LMS J. Comput. Math. 19 (1) (2016) 281–292
doi:10.1112/S1461157016000127

On the double Laplace transform of the truncated variation of a Brownian motion with drift

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Abstract

The aim of this paper is to find a formula for the double Laplace transform of the truncated variation of a Brownian motion with drift. In order to find the double Laplace transform, we also prove some identities for the Brownian motion with drift, which may be of independent interest.

1. *Introduction*

Let $X = (X_t)_{t\in[a;b]}$ be a real-valued stochastic process with càdlàg trajectories. In general, the total path variation of X, defined as

$$TV(X,[a;b]) = \sup_n \sup_{a\leqslant t_0<t_1<\ldots<t_n\leqslant b} \sum_{i=1}^n |X_{t_i} - X_{t_{i-1}}|,$$

may be (and in many most important cases is) almost surely infinite. However, in the neighborhood of every càdlàg path we may easily find a function, the total variation of which is finite.

Let $f : [a;b] \to \mathbb{R}$ be a càdlàg function and let $c > 0$. The natural question arises, what is the smallest possible value (or infimum) of the total variations of functions $g : [a;b] \to \mathbb{R}$ from the ball $\{g : \|f-g\|_\infty \leqslant \frac{1}{2}c\}$, where $\|f-g\|_\infty := \sup_{s\in[a;b]} |f(s)-g(s)|$. The bound from below reads as

$$TV(g,[a;b]) \geqslant TV^c(f,[a;b]),$$

where

$$TV^c(f,[a;b]) := \sup_n \sup_{a\leqslant t_0<t_1<\ldots<t_n\leqslant b} \sum_{i=1}^n \max\{|f(t_i)-f(t_{i-1})|-c,0\} \tag{1}$$

and follows immediately from the inequality

$$|g(t_i)-g(t_{i-1})| \geqslant \max\{|f(t_i)-f(t_{i-1})|-c,0\}$$

holding for any $t_{i-1}, t_i \in [a;b]$ and any function $g : [a;b] \to \mathbb{R}$ from the ball $\{g : \|f-g\|_\infty \leqslant \frac{1}{2}c\}$.

In fact, in [3], it was proven that we have the equality

$$\inf\{TV(g,[a;b]) : \|f-g\|_\infty \leqslant \tfrac{1}{2}c\} = TV^c(f,[a;b]). \tag{2}$$

Remark 1.1. Since we deal with càdlàg functions, a more natural setting of our problem would be the investigation of

$$\inf\{TV(g,[a;b]) : g - \text{càdlàg}, d_D(f,g) \leqslant \tfrac{1}{2}c\},$$

Received 15 September 2015; revised 28 December 2015.

2010 Mathematics Subject Classification 60E10 (primary), 60G15 (secondary).

where d_D denotes the Skorohod metric. Since the total variation does not depend on the (continuous and strictly increasing) change of the variable t and the function f^c minimizing $TV(g,[a;b])$ appears to be a càdlàg one, solutions of both problems coincide.

The bound (1) is called truncated variation. Moreover, for any $c \leqslant \sup_{s,u\in[a;b]} |f(s)-f(u)|$, there exists a unique càdlàg function $f^c : [a;b] \to \mathbb{R}$ such that $\|f-f^c\|_\infty \leqslant \frac{1}{2}c$ and, for any $s \in (a;b]$,

$$TV(f^c,[a;s]) = TV^c(f,[a;s]).$$

The function f^c is a càdlàg function with jumps possible only at the points where the function f has jumps and it may be represented in the following form:

$$f^c(s) = f^c(a) + UTV^c(f;[a;s]) - DTV^c(f;[a;s]),$$

where

$$UTV^c(f,[a;b]) := \sup_n \sup_{a\leqslant t_0<t_1<\ldots<t_n\leqslant b} \sum_{i=1}^{n} \max\{f(t_i)-f(t_{i-1})-c,0\}, \tag{3}$$

$$DTV^c(f,[a;b]) := \sup_n \sup_{a\leqslant t_0<t_1<\ldots<t_n\leqslant b} \sum_{i=1}^{n} \max\{f(t_{i-1})-f(t_i)-c,0\} \tag{4}$$

are called upward and downward truncated variations of the function f respectively (see the next section and, for more general results concerning regulated functions, see [4, Theorem 4]). We also have

$$TV^c(f,[a;b]) = UTV^c(f,[a;b]) + DTV^c(f,[a;b]). \tag{5}$$

Properties of the truncated variation and two other related quantities (upward and downward truncated variations) of trajectories of stochastic processes are up to some degree known (see [2, 5, 6]). In particular, in [7], an exact representation of the truncated variation of a (shifted) standard Brownian motion B in terms of its local times was given and in [2] there were calculated double Laplace transforms of $UTV^c(W,[0;S])$ and $DTV^c(W,[0;S])$ for $W_t = B_t + \mu t$ being a standard Brownian motion with drift μ and $[0;S]$ being a random interval, the length of which is exponentially distributed and independent from the underlying Brownian motion B.

REMARK 1.2. In [2], the functionals $UTV^c(\cdot,[a;b])$ and $DTV^c(\cdot,[a;b])$ were defined with slightly different formulas, but it is easy to see that both definitions coincide.

This also gives the full characterization of the distributions of $UTV^c(W,[0;T])$ and $DTV^c(W,[0;T])$ for deterministic time T. However, since the variables $UTV^c(W,[0;T])$ and $DTV^c(W,[0;T])$ are dependent, these results do not provide us with the full characterization of the distribution of $TV^c(W,[0;T])$ and the dependence structure between $UTV^c(W,[0;T])$ and $DTV^c(W,[0;T])$.

The aim of this paper is to find a formula for the Laplace transform of $TV^c(W,[0;S])$. In order to find this double Laplace transform, we will also prove some identities for the Brownian motion with drift, which may be of independent interest.

The paper is organized as follows. In the next section we introduce some necessary definitions, notation and results. In the last section we deal with the Laplace transform of the truncated variation of a standard Brownian motion with drift, its moments and the covariance between the upward and downward truncated variations of the Brownian motion with drift.

2. *Truncated variation, upward truncated variation and downward truncated variation of a càdlàg function, their optimality and other properties*

2.1. *Definitions and notation*

Let $-\infty < a < b < +\infty$ and let $f : [a; b] \to \mathbb{R}$ be a càdlàg function. For $c > 0$, applying the convention $\inf \emptyset = \infty$, we define two stopping times

$$T_D^c f = \inf\left\{ s \in [a; b] : \sup_{t\in[a;s]} f(t) - f(s) \geqslant c \right\},$$
$$T_U^c f = \inf\left\{ s \in [a; b] : f(s) - \inf_{t\in[a;s]} f(t) \geqslant c \right\}.$$

Assume $T_D^c f \geqslant T_U^c f$, that is, the first time f is at the distance greater than or equal to c above its running minimum, appears before the first time f is at the distance greater than or equal to c below its running maximum, or both events do not occur (both times are infinite). Note that in the case $T_D^c f < T_U^c f$, we may simply consider the function $-f$. Now we define sequences $(T_{U,k}^c)_{k=0}^{\infty}, (T_{D,k}^c)_{k=-1}^{\infty}$, in the following way: $T_{D,-1}^c = a$, $T_{U,0}^c = T_U^c f$ and, for $k = 0, 1, 2, \ldots,$

$$T_{D,k}^c = \begin{cases} \inf\left\{ s \in [T_{U,k}^c; b] : \sup_{t\in[T_{U,k}^c;s]} f(t) - f(s) \geqslant c \right\} & \text{if } T_{U,k}^c < b, \\ \infty & \text{if } T_{U,k}^c \geqslant b, \end{cases}$$
$$T_{U,k+1}^c = \begin{cases} \inf\left\{ s \in [T_{D,k}^c; b] : f(s) - \inf_{t\in[T_{D,k}^c;s]} f(t) \geqslant c \right\} & \text{if } T_{D,k}^c < b, \\ \infty & \text{if } T_{D,k}^c \geqslant b. \end{cases}$$

REMARK 2.1. Note that there exists such $K < \infty$ that $T_{U,K}^c = \infty$ or $T_{D,K}^c = \infty$. Otherwise we would obtain two infinite sequences $(s_k)_{k=1}^{\infty}, (S_k)_{k=1}^{\infty}$ such that $a \leqslant s_1 < S_1 < s_2 < S_2 < \ldots \leqslant b$ and $f(S_k) - f(s_k) \geqslant \frac{1}{2}c$. But this is a contradiction, since f is càdlàg and $(f(s_k))_{k=1}^{\infty}, (f(S_k))_{k=1}^{\infty}$ have a common limit.

Now let us define two sequences of non-decreasing functions $m_k^c : [T_{D,k-1}^c; T_{U,k}^c) \cap [a; b] \to \mathbb{R}$ and $M_k^c : [T_{U,k}^c; T_{D,k}^c) \cap [a; b] \to \mathbb{R}$ for such k that $T_{D,k-1}^c < \infty$ and $T_{U,k}^c < \infty$, respectively, with the formulas

$$m_k^c(s) = \inf_{t\in[T_{D,k-1}^c;s]} f(t), \; M_k^c(s) = \sup_{t\in[T_{U,k}^c;s]} f(t).$$

Next we define two finite sequences of real numbers (m_k^c) and (M_k^c) for such k that $T_{D,k-1}^c < \infty$ and $T_{U,k}^c < \infty$, respectively, with the formulas

$$m_k^c = m_k^c(T_{U,k}^c-) = \inf_{t\in[T_{D,k-1}^c;T_{U,k}^c)\cap[a;b]} f(t),$$
$$M_k^c = M_k^c(T_{D,k}^c-) = \sup_{t\in[T_{U,k}^c;T_{D,k}^c)\cap[a;b]} f(t).$$

Finally, let us define the function $f^c : [a; b] \to \mathbb{R}$ with the formulas

$$f^c(s) = \begin{cases} m_0^c + c/2 & \text{if } s \in [a; T_{U,0}^c); \\ M_k^c(s) - c/2 & \text{if } s \in [T_{U,k}^c; T_{D,k}^c), k = 0, 1, 2, \ldots; \\ m_{k+1}^c(s) + c/2 & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c), k = 0, 1, 2, \ldots. \end{cases}$$

REMARK 2.2. Note that due to Remark 2.1, b belongs to one of the intervals $[T_{U,k}^c; T_{D,k}^c)$ or $[T_{D,k}^c; T_{U,k+1}^c)$ for some $k = 0, 1, 2, \ldots$ and the function f^c is defined for every $s \in [a; b]$.

https://doi.org/10.1112/S1461157016000127 Published online by Cambridge University Press

The subtraction (respectively addition) of the term $c/2$ from $M_k^c(s)$ (respectively to $m_{k+1}^c(s)$) joins pieces of the graph of the function f^c so that it becomes continuous (in the case when f is continuous).

REMARK 2.3. One may think about the function f^c as the most 'lazy' function possible, which changes its value only if it is necessary for the relation $\|f - f^c\|_\infty \leqslant c/2$ to hold.

The function f^c has finite total variation since it is non-decreasing on the intervals $[T_{U,k}^c; T_{D,k}^c) \cap [a; b], k = 0, 1, 2, \dots$ and non-increasing on the intervals $[T_{D,k}^c; T_{U,k+1}^c) \cap [a; b], k = 0, 1, 2, \dots$ and, since f^c has finite total variation, we know that there exist two non-decreasing functions f_U^c and $f_D^c : [a; b] \to [0; +\infty)$ such that $f^c(t) = f^c(a) + f_U^c(t) - f_D^c(t)$ for $s \in [a; b]$. Setting $f_U^c(s) = f_D^c(s) = 0$ for $s \in [a; T_{U,0}^c)$,

$$f_U^c(s) = \begin{cases} \sum_{i=0}^{k-1} \{M_i^c - m_i^c - c\} + M_k^c(s) - m_k^c - c & \text{if } s \in [T_{U,k}^c; T_{D,k}^c); \\ \sum_{i=0}^{k} \{M_i^c - m_i^c - c\} & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c) \end{cases}$$

and

$$f_D^c(s) = \begin{cases} \sum_{i=0}^{k-1} \{M_i^c - m_{i+1}^c - c\} & \text{if } s \in [T_{U,k}^c; T_{D,k}^c); \\ \sum_{i=0}^{k-1} \{M_i^c - m_{i+1}^c - c\} + M_k^c - m_{k+1}^c(s) - c & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c), \end{cases}$$

we have

$$f^c(s) = f^c(a) + f_U^c(s) - f_D^c(s) \quad \text{for } s \in [a; b].$$

2.2. *Important result*

In the case $T_D^c f < T_U^c f$, we may apply the definitions from the previous subsection to the function $-f$ and simply define $f^c = -(-f)^c$, $f_U^c = -(-f)_U^c$ and $f_D^c = -(-f)_D^c$. In this way the mappings $f \mapsto f^c$, $f \mapsto f_U^c$ and $f \mapsto f_D^c$ are defined for any càdlàg function $f : [a; b] \to \mathbb{R}$.

We have the following result (cf. [**3**, Corollary 3.8 and Theorem 4.1]).

THEOREM 2.1. *The function f^c is optimal, that is, if $g : [a; b] \to \mathbb{R}$ is such that $\|f - g\|_\infty \leqslant c/2$ and has finite total variation, then, for every $s \in [a; b]$,*

$$TV(g, [a; s]) \geqslant TV(f^c, [a; s]).$$

It is unique in such a sense that if for every $s \in [a; b]$ the opposite inequality holds:

$$TV(g, [a; s]) \leqslant TV(f^c, [a; s])$$

and $c \leqslant \sup_{s,u \in [a;b]} |f(s) - f(u)|$, then $g = f^c$. Moreover, for any $s \in (a; b]$, the following equalities hold:

$$UTV^c(f, [a; s]) = f_U^c(s), \tag{6}$$
$$DTV^c(f, [a; s]) = f_D^c(s), \tag{7}$$
$$TV^c(f, [a; s]) = f_U^c(s) + f_D^c(s). \tag{8}$$

3. *The Laplace transform of truncated variation process of Brownian motion with drift stopped at exponential time*

Let B_t, $t \geqslant 0$, be a standard Brownian motion, $c > 0$, $\mu \in \mathbb{R}$ and $W_t = B_t + \mu t$ be a standard Brownian motion with the drift μ. For typographical reasons, instead of W_t we will sometimes write $W(t)$. In this section we will calculate the double Laplace transform of the truncated variation process of W, that is, the Laplace transform of the process $TV^c(W, s) := TV^c(W, [0; s])$, $s \geqslant 0$, stopped at (independent from W) exponentially distributed time S.

3.1. *The Laplace transform*

We begin with some auxiliary observations. First let us notice that by Theorem 2.1, on the set $\{T_D^c W \geqslant T_U^c W\}$, applying the definition of sequences $(T_{U,k}^c)_{k=0}^{\infty}$ and $(T_{D,k}^c)_{k=-1}^{\infty}$ (cf. § 2.1) to the function $f = W$, for $s \geqslant 0$ we obtain

$$TV^c(W,s) = \begin{cases} 0 & \text{if } s \in [0; T_{U,0}^c); \\ \displaystyle\sum_{i=0}^{k-1} \{M_i^c - m_i^c - c\} + \sum_{i=0}^{k-1} \{M_i^c - m_{i+1}^c - c\} & \\ \quad + M_k^c(s) - m_k^c - c & \text{if } s \in [T_{U,k}^c; T_{D,k}^c); \\ \displaystyle\sum_{i=0}^{k} \{M_i^c - m_i^c - c\} + \sum_{i=0}^{k-1} \{M_i^c - m_{i+1}^c - c\} & \\ \quad + M_k^c - m_{k+1}^c(s) - c & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c) \end{cases}$$

(although $T_{U,k}^c, T_{D,k}^c$ were defined for a function with a domain being the compact interval $[a; b]$, the extension of their definition to a function defined on a half-line is straightforward). By the continuity of Brownian paths, on the set $\{T_D^c W \geqslant T_U^c W\}$, we have

$$W(T_{U,k}^c) = m_k^c + c, W(T_{D,k}^c) = M_k^c - c$$

and hence

$$TV^c(W,s) = \begin{cases} 0 & \text{if } s \in [0; T_{U,0}^c); \\ \displaystyle\sum_{i=0}^{k-1} \{M_i^c - W(T_{U,i}^c)\} + \sum_{i=0}^{k-1} \{W(T_{D,i}^c) - m_{i+1}^c\} & \\ \quad + M_k^c(s) - W(T_{U,k}^c) & \text{if } s \in [T_{U,k}^c; T_{D,k}^c); \\ \displaystyle\sum_{i=0}^{k} \{M_i^c - W(T_{U,i}^c)\} + \sum_{i=0}^{k-1} \{W(T_{D,i}^c) - m_{i+1}^c\} & \\ \quad + W(T_{D,k}^c) - m_{k+1}^c(s) & \text{if } s \in [T_{D,k}^c; T_{U,k+1}^c). \end{cases}$$

Now let us define two sequences of stopping times $(\tilde{T}_{U,k}^c)_{k=0}^{\infty}, (\tilde{T}_{D,k}^c)_{k=0}^{\infty}$ in the following way: $\tilde{T}_{U,0}^c = T_U^c W$ and, for $k = 0, 1, 2, \ldots,$

$$\tilde{T}_{D,k}^c = \inf\left\{ s > \tilde{T}_{U,k}^c : \sup_{t \in [\tilde{T}_{U,k}^c; s]} W_t - W_s \geqslant c \right\},$$

$$\tilde{T}_{U,k+1}^c = \inf\left\{ s > \tilde{T}_{D,k}^c : W_s - \inf_{t \in [\tilde{T}_{D,k}^c; s]} W_t \geqslant c \right\}.$$

Notice that on the set $\{T_D^c W \geqslant T_U^c W\}$, the sequences $(T_{U,k}^c)_{k=0}^{\infty}$, $(T_{D,k}^c)_{k=0}^{\infty}$ and $(\tilde{T}_{U,k}^c)_{k=0}^{\infty}$, $(\tilde{T}_{D,k}^c)_{k=0}^{\infty}$ coincide.

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Now, for any $0 \leqslant a \leqslant b < +\infty$, we define two auxiliary functions

$$U[a;b] = \sup_{a \leqslant t \leqslant b} W_t - W_a,$$
$$D[a;b] = W_a - \inf_{a \leqslant t \leqslant b} W_t$$

and, for $s \geqslant 0$, we define two quantities

$$U^c(W,s) = \sum_{i=0}^{\infty} U[\tilde{T}^c_{U,i} \wedge s; \tilde{T}^c_{D,i} \wedge s] + \sum_{i=0}^{\infty} D[\tilde{T}^c_{D,i} \wedge s; \tilde{T}^c_{U,i+1} \wedge s],$$
$$D^c(W,s) = \sum_{i=0}^{\infty} D[\tilde{T}^c_{D,i} \wedge s; \tilde{T}^c_{U,i+1} \wedge s] + \sum_{i=0}^{\infty} U[\tilde{T}^c_{U,i+1} \wedge s; \tilde{T}^c_{D,i+2} \wedge s].$$

Notice that on the set $\{T^c_D W \geqslant T^c_U W\}$, we have

$$TV^c(W,s) = U^c(W,s).$$

Similarly, if $T^c_D W < T^c_U(W)$, then we change in the definitions of the sequences $(\tilde{T}^c_{U,k})_{k=0}^{\infty}$ and $(\tilde{T}^c_{D,k})_{k=0}^{\infty}$ W for $-W$ and obtain

$$TV^c(W,s) = U^c(-W,s).$$

What will be important to us is the fact that the conditional distribution of $U^c(W,s)$ given $T^c_D W \geqslant T^c_U(W)$, $\mathcal{L}(U^c(W,s)|T^c_D W \geqslant T^c_U(W))$, is the same as the distribution of $U^c(W,s)$. This follows from the strong Markov property and the independence of the increments of a Brownian motion.

Now let S be an exponential random variable, independent from W, with density $\nu e^{-\nu x}$. By $M_{TV^c(W,S)}(\lambda)$, we denote the moment generating function of $TV^c(W,S)$, that is,

$$M_{TV^c(W,S)}(\lambda) := \mathbb{E}[\exp(\lambda \cdot TV^c(W,S))].$$

We have the following equation:

$$\begin{aligned} M_{TV^c(W,S)}(\lambda) = {} & \mathbb{E}[\exp(\lambda \cdot U^c(W,S))|S \geqslant T^c_{U,0}, T^c_D W \geqslant T^c_U W] \times P(S \geqslant T^c_{U,0}, T^c_D W \geqslant T^c_U W) \\ & + \mathbb{E}[\exp(\lambda \cdot U^c(-W,S))|S \geqslant T^c_{U,0}, T^c_D W < T^c_U W] \\ & \times P(S \geqslant T^c_{U,0}, T^c_D W < T^c_U W) \\ & + \mathbb{P}(\min\{T^c_U W, T^c_D W\} > S). \end{aligned} \tag{9}$$

By the lack of memory of the exponential distribution, the strong Markov property and the independence of the increments of a Brownian motion, we have

$$\begin{aligned} & \mathbb{E}[\exp(\lambda \cdot U^c(W,S))|S \geqslant \tilde{T}^c_{U,0}, T^c_D W \geqslant T^c_U W] \\ & = \mathbb{E}\exp(\lambda \cdot U^c(W, S + \tilde{T}^c_{U,0})) \\ & = \mathbb{E}[\exp(\lambda \cdot U^c(W, S + \tilde{T}^c_{U,0})); S < \tilde{T}^c_{D,0} - \tilde{T}^c_{U,0}] \\ & \quad + \mathbb{E}[\exp(\lambda \cdot U^c(W, S + \tilde{T}^c_{U,0})); S \geqslant \tilde{T}^c_{D,0} - \tilde{T}^c_{U,0}] \\ & = \mathbb{E}[\exp(\lambda \cdot U[\tilde{T}^c_{U,0}; S + \tilde{T}^c_{U,0}]); S < \tilde{T}^c_{D,0} - \tilde{T}^c_{U,0}] \\ & \quad + \mathbb{E}[\exp\{\lambda \cdot U[\tilde{T}^c_{U,0}; \tilde{T}^c_{D,0}] + \lambda \cdot D^c(W, S + \tilde{T}^c_{U,0})\}; S \geqslant \tilde{T}^c_{D,0} - \tilde{T}^c_{U,0}] \\ & = \mathbb{E}[\exp(\lambda \cdot U[\tilde{T}^c_{U,0}; S + \tilde{T}^c_{U,0}]); S < \tilde{T}^c_{D,0} - \tilde{T}^c_{U,0}] \\ & \quad + \mathbb{E}[\exp(\lambda \cdot U[T^c_{U,0}; T^c_{D,0}]); S \geqslant \tilde{T}^c_{D,0} - \tilde{T}^c_{U,0}]\mathbb{E}\exp(\lambda \cdot D^c(W, S + \tilde{T}^c_{D,0})). \end{aligned} \tag{10}$$

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Notice that in all the calculations above, except the first line, the starting value of $\tilde{T}^c_{U,0} \geqslant 0$ is irrelevant: we need only to know the recursive definitions of $\tilde{T}^c_{D,0}, \tilde{T}^c_{U,1}, \ldots$; thus, we may set $\tilde{T}^c_{U,0} = 0$ and we have

$$\mathbb{E}[\exp(\lambda \cdot U^c(W, S + \tilde{T}^c_{U,0})); S < \tilde{T}^c_{D,0} - \tilde{T}^c_{U,0}] = \mathbb{E}\biggl[\exp\biggl(\lambda \cdot \sup_{0 \leqslant t \leqslant S} W_t\biggr); S < T^c_D W\biggr] \tag{11}$$

and

$$\mathbb{E}[\exp(\lambda \cdot U[T^c_{U,0}; T^c_{D,0}]); S \geqslant T^c_{D,0} - T^c_{U,0}] = \mathbb{E}\biggl[\exp\biggl(\lambda \cdot \sup_{0 \leqslant t \leqslant T^c_D W} W_t\biggr); S \geqslant T^c_D W\biggr]. \tag{12}$$

Similarly,

$$\begin{aligned}
&\mathbb{E}\exp(\lambda \cdot D^c(W, S + \tilde{T}^c_{D,0})) \\
&\quad = \mathbb{E}[\exp(\lambda \cdot D^c(W, S + \tilde{T}^c_{D,0})); S < \tilde{T}^c_{U,1} - \tilde{T}^c_{D,0}] \\
&\qquad + \mathbb{E}\exp(\lambda \cdot D[\tilde{T}^c_{D,0}; \tilde{T}^c_{U,1}]; S \geqslant \tilde{T}^c_{U,1} - \tilde{T}^c_{D,0})\mathbb{E}[\exp(\lambda \cdot U^c(W, S + \tilde{T}^c_{U,1}))],
\end{aligned} \tag{13}$$

from which we get

$$\mathbb{E}[\exp(\lambda \cdot D^c(W, S + \tilde{T}^c_{D,0})); S < \tilde{T}^c_{U,1} - \tilde{T}^c_{D,0}] = \mathbb{E}\biggl[\exp\biggl(-\lambda \cdot \inf_{0 \leqslant t \leqslant S} W_t\biggr); S < T^c_U W\biggr] \tag{14}$$

and

$$\mathbb{E}\biggl[\exp(\lambda \cdot D[T^c_{D,0}; T^c_{U,1}]); S \geqslant \tilde{T}^c_{U,1} - \tilde{T}^c_{D,0}\biggr] = \mathbb{E}\biggl[\exp\biggl(-\lambda \cdot \inf_{0 \leqslant t \leqslant T^c_U W} W_t\biggr); S \geqslant T^c_U W\biggr]. \tag{15}$$

Now, substituting in (10) the expression (13) for $\mathbb{E}\exp(\lambda \cdot D^c(W, S + T^c_{D,0}))$, and using (11)–(12) and (14)–(15), we get

$$\begin{aligned}
&\mathbb{E}[\exp(\lambda \cdot U^c(W, S)) | S \geqslant T^c_{U,0}, T^c_D W \geqslant T^c_U W] \\
&= \frac{\mathbb{E}[\exp(\lambda \cdot \sup_{0 \leqslant t \leqslant S} W_t); S < T^c_D W]}{1 - \mathbb{E}[\exp(\lambda \cdot \sup_{0 \leqslant t \leqslant T^c_D W} W_t); S \geqslant T^c_D W]\mathbb{E}[\exp(-\lambda \cdot \inf_{0 \leqslant t \leqslant T^c_U W} W_t); S \geqslant T^c_U W]} \\
&\quad + \frac{\mathbb{E}[\exp(\lambda \cdot \sup_{0 \leqslant t \leqslant T^c_D W} W_t); S \geqslant T^c_D W]\mathbb{E}[\exp(-\lambda \cdot \inf_{0 \leqslant t \leqslant S} W_t); S < T^c_U W]}{1 - \mathbb{E}[\exp(\lambda \cdot \sup_{0 \leqslant t \leqslant T^c_D W} W_t); S \geqslant T^c_D W]\mathbb{E}[\exp(-\lambda \cdot \inf_{0 \leqslant t \leqslant T^c_U W} W_t); S \geqslant T^c_U W]}.
\end{aligned} \tag{16}$$

Using results of [8], we will be able to calculate all quantities appearing in (16).

To calculate $\mathbb{E}[\exp(\lambda \cdot \sup_{0 \leqslant t \leqslant S} W_t); S < T^c_D W]$, we will use formulas appearing in [8, p. 236]. Denote $\tau(x) = \inf\{t \geqslant 0 : \sup_{0 \leqslant s \leqslant t} W_s = x\}$. We have the equality (note that in the notation of [8], S is denoted by ξ with parameter $\beta = \nu$, $T^c_D W$ is denoted by T and c is denoted by a)

$$\begin{aligned}
\mathbb{P}\biggl(\sup_{0 \leqslant t \leqslant S} W_t > x, S < T^c_D W\biggr) &= \mathbb{P}(\tau(x) < S < T^c_D W) = \mathbb{P}(\tau(x) < S \leqslant T^c_D W) \\
&= \mathbb{P}(\tau(x) \leqslant T^c_D W, \tau(x) < S) - \mathbb{P}(\tau(x) \leqslant T^c_D W < S) \\
&= \exp(-\theta_\mu(\nu)x)[1 - \mathbb{E}\exp(-\nu T^c_D W)] \\
&= \exp(-\theta_\mu(\nu)x)[1 - e^{-\mu c} V_\mu(\nu)/\theta_\mu(\nu)],
\end{aligned}$$

where we define

$$\theta_\mu(\nu) = \sqrt{\mu^2 + 2\nu}\coth(c\sqrt{\mu^2 + 2\nu}) - \mu$$

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and

$$V_\mu(\nu) = \frac{\sqrt{\mu^2+2\nu}}{\sinh(c\sqrt{\mu^2+2\nu})}\ .$$

Thus, for λ such that $\Re(\lambda) < \theta_\mu(\nu)$,

$$\mathbb{E}\left[\exp\left(\lambda \cdot \sup_{0\leqslant t\leqslant S} W_t\right); S < T_D^c W\right] = \frac{\theta_\mu(\nu) - e^{-\mu c}V_\mu(\nu)}{\theta_\mu(\nu)-\lambda}.$$

Further, by the definition of $T_D^c W$, we have $\sup_{0\leqslant t\leqslant T_D^c W} W_t = W_{T_D^c W} + c$. By this and by the independence of S from $T_D^c W$, we calculate

$$\begin{aligned}\mathbb{E}\left[\exp\left(\lambda \cdot \sup_{0\leqslant t\leqslant T_D^c W} W_t\right); S \geqslant T_D^c W\right] &= \mathbb{E}\left[\exp\left(\lambda \cdot \sup_{0\leqslant t\leqslant T_D^c W} W_t\right)\exp(-\nu T_D^c W)\right]\\ &= \mathbb{E}[\exp(\lambda\cdot(W_{T_D^c W}+c) - \nu T_D^c W)]\\ &= e^{\lambda c}\mathbb{E}[\exp(\lambda\cdot W_{T_D^c W} - \nu T_D^c W)].\end{aligned}$$

Now, utilizing the main result of [8], that is, equation (1.1), we have

$$e^{\lambda c}\mathbb{E}[\exp(\lambda\cdot W_{T_D^c W} - \nu T_D^c W)] = \frac{e^{-\mu c}V_\mu(\nu)}{\theta_\mu(\nu)-\lambda}.$$

Similarly, using symmetry, for λ such that $\Re(\lambda) < \theta_{-\mu}(\nu)$,

$$\mathbb{E}\left[\exp\left(-\lambda \cdot \inf_{0\leqslant t\leqslant S} W_t\right); S < T_U^c W\right] = \frac{\theta_{-\mu}(\nu) - e^{\mu c}V_\mu(\nu)}{\theta_{-\mu}(\nu)-\lambda}$$

and

$$\mathbb{E}\left[\exp\left(-\lambda \cdot \inf_{0\leqslant t\leqslant T_U^c W} W_t\right); S \geqslant T_U^c W\right] = \frac{e^{\mu c}V_\mu(\nu)}{\theta_{-\mu}(\nu)-\lambda}.$$

Substituting the above formulas into (16) and simplifying, for λ such that $\Re(\lambda) < \min\{\theta_\mu(\nu), \theta_{-\mu}(\nu)\}$, we obtain

$$\begin{aligned}&\mathbb{E}[\exp(\lambda\cdot U^c(W,S))|S\geqslant T_{U,0}^c, T_D^c W \geqslant T_U^c W]\\ &\quad= \frac{(\theta_\mu(\nu)-e^{-\mu c}V_\mu(\nu))/(\theta_\mu(\nu)-\lambda) + (e^{-\mu c}V_\mu(\nu)/(\theta_\mu(\nu)-\lambda))((\theta_{-\mu}(\nu)-e^{\mu c}V_\mu(\nu))/(\theta_{-\mu}(\nu)-\lambda))}{1-((\theta_\mu(\nu)-e^{-\mu c}V_\mu(\nu))/(\theta_\mu(\nu)-\lambda))((\theta_{-\mu}(\nu)-e^{\mu c}V_\mu(\nu))/(\theta_{-\mu}(\nu)-\lambda))}\\ &\quad= 1+\lambda\frac{\theta_{-\mu}(\nu)+e^{-\mu c}V_\mu(\nu)-\lambda}{\lambda^2+2\nu+2\lambda\mu-2\lambda\theta_{-\mu}(\nu)}.\end{aligned} \tag{17}$$

To obtain the formula for $\mathbb{E}[\exp(\lambda\cdot U^c(-W,S))|S\geqslant T_{U,0}^c, T_D^c W < T_U^c W]$, we need only change μ into $-\mu$ in the formula for $\mathbb{E}[\exp(\lambda\cdot U^c(W,S))|S\geqslant T_{U,0}^c, T_D^c W \geqslant T_U^c W]$.

To calculate the probabilities appearing in the expression (9) for $M_{TV^c(W,S)}(\lambda)$, that is,

$$\mathbb{P}(S\geqslant T_{U,0}^c, T_D^c W\geqslant T_U^c W) = \mathbb{P}(S\geqslant T_U^c W, T_D^c W > T_U^c W)$$

and

$$\mathbb{P}(S\geqslant T_{U,0}^c, T_D^c W < T_U^c W) = \mathbb{P}(S\geqslant T_D^c W, T_D^c W < T_U^c W),$$

we will use results of [2]. Since S is independent from $(T_D^c W, T_U^c W)$, we have

$$\mathbb{P}(S\geqslant T_U^c W, T_D^c W > T_U^c W) = \mathbb{E}e^{-\nu T_U^c W} I_{\{T_D^c W > T_U^c W\}},$$

$$\mathbb{P}(S\geqslant T_D^c W, T_D^c W < T_U^c W) = \mathbb{E}e^{-\nu T_D^c W} I_{\{T_U^c W > T_D^c W\}}.$$

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Using the formula just below formula (19) in [2], with $y=0$, we get

$$\mathbb{E}e^{-\nu T_D^c W} I_{\{T_U^c W > T_D^c W\}} = (1 - L_0^{-W}(\nu; c))\mathbb{E}e^{-\nu T_D^c W},$$

where (cf. [2, last but one formula on p. 389]) we have

$$L_0^{-W}(\nu; c) = \frac{\sqrt{\mu^2+2\nu}}{2\nu}\left\{\frac{e^{\mu c}\theta_\mu(\nu)}{\sinh(c\sqrt{\mu^2+2\nu})} - \frac{\sqrt{\mu^2+2\nu}}{\sinh(c\sqrt{\mu^2+2\nu})^2}\right\}$$
$$= \frac{V_\mu(\nu)}{2\nu}(e^{\mu c}\theta_\mu(\nu) - V_\mu(\nu)).$$

Thus,

$$\mathbb{P}(S \geqslant T_D^c W, T_D^c W < T_U^c W) = \left(1 - \frac{V_\mu(\nu)}{2\nu}(e^{\mu c}\theta_\mu(\nu) - V_\mu(\nu))\right)\frac{e^{-\mu c}V_\mu(\nu)}{\theta_\mu(\nu)}$$

and similarly

$$\mathbb{P}(S \geqslant T_U^c W, T_D^c W > T_U^c W) = \left(1 - \frac{V_\mu(\nu)}{2\nu}(e^{-\mu c}\theta_{-\mu}(\nu) - V_\mu(\nu))\right)\frac{e^{\mu c}V_\mu(\nu)}{\theta_{-\mu}(\nu)}.$$

Now, by (9), (17) and calculations above, we have

$$\begin{aligned}
&M_{TV^c(W,S)}(\lambda)\\
&= \left(1 + \lambda\frac{\theta_{-\mu}(\nu) + e^{-\mu c}V_\mu(\nu) - \lambda}{\lambda^2 + 2\nu + 2\lambda\mu - 2\lambda\theta_{-\mu}(\nu)}\right)P(S \geqslant T_U^c W, T_D^c W > T_U^c W)\\
&\quad + \left(1 + \lambda\frac{\theta_\mu(\nu) + e^{\mu c}V_\mu(\nu) - \lambda}{\lambda^2 + 2\nu - 2\lambda\mu - 2\lambda\theta_\mu(\nu)}\right)P(S \geqslant T_D^c W, T_D^c W < T_U^c W)\\
&\quad + 1 - \mathbb{P}(S \geqslant T_U^c W, T_D^c W > T_U^c W) - P(S \geqslant T_D^c W, T_D^c W < T_U^c W)\\
&= 1 + \lambda\frac{\theta_{-\mu}(\nu) + e^{-\mu c}V_\mu(\nu) - \lambda}{\lambda^2 + 2\nu + 2\lambda\mu - 2\lambda\theta_{-\mu}(\nu)}\left(e^{\mu c} - \frac{V_\mu(\nu)}{2\nu}\theta_{-\mu}(\nu) + e^{\mu c}\frac{V_\mu(\nu)^2}{2\nu}\right)\frac{V_\mu(\nu)}{\theta_{-\mu}(\nu)}\\
&\quad + \lambda\frac{\theta_\mu(\nu) + e^{\mu c}V_\mu(\nu) - \lambda}{\lambda^2 + 2\nu - 2\lambda\mu - 2\lambda\theta_\mu(\nu)}\left(e^{-\mu c} - \frac{V_\mu(\nu)}{2\nu}\theta_\mu(\nu) + e^{-\mu c}\frac{V_\mu(\nu)^2}{2\nu}\right)\frac{V_\mu(\nu)}{\theta_\mu(\nu)}.
\end{aligned}$$

Thus, we have obtained the following result.

THEOREM 3.1. *Let W be a standard Wiener process with drift μ and S be an exponential random variable with density $\nu e^{-\nu x}I_{\{x\geqslant 0\}}$, independent from W. For any complex λ such that $\Re(\lambda) < \min\{\theta_\mu(\nu), \theta_{-\mu}(\nu)\}$, one has*

$$\begin{aligned}
&\mathbb{E}[\exp(\lambda \cdot TV^c(W,S))]\\
&= 1 + \lambda\frac{\theta_{-\mu}(\nu) + e^{-\mu c}V_\mu(\nu) - \lambda}{\lambda^2 + 2\nu + 2\lambda\mu - 2\lambda\theta_{-\mu}(\nu)}\left(e^{\mu c} - \frac{V_\mu(\nu)}{2\nu}\theta_{-\mu}(\nu) + e^{\mu c}\frac{V_\mu(\nu)^2}{2\nu}\right)\frac{V_\mu(\nu)}{\theta_{-\mu}(\nu)}\\
&\quad + \lambda\frac{\theta_\mu(\nu) + e^{\mu c}V_\mu(\nu) - \lambda}{\lambda^2 + 2\nu - 2\lambda\mu - 2\lambda\theta_\mu(\nu)}\left(e^{-\mu c} - \frac{V_\mu(\nu)}{2\nu}\theta_\mu(\nu) + e^{-\mu c}\frac{V_\mu(\nu)^2}{2\nu}\right)\frac{V_\mu(\nu)}{\theta_\mu(\nu)}.
\end{aligned} \tag{18}$$

3.2. *Examples of applications of formula* (18)

3.2.1. *The first two moments of the truncated variation process of the Brownian motion with drift stopped at exponential time.* Differentiating formula (18), we obtain

$$\mathbb{E}TV^c(W,S) = \left[\frac{\partial}{\partial\lambda}M_{TV^c(W,S)}(\lambda)\right]_{\lambda=0} = \frac{V_\mu(\nu)}{\nu}\cosh(c\mu), \tag{19}$$

https://doi.org/10.1112/S1461157016000127 Published online by Cambridge University Press

which agrees with the relation

$$TV^{\mu}(W,S) = UTV^{c}(W,S) + DTV^{c}(W,S) \tag{20}$$

and formulas already obtained in [2]

$$\mathbb{E}UTV^{c}(W,S) = \frac{e^{\mu c}V_{\mu}(\nu)}{2\nu}, \quad \mathbb{E}DTV^{c}(W,S) = \frac{e^{-\mu c}V_{\mu}(\nu)}{2\nu}. \tag{21}$$

Similarly, we calculate

$$\begin{aligned}\mathbb{E}TV^{c}(W,S)^2 &= \left[\frac{\partial^2}{\partial\lambda^2}M_{TV^{c}(W,S)}(\lambda)\right]_{\lambda=0} \\ &= \frac{V_{\mu}(\nu)}{\nu^2}(V_{\mu}(\nu) + \cosh(\mu c)\theta_{\mu}(\nu) + e^{\mu c}\mu). \end{aligned}\tag{22}$$

REMARK 3.1. Inverting formulas (19) and (22), similarly as was done with formula (21) in [2, § 4.1], we may calculate the first and the second moments of $TV^{c}(W,T)$, where $T>0$ is deterministic. To invert (22), one may use formulas from [1, p. 642].

3.2.2. *Covariance of the upward and downward truncated variation processes of the Brownian motion with drift stopped at exponential time.* Using (20), (22) and (21) as well as results of [2, § 4.3], we are able to calculate the covariance of $UTV^{c}(W,S)$ and $DTV^{c}(W,S)$. Indeed, we have

$$\begin{aligned}\mathbb{E}(UTV^{c}(W,S)\cdot DTV^{c}(W,S)) &= \frac{1}{2}(\mathbb{E}TV^{c}(W,S)^2 - \mathbb{E}UTV^{c}(W,S)^2 - \mathbb{E}DTV^{c}(W,S)^2) \\ &= \frac{V_{\mu}(\nu)^2}{2\nu^2}, \end{aligned}\tag{23}$$

where we have used (22), the following formula (cf. [2, § 4.3]):

$$\begin{aligned}\mathbb{E}UTV^{c}(W,S)^2 &= \int_0^{\infty}\mathbb{E}UTV^{c}(W,t)^2P(S\in\, dt) \\ &= \nu\int_0^{\infty}e^{-\nu t}\mathbb{E}UTV^{c}(W,t)^2\,dt \\ &= \frac{e^{\mu c}V_{\mu}(\nu)(\mu^2+2\nu-\nu(1-\cosh(2c\sqrt{\mu^2+2\nu})))}{2\nu^2\theta_{\mu}(\nu)\sinh^2(c\sqrt{\mu^2+2\nu})} \\ &= \frac{e^{\mu c}V_{\mu}(\nu)\theta_{-\mu}(\nu)}{2\nu^2} \end{aligned}\tag{24}$$

and the symmetric formula

$$\mathbb{E}DTV^{c}(W,S)^2 = \frac{e^{-\mu c}V_{\mu}(\nu)\theta_{\mu}(\nu)}{2\nu^2}.$$

Now we have

$$\begin{aligned}&\mathrm{Cov}(UTV^{c}(W,S), DTV^{c}(W,S)) \\ &\quad= \mathbb{E}(UTV^{c}(W,S)\cdot DTV^{c}(W,S)) - \mathbb{E}UTV^{c}(W,S)\cdot\mathbb{E}DTV^{c}(W,S) \\ &\quad= \frac{V_{\mu}(\nu)^2}{2\nu^2} - \frac{e^{\mu c}V_{\mu}(\nu)}{2\nu}\frac{e^{-\mu c}V_{\mu}(\nu)}{2\nu}\end{aligned}$$

$$= \frac{V_\mu(\nu)^2}{4\nu^2}$$
$$= \frac{\mu^2 + 2\nu}{4\nu^2(\sinh(c\sqrt{\mu^2+2\nu}))^2} > 0.$$

Thus, we can observe that the correlation between $UTV^c(W,S)$ and $DTV^c(W,S)$ is positive. This is due to the fact that the magnitudes of $UTV^c(W,S)$ and $DTV^c(W,S)$ are highly dependent on the value of S.

3.2.3. *Covariance of UTV and DTV of the Brownian motion with drift.* Performing similar calculations to those in [**2**, § 4.1], we may simply obtain formulas for the covariance between $UTV^c(W,T)$ and $DTV^c(W,T)$, where T is deterministic. Indeed, denoting by $\mathcal{L}_\nu^{-1}(g)$ the inverse of the Laplace transform of the function $g(\nu) = \int_0^\infty e^{-\nu t} f(t)\,dt$, that is, the function $f(t)$, we get

$$\mathcal{L}_\nu^{-1}(\nu^{-3}) = t^2/2,$$
$$\mathcal{L}_\nu^{-1}\left(\frac{2\nu+\mu^2}{\sinh^2(c\sqrt{2\nu+\mu^2})}\right) = \mathcal{L}_\nu^{-1}\left(\frac{2(\nu+\mu^2/2)}{\sinh^2(c\sqrt{2(\nu+\mu^2/2)})}\right)$$
$$= e^{-\mu^2 t/2}\mathcal{L}_\nu^{-1}\left(\frac{2\nu}{\sinh^2(c\sqrt{2\nu})}\right)$$

and, by the second formula in [**1**, p. 642],

$$\mathcal{L}_\nu^{-1}\left(\frac{2\nu}{\sinh^2(c\sqrt{2\nu})}\right) = 4\sum_{k=0}^{\infty} \frac{\Gamma(2+k)e^{-(2c+2kc)^2/(4t)}}{\sqrt{2\pi}t^2\Gamma(2)k!} D_3\left(\frac{2c+2kc}{\sqrt{t}}\right)$$
$$= \frac{8c}{\sqrt{2\pi}}\sum_{k=0}^{\infty}(k+1)^2\frac{4(k+1)^2c^2-3t}{t^{7/2}}e^{-2(k+1)^2c^2/t},$$

where D_3 denotes the parabolic cylinder function of order 3.

Now, by (23) and the Borel convolution theorem, we obtain

$$\begin{aligned}
&\mathbb{E}(UTV^c(W,T)\cdot DTV^c(W,T))\\
&\quad= \mathcal{L}_\nu^{-1}\left(\frac{2\nu+\mu^2}{2\nu^3\sinh^2(c\sqrt{2\nu+\mu^2})}\right)\\
&\quad= \frac{1}{2}\int_0^T \frac{(T-t)^2}{2}\mathcal{L}_\nu^{-1}\left(\frac{2\nu+\mu^2}{\sinh^2(c\sqrt{2\nu+\mu^2})}\right)dt\\
&\quad= \frac{2c}{\sqrt{2\pi}}\sum_{k=0}^{\infty}(k+1)^2\int_0^T (T-t)^2\frac{4(k+1)^2c^2-3t}{t^{7/2}}e^{-\mu^2 t/2-2(k+1)^2c^2/t}\,dt. \qquad (25)
\end{aligned}$$

Finally, by (25) and [**2**, formula (28)] (notice that in [**2**, formula (28)] and in [**2**, formula (27)] the term $\mu^2 t$ shall be changed into $\mu^2 t/2$),

$$\begin{aligned}
&\mathrm{Cov}(UTV^c(W,T), DTV^c(W,T))\\
&\quad= \frac{2c}{\sqrt{2\pi}}\sum_{k=0}^{\infty}(k+1)^2\int_0^T (T-t)^2\frac{4(k+1)^2c^2-3t}{t^{7/2}}e^{-\mu^2 t/2-2(k+1)^2c^2/t}\,dt\\
&\qquad- \frac{1}{2\pi}\left(\sum_{k=0}^{\infty}\int_0^T (T-t)\frac{(2k+1)^2c^2-t}{t^{5/2}}e^{-\mu^2 t/2-(2k+1)^2c^2/(2t)}\,dt\right)^2. \qquad (26)
\end{aligned}$$

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Numerical experiments show that formula (26) gives negative numbers, but the strict proof of this fact is not known to the author.

REMARK 3.2. Inverting formula (24), using the second formula in [**1**, p. 642] and the just calculated covariance, it is possible to obtain a formula for $\operatorname{Var} UTV^c(W,T)$ and hence for $\operatorname{Cor}(UTV^c(W,T), DTV^c(W,T))$. However, numerical experiments show that the obtained formulas are rather unstable for small values of c. On the other hand, results of [**5**] give the exact value of $\operatorname{Cor}(UTV^c(W,T), DTV^c(W,T))$ as $c \to 0+$, namely,

$$\lim_{c\to 0+} \operatorname{Cor}(UTV^c(W,T), DTV^c(W,T)) = -\tfrac{1}{2}.$$

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