correspondences were placed in a categorical setting by the author in [7].

The key step in generalising these correspondences will be to generalise the coset semigroup $K(G)$ to arbitrary inverse semigroups. Although Nambooripad and Veeramony [10] have generalised $K(G)$ in one direction, the generalisation needed for our purposes is related to work of Schein [15]. For each inverse semigroup S we construct the *inverse semigroup of cosets* of S, denoted $K(S)$. This construction forms the basis of Section 1. Independently, Leech [8] has likewise introduced the semigroup $K(S)$ within the context of his theory of inverse algebras. In Section 2, we show how idempotent pure prehomomorphisms from S to semigroup of the form $K(U)$, for some inverse semigroup U, give rise to covers of S through U. In Section 3, we show how idempotent pure prehomomorphisms from S to semigroups of the form $K(U)$ give rise to embedding of S into inverse semigroups T, which contain U as an inverse subsemigroup. We refer the reader to Petrich [12] for the basic definitions from inverse semigroup theory. We list below some definitions and notation of particular importance to this paper.

Let S be an inverse semigroup and $x, y \in S$. The *trace product* of x and y, denoted $x.y$, is defined to be xy if $x^{-1}x = yy^{-1}$, and undefined otherwise. A *prehomomorphism* θ from an inverse semigroup S to an inverse semigroup T is a function $\theta: S \to T$ such that $\theta(xy) \leqq \theta(x)\theta(y)$ and $\theta(x^{-1}) = \theta(x)^{-1}$, for all elements $x, y \in S$. Here, and throughout this paper, the natural partial order on an inverse semigroup will be denoted by "$\leqq$". A prehomomorphism is a homomorphism precisely when it preserves the meet of any two idempotents. It is important to note that we use the term "prehomomorphism" in the sense of our paper [3], and not in Petrich's sense [12]. A prehomomorphism θ is said to be *idempotent pure* if $\theta(x)$ is an idempotent only when x is an idempotent. We denote the maximum idempotent pure congruence on an inverse semigroup by τ. $I(X)$ (resp. $I^*(X)$) is the symmetric inverse semigroup of all partial injective functions on the set X with arguments written on the right (resp. left). The identity relation on a set will always be denoted Δ. If A and B are any sets then π_1 and π_2 denote the projections of $A \times B$ to A and B respectively. If θ is a homomorphism then $\ker\theta$ denotes the equivalence relation induced on the domain of θ. Finally, if ρ and σ are two binary relations on a set X, their product is written $\rho \circ \sigma$.

1. Closet semigroups

Unless stated otherwise, U will always denote an inverse semigroup.

Definition. Let A be a subset of an inverse semigroup U. Define

$$[A]^{\uparrow} = \{u \in U : a \leqq u \text{ some } a \in A\} \text{ and } [A] = \{u \in U : u \leqq a \text{ some } a \in A\}.$$

The former set is called the *upper closure* of A and the latter the *lower closure* of A. Since this paper will deal almost exclusively with upper closures we shall use the word "closure" to mean "upper closure".

Definition. A subset A of U is said to be *closed* if $[A]^{\uparrow} = A$.

The operation of taking the closure of a subset of U has a number of elementary properties; we list them below, and leave the easy verifications to the reader.

Lemma 1.1. *Let A and B be arbitrary subsets of U. Then:*

(i) *If $A \subseteq B$ then $[A]^{\uparrow} \subseteq [B]^{\uparrow}$.*

(ii) $A \subseteq [A]^{\uparrow}$.

(iii) *If A is an inverse subsemigroup of U then $[A]^{\uparrow}$ is a closed inverse subsemigroup of U.*

Definition. A subset A of U is called an *inductive atlas* if $A = AA^{-1}A$.

Remark. Since no other kinds of atlas will be considered in this paper we shall use the word "atlas" rather than "inductive atlas". Ehresmann [1] introduced atlases in ordered groupoids for his work in differential geometry. See Schein [13] for the related notion of "groud". In checking that a subset A of an inverse semigroup U is an atlas, it is enough to show that $AA^{-1}A \subseteq A$, the reverse inequality holds automatically.

Lemma 1.2. *Let U be an inverse semigroup. Then:*

(i) *If A is an atlas in U then $[A]^{\uparrow}$ is a closed atlas.*

(ii) *If ρ is a congruence on U then $\rho(u)$ is an atlas for all $u \in U$.*

(iii) *If A is a closed atlas in U then so is A^{-1}.*

(iv) *If H is a closed inverse subsemigroup of U and $aa^{-1} \in H$ then $[Ha]^{\uparrow}$ is a closed atlas.*

(v) *Let $\theta: U \to V$ be a homomorphism of inverse semigroups. If A is a closed atlas of V then $B = \theta^{-1}(A)$ is a closed atlas of U.*

Proof. The proofs of (i), (ii) and (iii) are straightforward.

(iv) It is enough to show that $[Ha]^{\uparrow}$ is an atlas. Let $x, y, z \in [Ha]^{\uparrow}$. Then there exist $h, k, l \in H$ such that $ha \leqq x$, $ka \leqq y$ and $la \leqq z$. Thus $ha(ka)^{-1}la \leqq xy^{-1}z$. But $ha(ka)^{-1}la = (h(aa^{-1})k^{-1}l)a$, and $aa^{-1} \in H$ so that $h(aa^{-1})k^{-1}l \in H$. Thus $xy^{-1}z \in [Ha]^{\uparrow}$.

(v) B is an atlas, for if $x, y, z \in B$ then $\theta(x), \theta(y), \theta(z) \in A$. By assumption, A is an atlas and so $\theta(x)\theta(y)^{-1}\theta(z) \in A$. Thus $xy^{-1}z \in B$. B is closed, for if $x \in B$ and $x \leqq y$, then $\theta(x) \in A$ and $\theta(x) \leqq \theta(y)$. Thus $\theta(y) \in A$ and so $y \in B$. □

Definition. A subset A of an inverse semigroup U is called a *coset* if it is a closed atlas. The set of cosets of U will be denoted by $K(U)$.

The following result is central to the correspondence we shall set up in Section 2.

Proposition 1.3. *Let U, Q and S be inverse semigroups and let $\theta: Q \to S$ and $\phi: Q \to U$ be surjective homomorphisms. Define a map $\psi: S \to P(U)$ by $\psi(s) = \phi(\theta^{-1}(s))$. Then*

(i) *$\psi(s)$ is an atlas,*

(ii) *$\psi(s)$ is a coset for all $s \in S$ if and only if* $(\ker \phi \circ \leqq) \subseteq \ker \theta \circ \ker \phi$.

Proof. Since (i) is easily verified we prove (ii) only. We begin by assuming that $\psi(s)$

is a coset for all elements s of S. Suppose that $(p,q)\in\ker\phi\circ\leqq$. By definition there exists an elements q' in Q such that

$$\phi(p)=\phi(q') \text{ and } q'\leqq q.$$

Since ϕ is order preserving we have that $\phi(q')\leqq\phi(q)$ in U. Let $\theta(p)=s$. Then, by the definition of ψ, $\phi(p)$ is an element of $\psi(s)$. By assumption, $\psi(s)$ is a coset, and $\phi(p)\leqq\phi(q)$. Thus $\phi(q)$ also belongs to $\psi(s)$. From the definition of ψ, there must therefore exist an element q'' in Q such that

$$\theta(q'')=s \text{ and } \phi(q'')=\phi(q).$$

But now we see that $\theta(p)=s=\theta(q'')$ so that $(p,q'')\in\ker\theta$, and $\phi(q'')=\phi(q)$ so that $(q'',q)\in\ker\phi$. Thus $(p,q)\in\ker\theta\circ\ker\phi$ as required.

Conversely, suppose that $(\ker\phi\circ\leqq)\subseteq\ker\theta\circ\ker\phi$. We shall prove that $\psi(s)$ is a coset for all elements s of S. By (i), it is enough to show that $\psi(s)$ is closed. Let $u\in\psi(s)$ and suppose that $u\leqq v$. By definition of ψ there exists $p\in Q$ such that $\phi(p)=u$ and $\theta(p)=s$. Since ϕ is surjective there exists $q\in Q$ such that $\phi(q)=v$. In particular, $\phi(p)\leqq\phi(q)$. Put $e=pp^{-1}$. Then $\phi(p)=\phi(e)\phi(q)=\phi(eq)$. Thus $\phi(p)=\phi(eq)$ and $eq\leqq q$, which means that $(p,q)\in\ker\phi\circ\leqq$. By assumption, $(p,q)\in\ker\theta\circ\ker\phi$ so that there exists $q'\in Q$ such that

$$\theta(p)=\theta(q') \text{ and } \phi(q')=\phi(q).$$

If follows that

$$s=\theta(p)=\theta(q') \text{ and } \phi(q')=\phi(q)=v.$$

In particular,

$$\phi(q')=v \text{ and } \theta(q')=s.$$

Thus $v\in\psi(s)$. Hence $\psi(s)$ is a coset. □

In the remainder of this section we shall prove that, with respect to a suitable definition of multiplication, $K(U)$ is an inverse semigroup. Some of the following results could be deduced from Schein [15] but we have preferred to prove them directly, for the sake of completeness.

Proposition 1.4. *Let A be a coset in U.*

(i) *Put $H=[AA^{-1}]^{\uparrow}$ and let $a\in A$. Then H is a closed inverse subsemigroup of U and $A=[Ha]^{\uparrow}$.*

(ii) *Put $K=[A^{-1}A]^{\uparrow}$ and let $b\in A$. Then K is a closed inverse subsemigroup of U and $A=[bK]^{\uparrow}$.*

Proof. We shall prove (i), the proof of (ii) is similar. It is easy to check that AA^{-1} is an inverse subsemigroup of U. By Lemma 1.1, $H=[AA^{-1}]^{\uparrow}$ is a closed inverse subsemigroup of U. We claim that $Ha\subseteq A$. Indeed, let $h\in H$ and consider the product ha. By the definition of H there exist elements $b,c\in A$ such that $bc^{-1}\leqq h$. But then $bc^{-1}a\leqq ha$. Since A is an atlas $bc^{-1}a\in A$. However, A is closed so that $ha\in A$. Thus $Ha\subseteq A$, as required. By Lemma 1.1, we have that $[Ha]^{\uparrow}\subseteq[A]^{\uparrow}=A$. We now show that the reverse inclusion holds. Let $b\in A$. Then $ba^{-1}a\leqq b$. Now $ba^{-1}\in AA^{-1}$ and so $ba^{-1}\in H$. Thus $ba^{-1}a\in Ha$ and so $ba^{-1}a\in[Ha]^{\uparrow}$. But $[Ha]^{\uparrow}$ is closed and so contains the element b, thus $A\subseteq[Ha]^{\uparrow}$. □

Remark. The above proposition together with Lemma 1.2(iv) shows that our definition of coset coincides with Leech's [**8**].

Proposition 1.5. *Let A be a coset of U which contains an idempotent. Then A is a closed inverse subsemigroup of U.*

Proof. Let e be an idempotent in A. Put $H=[AA^{-1}]$. Then by Proposition 1.4(i), we have that $A=[He]^{\uparrow}$. Now $He\subseteq H$, since $e\in H$. Thus by Lemma 1.1(i), we obtain that $A\subseteq H$. Now let $h\in H$. Then $he\leqq h$, and so $h\in[He]^{\uparrow}=A$. Thus $h\in A$. It follows that $A=H$. □

Our definition of a suitable multiplication on $K(U)$ will be achieved by means of quantales, defined below. Although the multiplication can just as easily be defined without them, our approach may have applications elsewhere in semigroup theory.

Definition. A *quantale* Q is a lattice-ordered semigroup which is join complete as a lattice and satisfies the following distributivity conditions

$$a(\bigvee b_i)=\bigvee(ab_i) \text{ and } (\bigvee b_i)a=\bigvee(b_ia)$$

for all $a\in Q$ and $\{b_i\}\subseteq Q$. A *homomorphism* of quantales is a semigroup map which preserves arbitrary sups.

Homomorphisms on quantales are most easily described by means of nuclei.

Definition. Let Q be a quantale. A *quantic nucleus* is a map $j:Q\rightarrow Q$ satisfying the following conditions:

$(QN1)$ $a\leqq j(a)$ for all $a\in Q$.

$(QN2)$ $a\leqq b$ implies $j(a)\leqq j(b)$ for all $a,b\in Q$.

$(QN3)$ $j^2(a)=j(a)$ for all $a\in Q$.

$(QN4)$ $j(a)j(b)\leqq j(ab)$ for all $a,b\in Q$.

Remark. Observe that if j is a quantic nucleus then

$$j(ab) = j(aj(b)) = j(j(a)b) = j(j(a)\,j(b)),$$

a result which we shall use many times in the sequel.

Definition. If j is any quantic nucleus on Q, put $Q_j = \{a \in Q : j(a) = a\}$. Define an operation "$\circ$" on Q_j by $a \circ b = j(ab)$. It can be shown that $(Q_j, \circ)$ is a quantale and $j: Q \to Q_j$ is a surjective quantale homomorphism. We refer the reader to [**11**] for details.

Definition. If Q is a quantale and a and b are arbitrary elements of Q define two new operations '/' and '\' as follows:

$$a/b = \bigvee \{q \in Q : bq \leqq a\} \text{ and } a\backslash b = \bigvee \{q \in Q : qb \leqq a\}.$$

Observe that for any element c of Q

$$bc \leqq a \Leftrightarrow c \leqq a/b \text{ and } cb \leqq a \Leftrightarrow c \leqq a\backslash b.$$

Remark. Clearly, underlying every quantale is a semigroup and underlying every quantale map is a semigroup homomorphism. Thus there is a forgetful functor from the category of quantales and quantale maps to the category of semigroups and semigroup homomorphisms. This functor has a left adjoint which takes a semigroup S to its power semigroup $P(S)$, which is clearly a quantale, and lifts semigroup homomorphisms to the obvious map between power semigroups [**11**].

The following lemma provides a way of constructing quantic nuclei on quantales of the form $P(S)$.

Lemma 1.6. *Let $\mathscr{I} = \{A_i : i \in I\}$ be a family of subsets of S, satisfying the following three conditions:*

(i) *For each $X \in P(S)$ there exists an $A_i \in \mathscr{I}$ such that $X \subseteq A_i$.*
(ii) *For all $J \subseteq I$, the intersection $\bigcap \{A_i : i \in J\}$ belongs to $\mathscr{I}$.*
(iii) *For all $X \in P(S)$ the elements $A_i \backslash X$ and A_i/X belong to $\mathscr{I}$.*

Define a map $j: P(S) \to P(S)$ by

$$j(X) = \bigcap \{A_i : X \subseteq A_i \text{ and } A_i \in \mathscr{I}\}.$$

Then j is a nucleus on $P(S)$.

Proof. It is easy to see that $(QN1)$, $(QN2)$ and $(QN3)$ hold, so that it only remains to show that $(QN4)$ holds. Let X and Y be arbitrary subsets of S. To prove that $(QN4)$ holds it is enough to show that every element C of $\mathscr{I}$ which contains XY must also contain $j(X)j(Y)$. To this end, let $XY \subseteq C$, where C is an element of $\mathscr{I}$. Then $X \subseteq C \backslash Y$.

By assumption, $C\backslash Y\in\mathscr{I}$. Thus $j(X)\subseteq C\backslash Y$ and so $j(X)Y\subseteq C$. But then $Y\subseteq C/j(X)$. By assumption, $C/j(X)\in\mathscr{I}$. Thus $j(Y)\subseteq C/j(X)$, and so $j(X)j(Y)\subseteq C$, as required. □

As an application of the above result we have the following.

Proposition 1.7. *Let* $\mathscr{I}=\{A: A\in K(U)\}\cup\{\varnothing\}$. *Then* $\mathscr{I}$ *satisfies the conditions of Lemma 1.6.*

Proof. Conditions (i) and (ii) are easy to verify. We shall show that (iii) holds. Specifically, we shall prove that if A is a coset and X an arbitrary non-empty set such that $A\backslash X$ is non-empty, then $A\backslash X$ is also a coset. The proof that, if A/X is non-empty it is also a coset, is similar. We begin by showing the $A\backslash X$ is an atlas. Let $u,v,w\in A\backslash X$ and let $x\in X$. By definition we have that

$$ux=a, vx=b \text{ and } wx=c,$$

for some $a,b,c\in A$. Now

$$ab^{-1}c=(ux)(vx)^{-1}(wx)=uxx^{-1}v^{-1}wx\leqq uv^{-1}wx.$$

Such A is an atlas the product $ab^{-1}c\in A$. Also, A is closed so that the product $uv^{-1}wx\in A$ for all $x\in X$. Thus $uv^{-1}w\in A\backslash X$, and $A\backslash X$ is an atlas. It now only remains to show that $A\backslash X$ is closed. Let $u\in A\backslash X$ and $u\leqq v$. If $x\in X$ then $ux=a$, some $a\in A$. It follows that $ux\leqq vx$. But $ux=a$ is an element of A, which is a closed set. Thus $vx\in A$ and the result follows. □

It is now immediate from Lemma 1.6 and Proposition 1.7 that the function $j:P(U)\to P(U)$ defined by

$$j(X)=\bigcap\{A: A\in K(U)\cup\{\varnothing\} \text{ and } X\subseteq A\}$$

is a nucleus. Thus the set $P(U)_j=K(U)\cup\{\varnothing\}$ becomes a quantale when we define a product $\otimes$ by

$$A\otimes B=j(AB).$$

It is clear that if A and B are non-empty then $A\otimes B$ is also non-empty. Thus $(K(U),\otimes)$ is a semigroup; that it is also an inverse semigroup is a consequence of the next result.

Remark. For the remainder of this paper "j" will always denote the particular nucleus defined above.

Theorem 1.8. $(K(U),\otimes)$ *is an inverse semigroup having the following properties*:

(i) *The idempotents are the closed inverse subsemigroups of* U.

(ii) *The natural partial order is reverse inclusion.*

(iii) *Put* $U'=\{[u]^{\uparrow}:u\in U\}$. *Then* U' *is an inverse subsemigroup of* $K(U)$ *isomorphic to* U, *such that* $[U']=K(U)$.

(iv) *Let* $A,B\in K(U)$. *Let* $H=[AA^{-1}]^{\uparrow}$ *and* $a\in A$, *and let* $K=[BB^{-1}]^{\uparrow}$ *and* $b\in B$. *Then* $A\otimes B=[[\langle H,aKa^{-1}\rangle]^{\uparrow}ab]^{\uparrow}$, *where* $\langle H,aKa^{-1}\rangle$ *is the inverse subsemigroup generated by* H *and* aKa^{-1}.

Proof. It is clear that $K(U)$ is a regular semigroup, since

$$A\otimes A^{-1}\otimes A=j(AA^{-1}A)=j(A)=A$$

for all $A\in K(U)$. To prove that $K(U)$ is inverse, it is enough to prove that the idempotents commute. The first step is to characterise the idempotents.

(i) Let $A\otimes A=A$ in $K(U)$. We shall show that A contains an idempotent, the result will then follow from Proposition 1.5. Let $a\in A$ and $H=[AA^{-1}]^{\uparrow}$. Then $A=[Ha]^{\uparrow}$ by Proposition 1.4. A is a subsemigroup of U, since $AA\subseteq j(AA)=A$, and so $a^2\in A$. Since $a^2\in[Ha]^{\uparrow}$, there exists $h\in H$ such that $ha\leqq aa$. Thus $(ha)a^{-1}\leqq a(aa^{-1})\leqq a$. Hence $e=a^{-1}(ha)a^{-1}\leqq a^{-1}a$ is an idempotent. But $e\in A^{-1}AA^{-1}=A^{-1}$. Thus $e\in A$. Conversely, let A be a closed inverse subsemigroup of U. Then $A\otimes A=j(AA)=j(A)=A$ since $AA=A$. We can now show that the idempotents commute. Let A and B be arbitrary closed inverse subsemigroups of U. Their product $A\otimes B$ is given by

$$A\otimes B=j(AB)=\bigcap\{C:C\in K(U)\text{ and }AB\subseteq C\}.$$

It is clear that, since A and B contain idempotents, so does AB. Thus by Proposition 1.5, any coset which contains AB must be a closed, inverse subsemigroup of U. Furthermore, $AB\subseteq C\Leftrightarrow B^{-1}A^{-1}\subseteq C^{-1}$. Since A,B and C are all inverse subsemigroups, it follows that $AB\subseteq C\Leftrightarrow BA\subseteq C$. Thus $A\otimes B=B\otimes A$ is an idempotent.

(ii) Suppose that $X\leqq Y$, where $\leqq$ is the natural partial order on $K(U)$. By the definition of the natural partial order, we have that $X=j(XX^{-1}Y)$ and so $XX^{-1}Y\subseteq X$. Let x and y be arbitrary elements of X and Y respectively and let $u=xx^{-1}y\leqq y$. The element u belongs to $XX^{-1}Y$ and so to X. But X is closed and thus also contains y. Since y was chosen arbitrarily, we have shown that $Y\subseteq X$. Conversely, suppose that $Y\subseteq X$. Then, clearly, $XX^{-1}Y\subseteq X$ and so $j(XX^{-1}Y)\subseteq X$. Let x and y be chosen arbitrarily from X and Y respectively. Consider the product $u=xy^{-1}y$. It is clear that $u\in XX^{-1}Y$. Since $u\leqq x$, it follows that $x\in j(XX^{-1}Y)$. Since x was chosen arbitrarily in X, we have shown that $X\subseteq j(XX^{-1}Y)$.

(iii) We begin by showing that U' is an inverse subsemigroup of $K(U)$. It is clear that U' is closed under inverses. To establish closure under $\otimes$, observe that

$$[s]^{\uparrow}\otimes[t]^{\uparrow}=j([s]^{\uparrow}[t]^{\uparrow})=j(j\{s\}j\{t\})=j(\{st\})=[st]^{\uparrow}.$$

It is clear that $(U',\otimes)$ is isomorphic to U. Finally, let $A\in K(U)$ and $s\in A$. Then $[s]^{\uparrow}\subseteq A$, so that $A\leqq[s]^{\uparrow}$. Thus $[U']=K(U)$.

(iv) By Lemma 1.2, $[[\langle H,aKa^{-1}\rangle]^{\uparrow}ab]^{\uparrow}$ is a coset. A simple calculation shows that $A\otimes B=j(HaKb)$. We shall show that $j(HaKb)=[[\langle H,aKa^{-1}\rangle]^{\uparrow}ab]^{\uparrow}$. Let $HaKb\subseteq C$, where C is any coset. Observe that

$$ab=(aa^{-1})a(bb^{-1})b\in HaKb,$$

so that $ab\in C$. Put $D=[CC^{-1}]^{\uparrow}$. Then $C=[Dab]^{\uparrow}$ by Proposition 1.4. Since $HaKb\subseteq C$ we obtain that $HaKa^{-1}H\subseteq D$. It follows easily from this that $H,aKa^{-1}\subseteq D$. But D is a closed, inverse subsemigroup and so $[\langle H,aKa^{-1}\rangle]^{\uparrow}\subseteq D$. Thus $[[\langle H,aKa^{-1}\rangle]^{\uparrow}ab]^{\uparrow}\subseteq C$. □

Remark. Observe that S is embedded in $K(S)$ via the map $v:S\to K(S)$ given by $v(s)=[s]^{\uparrow}$. This makes $K(S)$ look like a kind of completion of S; what kind of completion is made precise by Leech [8].

Theorem 1.8 can be extended to encompass homomorphisms between inverse semigroups.

Theorem 1.9. *There is a faithful endofunctor* $\mathbf{K}$ *on the category of inverse semigroups and homomorphisms, which takes each inverse semigroup* S *to* $K(S)$.

Proof. Let $\theta:S\to T$ be a homomorphism between inverse semigroups. We define

$$\mathbf{K}(\theta):K(S)\to K(T) \text{ by } \mathbf{K}(\theta)(A)=j(\theta(A)) \text{ for all } A\in K(S).$$

We shall usually write $\mathbf{K}(\theta)=\theta^*$, to extend the notation used by Joubert [2] in the case where S is a group. To show that θ^* is a homomorphism, we have to show that for all $A,B\in K(S)$,

$$\theta^*(A\otimes B)=\theta^*(A)\otimes\theta^*(B),$$

which is equivalent to showing that $j(\theta(j(AB)))=j(\theta(AB))$. First observe that since $AB\subseteq j(AB)$, we have that $\theta(AB)\subseteq\theta(j(AB))$. Thus

$$\theta(AB)\subseteq\theta(j(AB))\subseteq j(\theta(j(AB))).$$

Hence $j(\theta(AB))\subseteq j(\theta(j(AB)))$. This shows that θ^* is a prehomomorphism. To show that θ^* is a homomorphism it remains to show that $j(\theta(j(AB)))\subseteq j(\theta(AB))$. Let $C\in K(T)$ be such that $\theta(AB)\subseteq C$. Clearly, $AB\subseteq\theta^{-1}(\theta(AB))\subseteq\theta^{-1}(C)$. But $\theta^{-1}(C)\in K(S)$ by Lemma 1.2(v), so that $j(AB)\subseteq\theta^{-1}(C)$. Thus $\theta(j(AB))\subseteq C$, and so $j(\theta(j(AB)))\subseteq C$. Hence $j(\theta(j(AB)))\subseteq j(\theta(AB))$. It is straightforward to check the $\mathbf{K}$ is a functor. It only remains to prove that $\mathbf{K}$ is faithful. Let θ and ψ be homomorphisms from S to T and suppose

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that $\theta^* = \psi^*$. Then for all $s \in S$, $\theta^*([s]^\uparrow) = \psi^*([s]^\uparrow)$. But $\theta^*([s]^\uparrow) = [\theta(s)]^\uparrow$ and $\psi^*([s]^\uparrow) = [\psi(s)]^\uparrow$. Thus $\theta(s) = \psi(s)$ for all $s \in S$. □

We now describe the form taken by the trace product in $K(U)$.

Proposition 1.10. *Let A and B be cosets in $K(U)$ such that $A^{-1} \otimes A = B \otimes B^{-1}$. Then $A.B = j(ab(B^{-1}B))$, where $a \in A$ and $b \in B$.*

Proof. By Proposition 1.4, we have that

$$A = [aK]^\uparrow \text{ where } K = [A^{-1}A]^\uparrow \text{ and } a \in A$$

and

$$B = [Hb]^\uparrow \text{ where } H = [BB^{-1}]^\uparrow \text{ and } b \in B.$$

By assumption, $K = H$. Observe that $A \otimes B = j(aHb)$, for

$$j(AB) = j(j(aH)j(Hb)) = j((aH)(Hb)) = j(aH^2b) = j(aHb)$$

where $H^2 = H$, since H is an inverse subsemigroup. We now prove that $j(aHb) = j(ab(B^{-1}B))$. Let $x \in j(aHb)$. Then $ahb \leqq x$ for some $h \in H$. Thus there exist $c, d \in B$ such that $cd^{-1} \leqq h$, and so $acd^{-1}b \leqq ahb \leqq x$. But $ab(b^{-1}c)(d^{-1}b) \leqq acd^{-1}b \leqq x$, and $ab(b^{-1}c)(d^{-1}b) \in ab(B^{-1}B)$. Thus $x \in j(ab(B^{-1}B))$. Conversely, if $x \in j(ab(B^{-1}B))$ then there exist $c, d \in B$ such that $abc^{-1}d \leqq x$. But $a(bc^{-1}db^{-1})b \leqq abc^{-1}d \leqq x$, and $a(bc^{-1}db^{-1})b \in aHb$. Thus $x \in j(aHb)$. □

Definition. A *representation* of an inverse semigroup S by partial injections is a homomorphism $\alpha: S \to I(X)$. The representation is said to be *effective* if every element of X belongs to the domain of some partial function in $\alpha(S)$. The representation is said to be an *effective, transitive* representation if it is effective and for any two element $x, y \in X$ there exists $s \in S$ such that $\alpha(s)(x) = y$. Similar definitions apply to $I^*(X)$.

The significance of the next lemma will be explained in the Remark which follows it.

Lemma 1.11. *Let K and S be inverse semigroups such that S is a subsemigroup of K and $[S] = K$. Let e be an idempotent in K. Then:*

(i) *There is an effective, transitive representation ϕ of S in $I(L_e)$ given by $\phi(s) = \{(x, y) \in L_e \times L_e : sx = y\}$.*

(ii) *There is an effective, transitive representation θ of S in $I^*(R_e)$ given by $\theta(s) = \{(x, y) \in R_e \times R_e : xs = y\}$.*

Proof. We shall prove (i), the proof of (ii) is similar. $\phi(s)$ is injective on its domain:

indeed, suppose that $(x,sx),(y,sy)\in\phi(s)$ and $sx=sy$. Thus $s^{-1}sx=s^{-1}sy$. But $s^{-1}sx=x$ and $s^{-1}sy=y$, and so $\phi(s)$ is injective. Next, we claim that for all $x,y\in L_e$ there exists $s\in S$ such that $sx=y$: indeed, since xLy there exists $k\in K$ such that $kx=y$. But $[S]=K$, so that there exists $s\in S$ such that $k\leqq s$. Clearly, $y\leqq sx$. But $sxLy$ since $y^{-1}y\leqq (sx)^{-1}sx\leqq x^{-1}x$. Thus $y=sx$. It remains to show that ϕ defines a semigroup homomorphism. Observe that

$$x\in\operatorname{dom}(\phi(s)\phi(t))\Leftrightarrow x\in L_e, tx\in L_e \text{ and } stx\in L_e.$$

On the other hand, $x\in\operatorname{dom}\phi(st)\Leftrightarrow x\in L_e$ and $stx\in L_e$. It is easy to see that this implies $tx\in L_e$. It is now evident that $\phi(st)=\phi(s)\phi(t)$. □

Remark. The above lemma enables us to show that the semigroup $K(S)$ contains all the information about the effective transitive representations of S by partial one-to-one maps. Let H be an idempotent in $K(S)$. The $\mathscr{R}$-class of H in $K(S)$ consists of all those cosets A in $K(S)$ such that $A\otimes A^{-1}=H$. Thus

$$R_H=\{[Ha]^{\uparrow}: a\in S \text{ and } aa^{-1}\in H\}.$$

The elements of R_H are precisely the "right ω-cosets of H in S" [**12**]. Observe that if $s\in S$ and $A\in K(S)$ and $A\otimes v(s)\mathscr{R}_A$ then the product $A\otimes v(s)$ is given by

$$A\otimes v(s)=j(Av(s))=j(j(Ha)v(s))=j(Has)=[Has]^{\uparrow}.$$

By Lemma 1.11, each element s of S determines a partial one-to-one map $\phi(s)$ of R_H by

$$\phi(s)=\{(A,B)\in R_H\times R_H: A\otimes v(s)=B\}.$$

Thus the map $\phi:S\to I^*(R_H)$ is an effective, transitive representation of S. From the theory of inverse semigroup representations every effective, transitive representation of S is isomorphic to one obtained in this way (see [**12**]).

2. Coverings of inverse semigroups

Our first definition can be regarded as a generalisation of certain properties of factorisable inverse monoids: if F is factorisable with group of units G then (F,G) is a "covering pair" in the sense of the following.

Definition. Let U be an inverse subsemigroup of the inverse semigroup T. We suppose in addition that the following two axioms hold:

$(CP1)$ $[U]^{\uparrow}=U$.

$(CP2)$ $[U]=T$.

Then we call (T, U) a *covering pair*.

Definition. Let (T, U) be a covering pair and let $\kappa: S \to T$ be an embedding of S in T. Suppose, in addition, that for each $u \in U$ there exists $s \in S$ such that $\kappa(s) \leqq u$. Then we say that *S is covered by U in T*. The map κ is called a *covered embedding* (*of S in T by U*).

We now show that every such *covered embedding* of S by U gives rise to a *cover* of S through U.

Theorem 2.1. *With the notation above, put*

$$Q = \{(s, u) \in S \times U : \kappa(s) \leqq u\}.$$

Then i

(i) *Q is an inverse subsemigroup of $S \times U$.*
(ii) *$\pi_1 : Q \to S$ is a surjective homomorphism.*
(iii) *$\pi_2 : Q \to U$ is a surjective idempotent pure homomorphism.*
(iv) $\ker \pi_1 \cap \ker \pi_2 = \Delta$.
(v) $(\ker \pi_2 \circ \leqq) \subseteq \ker \pi_1 \circ \ker \pi_2$.

Proof. (i) Straightfoward.

(ii) That π_1 is a homomorphism is easy to check. Surjectivity follows from $(CP2)$.

(iii) That π_2 is a homomorphism is easy to check. Surjectivity follows from the fact that for every $u \in U$ there exists $s \in S$ such that $\kappa(s) \leqq u$. It is also clear that π_2 is idempotent pure.

(iv) Immediate.

(v) Let $((s, u), (t, v)) \in \ker \pi_2 \circ \leqq$. Then there exists an element $(s', u') \in Q$ such that

$$((s, u), (s', u')) \in \ker \pi_2 \text{ and } (s', u') \leqq (t, v).$$

Thus $u = u'$, $s' \leqq t$ and $u \leqq v$. Since $\kappa(s) \leqq u$ and $u \leqq v$ then $\kappa(s) \leqq v$. The pair (s, v) is therefore an element of Q. But then

$$\pi_1(s, u) = \pi_1(s, v) \text{ and } \pi_2(s, v) = \pi_2(t, v).$$

Thus $((s, u), (t, v)) \in \ker \pi_1 \circ \ker \pi_2$. $\square$

In the result below recall that τ denotes the maximum idempotent pure congruence.

Corollary 2.2 *With the notation of Theorem 2.1, we have*

(i) *if $((s, u), (t, v)) \in \tau$ in Q then $(u, v) \in \tau$ in U,*
(ii) *U is E-disjunctive if and only if* $\ker \pi_2 = \tau$.

Proof. (i) Let x and y be elements of U such that $xuy \in E(U)$. By the definition of covered embeddings, there exist $p, q \in S$ such that $i(p) \leqq x$ and $i(q) \leqq y$. Thus both (p, x) and (q, y) belong to Q. Now

$$(p,x)(s,u)(q,y)=(psq,xuy) \text{ and } (p,x)(t,v)(q,y)=(ptq,xvy).$$

By assumption, xuy is an idempotent. Thus psq is an idempotent. We have shown that $(psq,xuy)\in E(Q)$. Since $((s,u),(t,v))\in\tau$, we have that $(ptq,xvy)\in E(Q)$. Thus xvy is an idempotent. A similar argument shows that if xvy is an idempotent then xuy is an idempotent. It can similarly be shown that

$$xu\in E(U)\Leftrightarrow xv\in E(U) \text{ and } uy\in E(U)\Leftrightarrow vy\in E(U).$$

Thus $((s,u),(t,v))\in\tau$ implies $(u,v)\in\tau$ in U.
(ii) If U is E-disjunctive, then $\tau=\Delta$ on U. Suppose that $((s,u),(t,v))\in\tau$ in Q. Then by the above, $(u,v)\in\tau$ in U. Thus $u=v$. It follows that $((s,u),(t,v))\in\ker\pi_2$, and so $\tau\subseteq\ker\pi_2$. Conversely, since $\ker\pi_2$ is idempotent pure $\ker\pi_2\subseteq\tau$. Thus $\ker\pi_2=\tau$. The converse is immediate, since τ is the maximum idempotent pure congruence on Q. □

In view of Theorem 2.1, we now make the following definition.

Definition. Let S, Q and U be inverse semigroups. A *special cover* of S is a pair of homomorphisms (θ,ϕ), where $\theta:Q\to S$ and $\phi:Q\to U$, satisfying the following conditions:

$(SC1)$ θ and ϕ are surjective.

$(SC2)$ ϕ is idempotent pure.

$(SC3)$ $\ker\theta\cap\ker\phi=\Delta$.

$(SC4)$ $(\ker\phi\circ\leqq)\subseteq\ker\theta\circ\ker\phi$.

We say that *Q is a special cover of S through U.*

Definition. A *representation of S by cosets of U* is a function $\psi:S\to K(U)$ satisfying the following conditions:

$(R1)$ ψ is a prehomomorphism.

$(R2)$ ψ is idempotent pure.

$(R3)$ $U=\cup\{\psi(s):s\in S\}$.

Theorem 2.3. *There is a correspondence between special covers of S through U and representations of S in $K(U)$.*

Proof. We proceed in the following way. From each special cover (θ,ϕ) of S we shall construct a representation $R(\theta,\phi)$ of S in $K(U)$. Similarly, for each representation ψ of S in $K(U)$, we construct a cover, $C(\psi)$ of S, through U. Finally, we prove that $C(R(\theta,\phi))$ is isomorphic to (θ,ϕ) and $R(C(\psi))=\psi$. It is in this sense that we understand the word "correspondence". Let (θ,ϕ) be a special cover of S through U. Define $R(\theta,\phi)$ to be the function $\psi:S\to K(U)$ given by $\psi(s)=\phi(\theta^{-1}(s))$. This function is well-defined by Proposition 1.3 and $(SC4)$. $(R1)$ holds: indeed let $u\in\psi(s)$ and $v\in\psi(t)$. Then there exist elements $p,q\in Q$ such that

$$\theta(p)=s \text{ and } \theta(q)=t \text{ and } \phi(p)=u \text{ and } \phi(q)=v.$$

Now, $\theta(pq)=st$ and $\phi(pq)=uv$. Thus $uv\in\psi(st)$. $(R2)$ holds: indeed suppose that $\psi(s)$ is an idempotent in $K(U)$. By Theorem 1.8, $\psi(s)$ is a closed inverse subsemigroup of U. In particular, U contains idempotents. Let $e\in\psi(s)\cap E(U)$. Then there exists a $p\in Q$ such that $\phi(p)=e$ and $\theta(p)=s$. But ϕ is idempotent pure by $(SC2)$, so that p is an idempotent. Hence s is an idempotent. $(R3)$ holds by $(SC1)$.

We now show how to construct a special cover of S through U, given a representation $\psi:S\to K(U)$. Put

$$Q=\{(s,u)\in S\times U:u\in\psi(s)\}$$

and let $\pi_1:Q\to S$ and $\pi_2:Q\to U$ be the projection maps. Then we claim that $C(\psi)=(\pi_1,\pi_2)$ is a special cover through U. Note that Q is easily seen to be inverse from the properties of prehomomorphisms. It is also evident that π_1 and π_2 are homomorphisms. $(SC1)$ holds: it is immediate that π_1 is surjective. Surjectivity of π_2 is a consequence of $(R3)$. $(SC2)$ holds: indeed suppose that $\pi_2(s,u)$ is an idempotent. Then u is an idempotent and $u\in\psi(s)$. By Proposition 1.5, $\psi(s)$ is a closed, inverse subsemigroup of U. In particular, $\psi(s)$ is an idempotent in $K(U)$. But by $(R2)$, ψ is idempotent pure and so s is an idempotent. It follows that (s,u) is an idempotent in Q, as required. That $(SC3)$ holds is immediate from the definitions. To see that $(SC4)$ holds: observe that

$$\pi_2(\pi_1^{-1}(s))=\pi_2(\{(s,u)\in Q:u\in\psi(s)\})$$

which is just $\psi(s)$. Thus by Proposition 1.3, $(SC4)$ holds. Notice that we have also proved that $R(C(\psi))=\psi$.

Finally we shall prove that $C(R(\theta,\phi))$ is isomorphic to (θ,ϕ). Let (θ,ϕ) be a special cover of S through U. Define $R(\theta,\phi)=\psi$, the associated representation as above. Now define an inverse semigroup Q' by

$$Q'=\{(s,u)\in S\times U:u\in\psi(s)\}.$$

Then $C(R(\theta,\phi))=(\pi_1,\pi_2)$ is a cover of S through U. Define a map $\alpha:Q\to Q'$ by $\alpha(q)=(\theta(q),\phi(q))$. We shall prove that α is an isomorphism, such that

$$\pi_1\alpha=\theta \text{ and } \pi_2\alpha=\phi.$$

It is evident that it is in fact enough to prove that α is an isomorphism, as the rest of the result follows from the definition of α. That α is injective is immediate from $(SC3)$. That α is surjective follows from the definition of ψ. It is clear that α is a homomorphism. □

3. Embeddings of inverse semigroups

In Theorem 2.1, we showed that covered embeddings give rise to special covers, whereas in Theorem 2.3, we showed that there is a correspondence between special covers and representations. In this section, we shall complete the circle by showing that there is a correspondence between representations and covered embeddings. It will be convenient to make use of the category of ordered groupoids and ordered functors. We refer the reader to [3] for more information, but to make this paper reasonably self-contained we provide below the main definitions.

A *groupoid* G is a (small) category in which every morphism is an isomorphism. If $x \in G$ then x^{-1} will be the unique *inverse* of x. We shall usually denote the partial product in G by concatenation, except when we denote it by "." for the sake of emphasis. An *identity* in G is any element e such that whenever $e.x$ (resp. $x.e$) is defined it equals x. The (unique) left and right identities of x are written $\mathbf{r}(x)$ and $\mathbf{d}(x)$ respectively. The set of identities is written G_0. For us, xy will be defined iff $\mathbf{d}(x) = \mathbf{r}(y)$, the usual convention in category theory. The *star* of a groupoid G at the identity e is the set $L_e = \{x \in G : \mathbf{d}(x) = e\}$. If θ is a functor from G to H then θ maps L_e to L_f, where $f = \theta(e)$. Any other definitions we use from category theory will be standard.

A groupoid G is said to be *ordered* if it is equipped with an order relation $\leqq$ such that the following axioms are satisfied:

(G) If $x \leqq y$ then $x^{-1} \leqq y^{-1}$.

$(OC3)$ If $x \leqq y$ and $x' \leqq y'$ and the products xx' and yy' exist then $xx' \leqq yy'$.

$(OC8)$(i) If $x \in G$ and $e \leqq \mathbf{d}(x)$, where $e \in G_0$ then there exists a unique element $(x \mid e)$, called the *restriction of x to e*, such that $(x \mid e) \leqq x$ and $\mathbf{d}(x \mid e) = e$.

$(OC8)$(ii) If $x \in G$ and $e \leqq \mathbf{r}(x)$, where $e \in G_0$ then there exists a unique element $(e \mid x)$, called the *corestriction of x to e*, such that $(e \mid x) \leqq x$ and $\mathbf{r}(e \mid x) = e$.

In [3] we followed Ehresmann's usage [1] and called groupoids satisfying the axioms above *functorially ordered*. We usually refer to "the ordered groupoid G" rather than to "the ordered groupoid $(G, ., \leqq)$". We shall have cause later to refer to the following two axioms both of which hold in ordered groupoids (the first trivially so):

$(OC5)$(i) If $x \in G$ and $e \leqq \mathbf{d}(x)$ there exists an element x' in G such that $x' \leqq x$ and $\mathbf{d}(x') = e$.

(OI) G_0 is an order ideal of G.

In an ordered groupoid G a pair of elements x and y, for which the greatest lower bound e of $\mathbf{d}(x)$ and $\mathbf{r}(y)$ exists, are said to have a *pseudoproduct* $x \otimes y = (x \mid e).(e \mid y)$. A subgroupoid G of an ordered groupoid H is called an *ordered subgroupoid* if for each $x \in G$ and $e \in G_0$ such that $e \leqq \mathbf{d}(x)$ the restriction $(x \mid e) \in G$; it is clear that with respect to the induced order G is an ordered groupoid in its own right.

An *ordered functor* between ordered groupoids is simply a functor which also preserves the partial orders. An *ordering embedding* $\iota : G \to H$ is an injective ordered functor ι such that whenever $\iota(x) \leqq \iota(y)$ then $x \leqq y$. An ordered functor $\theta : G \to H$ is said to be *star injective* (resp. *star surjective*) if for each identity e in G the restriction functions $(\theta \mid L_e)$ from L_e to L_f, where $f = \theta(e)$, are injective (resp. surjective). An *ordered covering functor* is an ordered functor which is both star injective and star surjective.

An ordered groupoid G is said to be *inductive* if the set G_0 with the induced order is a meet semilattice. An *inductive functor* between inductive groupoids is an ordered functor which preserves meets when restricted to the semilattice of identities. In an inductive groupoid G the pseudoproduct is everywhere defined and $(G,\otimes)$ is an inverse semigroup. Conversely, every inverse semigroup S gives rise to an inductive groupoid $(S,.,\leqq)$ where "." is the trace product and $\leqq$ the natural partial order. These correspondences form part of an isomorphism of categories called the Ehresmann–Schein Theorem (Theorem 3.5 of [3]) which asserts that the category of inverse semigroups and prehomomorphisms (resp. homomorphisms) is isomorphic to the category of inductive groupoids and ordered functors (resp. inductive functors). Under this correspondence, idempotent pure prehomomorphisms correspond to star injective ordered functors.

Lemma 3.1. *Let G be an ordered groupoid. If $x\leqq y$ and $u\leqq v$ and the pseudoproducts $x\otimes u$ and $y\otimes v$ are defined then $x\otimes u\leqq y\otimes v$.*

Proof. Put $e=\mathbf{d}(x)\wedge\mathbf{r}(\mathbf{u})$ and $f=\mathbf{d}(y)\wedge\mathbf{r}(v)$. Then by definition

$$x\otimes u=(x\,|\,e).(e\,|\,u) \text{ and } y\otimes v=(y\,|\,f).(f\,|\,v).$$

From $x\leqq y$ we have that $\mathbf{d}(x)\leqq\mathbf{d}(y)$. Similarly, from $u\leqq v$ it follows that $\mathbf{r}(u)\leqq\mathbf{r}(v)$. Thus $e\leqq f$. But then by Lemma 1.10 [7]

$$(x\,|\,e)\leqq(y\,|\,f) \text{ and } (e\,|\,u)\leqq(f\,|\,v).$$

Thus the result holds by $(OC3)$. □

We now extend the definition of covering pair given in Section 2 to the ordered groupoid case.

Definition. Let G be an ordered groupoid and U a subset of G satisfying the following conditions:

$(GCP1)$ U is an inductive groupoid with respect to the pseudoproduct in G.

$(GCP2)$ $[U]^{\uparrow}=U$.

$(GCP3)$ $[U]=G$.

Then we say that (G,U) is a *covering pair.*

Next we generalise a result of Joubert [2]; in the semigroup case it simply says that covering pairs give rise to a class of representations.

Theorem 3.2. *Let G be an ordered groupoid and (G,U) a covering pair. Define a function $\phi:G\to K(U)$ by $\phi(g)=\{u\in U:g\leqq u\}$. Then ϕ is a well-defined, ordered covering functor such that $U=\bigcup\{\phi(g):g\in G\}$.*

Proof. In this proof, we shall represent the pseudoproduct in U by concatenation. It is straightforward to show that $\phi(x)$ is a coset of U on the basis of Lemma 3.1. ϕ is a functor: indeed, it is clear by Theorem 1.8 that if e is an identity in G then $\phi(e)$ is an identity in $K(U)$. It therefore remains for us to prove that if the product xy is defined in G then $\phi(x).\phi(y)$ is defined in $K(U)$ and $\phi(xy)=\phi(x).\phi(y)$. We prove first that $\mathbf{d}(\phi(x))=\phi(\mathbf{d}(x))$. Clearly $\mathbf{d}(\phi(x))=[\phi(x^{-1})\phi(x)]^{\uparrow}$. Let $u\in\phi(\mathbf{d}(x))$ and $v\in\phi(x)$ be arbitrary elements. Since the pseudoproduct is everywhere defined in U, we have that $(v^{-1}v)(u^{-1}u)u\leqq u$. Now, $x\leqq v$ and $x^{-1}x\leqq$ u, so that $x^{-1}\leqq v^{-1}$ and $x\leqq v(u^{-1}u)u$. Hence $x^{-1}x\leqq v^{-1}(vu^{-1}uu)\leqq u$. Thus $u\in[\phi(x)^{-1}\phi(x)]^{\uparrow}$, and so $\phi(\mathbf{d}(x))\subseteq\mathbf{d}(\phi(x))$. The reverse inclusion is immediate. We may similarly show that $\mathbf{r}(\phi(x))=\phi(\mathbf{r}(x))$. We may now complete the proof that ϕ is a functor. Suppose that the product xy is defined in G. Then $\mathbf{d}(x)=\mathbf{r}(y)$, so that $\phi(\mathbf{d}(x))=\phi(\mathbf{r}(y))$. But by the result above $\phi(\mathbf{d}(x))=\mathbf{d}(\phi(x))$ and $\phi(\mathbf{r}(y))=\mathbf{r}(\phi(y))$, so that the product $\phi(x).\phi(y)$ is defined in $K(U)$. Finally, we show that $\phi(xy)=\phi(x).\phi(y)$. By Proposition 1.10, $\phi(x).\phi(y)=j(uv\mathbf{d}(\phi(y)))$ where $u\in\phi(x)$ and $v\in\phi(y)$. Now, $u\in\phi(x)$ and $v\in\phi(y)$ imply that $uv\in\phi(xy)$ by Lemma 3.1. Thus

$$\phi(xy)=j(uv\phi(\mathbf{d}(xy)))=j(uv\phi(\mathbf{d}(y))),$$

and so $\phi(x)\phi(y)=\phi(xy)$.

It is easy to check that ϕ preserves the order and that ϕ is star injective. We shall prove that ϕ is star surjective. Let e be an identity in G, so that $\phi(e)=A$, a closed inverse subsemigroup of U. Let B be a coset in $K(U)$ such that $\mathbf{d}(B)=A$, and let $b\in B$. Then $b^{-1}b\in A$. In particular, $e\leqq b^{-1}b$. Let $g=(b\,|\,e)$. Then $\mathbf{d}(g)=e$,

$$\mathbf{d}(\phi(g))=\phi(\mathbf{d}(g))=\phi(e)=A,$$

and $b\in\phi(g)$. Thus $\phi(g)=j(bA)=B$. □

Remark. We call the functor ϕ defined above the *majorising functor* associated with the covering pair (G,U). If (S,U) is a covering pair of inverse semigroups then the function ϕ is called the *majorising prehomomorphism* associated with (S,U).

We now lead up to the proof of Theorem 3.6 by proving a number of auxiliary results.

Theorem 3.3. *Let G be an ordered groupoid and let U be an inductive groupoid. Let $\theta:G\to K(U)$ be an ordered covering functor such that $U=\bigcup\{\theta(g):g\in G\}$. Put $H=G\amalg U$ as a groupoid. Define a relation $\leqq$ on H by:*

$$x\leqq y\Leftrightarrow\begin{cases} x,y\in U \text{ and } x\leqq y\\ x,y\in G \text{ and } x\leqq y\\ x\in G,\ y\in U \text{ and } y\in\theta(x).\end{cases}$$

Then

(i) (H,U) *is a covering pair.*

(ii) *Define a map* $\theta':H\rightarrow K(U)$ *by*

$$\theta'(h)=\begin{cases}\theta(h) \text{ if } h\in G\\ [h]^{\uparrow} \text{ if } h\in U.\end{cases}$$

Then θ' *is the majorising functor associated with the pair* (H,U).

Proof. (i) By Proposition 2.2(iii) of [3] it is only necessary to check that the axioms $(G),(OC5)$(i) and (OI) hold. It is clear that H is a groupoid and easy to check that $\leqq$ is a partial order. Since θ is a functor, (G) holds. That (OI) holds follows by Proposition 1.5 and Theorem 1.8 and the fact that θ is a covering functor. $(OC3)$ is easy to check. To check that $(OC5)$(i) holds, we consider the one non-trivial case. Let $y\in U$ and e be an identity in G such that $e\leqq\mathbf{d}(y)$. By definition of the order in H, $\mathbf{d}(y)\in\theta(e)$. Thus $\theta(e)\leqq[\mathbf{d}(y)]^{\uparrow}$. It follows that the restriction $([y]^{\uparrow}|\theta(e))$ is well-defined in $K(U)$. But θ is an ordered covering functor, so that there exists a (unique) element y' in G such that $\mathbf{d}(y')=e$ and $\theta(y')=([y]^{\uparrow}|\theta(e))$. Thus $\theta(y')\leqq[y]^{\uparrow}$, and so $y\in\theta(y')$. By definition $y'\leqq y$. Hence $(OC5)$(i) holds. To prove that (H,U) is a covering pair, we check the axioms $(GCP1)$, $(GCP2)$ and $(GCP3)$. $(GCP1)$ holds: indeed let $i\in G_0$, $e,f\in U_0$ and $i\leqq e,f$. Then $e,f\in\theta(\mathrm{i})$. But $\theta(\mathrm{i})$ is an inverse subsemigroup of U and so $e\otimes f\in\theta(\mathrm{i})$. Thus $i\leqq e\otimes f$ in H. Hence the meet of any two identities in U belongs to U, and so the pseudoproduct in U is the same as the pseudoproduct in H. It is immediate from the definition of the order in H that $[U]^{\uparrow}=U$, thus $(GCP2)$ holds. $(GCP3)$ holds: indeed, let g be an arbitrary element of G and $u\in\theta(g)$. Then $g\leqq u$ in H.

(ii) The majorising functor ϕ associated with the pair (H,U) is defined by $\phi(h)=\{u\in U:h\leqq u\}$. If $h\in U$ then $\phi(h)=[h]^{\uparrow}$, and if $h\in G$ then

$$\phi(h)=\{u\in U:\theta(h)\leqq[u]^{\uparrow}\}=\theta(h).$$

Thus $\phi=\theta'$. □

Suppose that U is an inverse subsemigroup of the inverse semigroup S and that $(CP2)$ holds. We show now that we can cut S down to a subsemigroup T such that (T,U) is a covering pair.

Lemma 3.4. *Let S be an inverse semigroup and U an inverse subsemigroup of S such that* $[U]=S$. *Let*

$$T'=\{s\in S\backslash U:u\leqq s \text{ some } u\in U\}.$$

Put $T=S\backslash T'$. *Then T is an inverse semigroup containing U. Furthermore,* (T,U) *is a covering pair.*

Proof. Clearly $U \subseteq T$. Suppose that $s, t \in T$. We shall prove that $st \in T$. Suppose to the contrary that $st \in T'$. Then there exists $u \in U$ such that $u \leqq st$. Let $st = s'.t'$, where $s' \leqq s$ and $t' \leqq t$ and $s'.t'$ is a trace product. By Proposition 2.2 of [3], there exist elements $u', u'' \in S$ such that $u = u'.u''$ and $u' \leqq s'$ and $u'' \leqq t'$. Since $[U] = S$, there exists $v' \in U$ such that $t \leqq v'$. Thus $u'' \leqq t' \leqq t \leqq v'$. We claim that $u'' \in U$. Indeed, it is clear that $u^{-1}u = (u'')^{-1}u''$ and $v'u^{-1}u \leqq v'$ and so $u'' = v'u^{-1}u \in U$. Thus $t \in T'$, which contradicts $t \in T$. Hence $st \in T$. If $s \in T$ then it is immediate that $s^{-1} \in T$. Thus T is an inverse semigroup containing U. It is now straightforward to check that (T, U) is a covering pair. □

The next result will enable us to convert from ordered groupoids to inverse semigroups.

Proposition 3.5. *Let (G, U) be a covering pair, where G is an ordered groupoid, with majorising functor ϕ'. Then there exists an inductive groupoid T and an embedding $\iota: G \to T$ such that (T, U) is a covering pair. If the majorising functor associated with (T, U) is ϕ then $\phi\iota = \phi'$.*

Proof. By the generalisation of the Vagner–Preston Representation Theorem [6], there exists an ordered embedding $\iota: G \to I(G)$. Put $S = [\iota(U)]$ in $I(G)$. Then S is an inductive groupoid which contains $\iota(G)$. We now apply Lemma 3.4 to the inductive groupoid S and its inductive subgroupoid $\iota(U)$. In this way, we obtain an inductive groupoid T which is an inductive subgroupoid of S and which contains $\iota(U)$. We claim that $\iota(G) \subseteq T$: for suppose $g \in G$ is such that $\iota(u) \leqq \iota(g)$, where $u \in U$. But by [6], $\iota(u) \leqq \iota(g)$ implies that $u \leqq g$. Thus $g \in U$, since (G, U) is a covering pair. Hence $\iota(G) \subseteq T$ as required. We may easily relabel the elements of $\iota(U)$ so that $U = \iota(U)$. Finally, we calculate $\phi(\iota(g))$, where $g \in G$. By definition, $u \in \phi(\iota(g))$ if, and only if, $\iota(g) \leqq u$ in T. Thus $g \leqq u$ in G. Hence $\phi(\iota(g)) = \phi'(g)$. □

We now come to the main result of this section.

Theorem 3.6. (i) *Let (T, U) be a covering pair, where T and U are inverse semigroups, and $\kappa: S \to T$ a covered embedding. If ϕ is the majorising prehomomorphism associated with the pair (T, U) then $\phi\kappa: S \to K(U)$ is a representation.*

(ii) *Let S and U be inverse semigroups and let $\psi: S \to K(U)$ be a representation. Then there exists a covering pair (T, U), where T is also inverse, and a covered embedding $\kappa: S \to T$ such that $\phi\kappa = \psi$, where ϕ is the majorising prehomomorphism associated with the pair (T, U).*

Proof. (i) Immediate by Theorem 3.2.

(ii) We begin by thinking of S and $K(U)$ as inductive groupoids and ψ as an ordered functor. Since ψ is star injective we may apply Theorem 2.3 of [7], the Maximum Enlargement Theorem. Thus there exists an embedding $\zeta: S \to G$ into an ordered groupoid G and an ordered covering functor $\psi': G \to K(U)$ such that $\psi'\zeta = \psi$ and $\zeta(S)$ is a full, reflective subgroupoid and order ideal of G. We now apply the construction of Theorem 3.3 to the ordered functor ψ'. We obtain an ordered groupoid H such that

(H, U) is a covering pair. G is contained in H and the restriction to G of the majorising functor ψ'' associated with (H, U) is ψ'. By Proposition 3.5 there is an embedding $\iota: H \to T$ into an inductive groupoid T, where (T, U) is a covering pair with majorising prehomomorphism ϕ. Define $\kappa = \iota\zeta: S \to T$, which is an embedding. It follows from the construction that $\phi\kappa = \psi$. Finally, κ is a covered embedding by: let $u \in U$. Then $u \in \psi(s)$ some $s \in S$, since ψ is a representation. Thus $u \in \phi(\kappa(s))$, and so $\kappa(s) \leqq u$ in T. □

We now summarise the results of this paper. Consider the following three classes of morphisms:

(I) *Special covers* (θ, ϕ) *of* S *through* U, where $\theta: Q \to S$ and $\phi: Q \to U$ are surjective homomorphisms between inverse semigroups, ϕ is idempotent pure, $\ker\theta \cap \ker\phi = \Delta$ and $(\ker\phi \circ \leqq) \subseteq \ker\theta \circ \ker\phi$.

(II) *Representations* ψ *of* S *in* $K(U)$, where ψ is an idempotent pure prehomomorphism and $U = \cup\{\psi(s): s \in S\}$.

(III) *Covered embeddings of* S *in* (T, U), where $\kappa: S \to T$ is an embedding, U is a subsemigroup of T, $[U] = T$, $[U]^{\dagger} = U$ and for each $u \in U$ there exists $s \in S$ such that $\kappa(s) \leqq u$.

The correspondence between special covers and representations is the subject of Theorem 2.3. The correspondence between representations and covered embeddings is the subject of Theorem 3.6. A noteworthy special case is obtained, according to Corollary 2.2, by taking ϕ to be the homomorphism associated with τ, the maximum idempotent pure congruence on Q. This forces U to be E-disjunctive.

The classical case of E-unitary covers of inverse semigroups, discussed in the Introduction to this paper, fits into the general theory we have developed. Let Q be an E-unitary inverse semigroup and let $G = Q/\sigma$. By Proposition III.7.2 of **[12]**, we have that $\sigma = \tau$. In particular, σ is idempotent pure. The homomorphisms $\theta: Q \to S$ for which $(\sigma^{\natural}, \theta)$ is a special cover of S are easy to characterise: $(SC3)$ holds precisely when θ is idempotent separating, whereas $(SC4)$ holds automatically, since G is a group. McAlister's covering theorem tells us that every inverse semigroup S has a special cover of the above form. Thus by Theorem 3.6, every inverse semigroup S can be embedded in a factorisable inverse monoid T by a map $\iota: S \to T$, in such a way that each g in $U(T)$ lies above some element $\iota(s)$ of $\iota(S)$, as originally proved in **[9]**.

Notes added in proof. (1) In the application of the generalisation of the Vagner–Preston Representation Theorem used in the proof of Proposition 3.5, it is important to note that the ordered embedding $\iota: G \to I(G)$ preserves any pseudoproducts in G.

(2) Boris M. Schein proves explicitly that $K(U)$ is an inverse semigroup, when U is inverse, in his 1966 paper "Semigroups of strong subsets" published in volume 4 of Volzhskii Matem. Sbornik in pages 180–186.

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