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DERIVATIVES OF KERNELS ASSOCIATED TO COMPLEX SUBELLIPTIC OPERATORS

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We prove large time Gaussian bounds for the derivatives of the semigroup kernel associated with complex, second-order, subelliptic operators on Lie groups of polynomial growth.

1. INTRODUCTION

It is well established that the kernel K of the semigroup generated by a right invariant sublaplacian on a connected Lie group with polynomial growth satisfies global Gaussian upper bounds. (See [**15**, Theorem VIII.2.9], or [**12**, Theorem IV.4.16].) Moreover, multiple subelliptic derivatives of the kernel satisfy small time Gaussian upper bounds with each derivative contributing an extra $t^{-1/2}$-singularity ([**7**]). Saloff-Coste [**13**] also proved similar bounds for single subelliptic derivatives of K and large t. Alexopoulos [**2**], however, gave an example of a solvable Lie group of polynomial growth for which certain second-order subelliptic derivatives of the kernel have a $t^{-1/2}$, and not a t^{-1}, asymptotic behaviour. Nevertheless the higher order derivatives of the kernel do have global Gaussian bounds with an additional $t^{-1/2}$-singularity for each derivative if the group is nilpotent ([**15**, **10**]). Moreover, for compact groups spectral arguments show that one has an exponential decrease for large t. The situation was clarified by the proof ([**11**]) that a t^{-1}-asymptotic behaviour is valid for the second-order subelliptic derivatives of the kernel if, and only if, the group is the (local) direct product of a compact group and a nilpotent group, and then the higher-order derivatives have a similar canonical behaviour. Despite this natural limitation Alexopoulos [**1**, Corollary of Theorem 7.7], showed that a general first-order derivative and some second-order derivatives of the kernel do satisfy the canonical large t Gaussian bounds. The situation then developed with a recent paper

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of [**3**] which proved canonical large time Gaussian bounds for multiple derivatives in the directions of the nilradical of G. Dungey's results were derived by combination of the Gaussian bounds on the kernel and L_2-bounds on its multiple derivatives. Since Gaussian bounds are now known for the semigroup kernels associated with complex second-order subelliptic operators ([**9**]) Dungey's results extend to this case. Dungey also established a $t^{-1/2}$-decay for L_2-bounds for any right invariant derivative of the semigroup ([**4**]) using an elaborate transference argument. In this paper we give a simpler proof of Dungey's results for multiple derivatives in the directions of the nilradical, based on interpolation arguments, almost as a corollary prove a $t^{-1/2}$-decay for any right invariant derivative of the kernel, extend Alexopoulos' result on general derivatives to the complex setting and in addition consider higher order derivatives.

Let G be a connected Lie group with polynomial growth and Lie algebra $\mathfrak{g}$. In order to describe the main theorem we need some decomposition theory, which can be found in [**1**, Sections 2 and 3]. Let $\mathfrak{q}$, $\mathfrak{n}$ and $\mathfrak{m}$ be the radical, nilradical and a Levi subalgebra for $\mathfrak{g}$. For all $a \in \mathfrak{g}$ let $S(a)$ and $K(a)$ denote the semisimple and nilpotent part in the Jordan decomposition of $\mathrm{ad}a$. Then there exists a subspace $\mathfrak{v}$ of $\mathfrak{q}$ such that $\mathfrak{q} = \mathfrak{v} \oplus \mathfrak{n}$, $[\mathfrak{m}, \mathfrak{v}] = \{0\}$ and $S(\mathfrak{v})\mathfrak{v} = \{0\}$. Then the nilshadow of $\mathfrak{q}$ is the nilpotent Lie algebra $\mathfrak{q}_N = \big(\mathfrak{q}, [\,\cdot\,,\,\cdot\,]_N\big)$ where

$$[a,b]_N = [a,b] - S(a_{\mathfrak{v}})b + S(b_{\mathfrak{v}})a$$

with $a_{\mathfrak{v}}, b_{\mathfrak{v}}$ the $\mathfrak{v}$-components of $a, b \in \mathfrak{q}$. If $\{\mathfrak{q}_{N;k}\}$ denotes the lower central series of $\mathfrak{q}_N$, that is, $\mathfrak{q}_{N;1} = \mathfrak{q}_N$ and $\mathfrak{q}_{N;k+1} = [\mathfrak{q}_N, \mathfrak{q}_{N;k}]_N$ for all $k \in \mathbf{N}$, then there exist vector subspaces $\mathfrak{k}, \mathfrak{h}_1, \ldots, \mathfrak{h}_r$ of $\mathfrak{q}$ such that $\mathfrak{h}_1 = \mathfrak{v} \oplus \mathfrak{k}$, $\mathfrak{n} = \mathfrak{k} \oplus \mathfrak{q}_{N;2}$ and $\mathfrak{q}_{N;k} = \mathfrak{q}_{N;k+1} \oplus \mathfrak{h}_k$ for all $k \in \{1, \ldots, r\}$, where r is the rank of $\mathfrak{q}_N$. Let $b_1, \ldots, b_d$ be a basis for $\mathfrak{g}$ passing through $\mathfrak{v}, \mathfrak{k}, \mathfrak{h}_2, \ldots, \mathfrak{h}_r, \mathfrak{m}$. For all $i \in \{1, \ldots, d\}$ define the weight $\omega_i = 0$ if $b_i \in \mathfrak{m} \oplus \mathfrak{v}$, $\omega_i = 1$ if $b_i \in \mathfrak{k}$ and $\omega_i = k$ if $b_i \in \mathfrak{h}_k$ with $k \geqslant 2$.

For all $a \in \mathfrak{g}$ let $dL_G(a)$ denote the generator of the one parameter group $t \mapsto L_G\big(\exp_G(-ta)\big)$, where L_G is the left regular representation in G and $\exp_G$ is the exponential map. We set $B_i = dL_G(b_i)$. We also need multi-index notation. Set $J(d) = \bigcup\limits_{n=0}^{\infty} \{1, \ldots, d\}^n$ and if $\alpha = (i_1, \ldots, i_n) \in J(d)$ set $|\alpha| = n$, $\|\alpha\| = \omega_{i_1} + \ldots + \omega_{i_n}$ and $B^\alpha = B_{i_1} \ldots B_{i_n}$.

Let $a_1, \ldots, a_{d'}$ be an algebraic basis for $\mathfrak{g}$ and let $C = (c_{kl})$ be a $d' \times d'$-matrix of complex coefficients. Assume $2^{-1}(C + C^*) \geqslant \mu I$ for some $\mu > 0$. Set $H = -\sum\limits_{i,j=1}^{d'} c_{kl} A_k A_l$, where $A_i = dL_G(a_i)$. Then it follows from [7] that the closure of the subelliptic operator H generates a holomorphic semigroup S which has a smooth kernel K. If $|\cdot|'$ is the modulus on G associated to the algebraic basis $a_1, \ldots, a_{d'}$ and $V'(\rho)$ denotes the Haar measure of the ball $\{g \in G : |g|' < \rho\}$ then it follows from [**9**] that K satisfies good

Gaussian bounds, that is, there exist $b, c > 0$ such that

$$|K_t| \leqslant c\, G_{b,t} \tag{1}$$

for all $t > 0$, where $G_{b,t}(g) = V'(t)^{-1/2} e^{-b(|g|')^2 t^{-1}}$.

The main result of this paper is that the derivatives of K satisfy the following Gaussian upper bounds.

Theorem 1.1. *If $\alpha \in J(d)$ and $i \in \{1, \ldots, d\}$ then there exist $b, c > 0$ such that*

$$|B^\alpha K_t| \leqslant c\, t^{-\|\alpha\|/2}\, G_{b,t} \quad \textit{and} \quad |B^\alpha B_i K_t| \leqslant c\, t^{-(\|\alpha\|+1)/2}\, G_{b,t}$$

for all $t \geqslant 1$.

Dungey [3] proved the Gaussian bounds $|B^\alpha K_t| \leqslant c\, t^{-\|\alpha\|/2} G_{b,t}$ and $|B^\alpha A_k K_t| \leqslant c\, t^{-(\|\alpha\|+1)/2} G_{b,t}$ for multi-indices α in the nilradical directions and in the directions of a subalgebra $\mathfrak{s}$ of $\mathfrak{m}$ for which there exists a subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{g}_0$ as Lie algebras. In addition Dungey proved exponential decay $e^{-\omega t}$ for any (higher-order) derivative which contains at least one derivative in the directions of $\mathfrak{s}$.

As in [3] we first prove first L_2-bounds for $B^\alpha S_t$ with derivatives in the nilpotent directions. Since our proof is shorter, we include it here.

The outline of the proof is as follows. First, we may assume that G is simply connected, since the general case follows from the simply connected case by a transference as in [11, p. 201]. So from now on G is simply connected. Secondly, we prove L_2-bounds on the derivatives of the semigroup in the nilpotent directions. These easily transform into L_∞-bounds on nilpotent derivatives of the kernel. Thirdly, by interpolation as in [8, Lemmas 4.2 and 4.3], one obtains Gaussian bounds on the derivatives of the kernel in the nilpotent directions. Fourthly, by the convolution property $K_{2t} = K_t * K_t$ one can move an additional derivative in any direction to the kernel on the right. Then, by induction, the first bounds of Theorem 1.1 follow. The second bounds follow similarly once one has the correct Gaussian bounds on the derivatives $A_i K_t$ in the algebraic directions.

In Section 2 we introduce some more structure theory for Lie groups with polynomial growth and prove Theorem 1.1 for derivatives in the direction of $\mathfrak{n}$. Then in Section 3 we prove Theorem 1.1 in full.

2. Nilpotent derivatives

The main difficulty in the proof of Theorem 1.1 is to prove Gaussian bounds for one derivative $B_i K_t$ with $b_i \in \mathfrak{n}$. We shall prove these bounds in this section. First we introduce more structure theory on Lie groups with polynomial growth and in addition we introduce a scale of Lie groups and subelliptic operators.

Let M and Q be the connected simply connected subgroups of G which have Lie algebras $\mathfrak{m}$ and $\mathfrak{q}$, respectively. Then $M \cap Q = \{e\}$ and M is compact since G has polynomial growth. The bounds of Theorem 1.1 are independent of the choice of $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$. By

[**1**, Proposition 2.3], one can, however choose $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ such that $\mathfrak{h}_k$ is invariant under $S(\mathfrak{v})$ and ad$\mathfrak{m}$. By [**1**, Proposition 2.4], there exists an inner product $(\cdot\,,\cdot)$ on $\mathfrak{g}$ such that the subspaces $\mathfrak{v}, \mathfrak{k}, \mathfrak{h}_2, \ldots, \mathfrak{h}_r, \mathfrak{m}$ are mutually orthogonal and the operators $S(v)$ are skew-symmetric for all $v \in \mathfrak{v}$. Define the inner product $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{g}$ by

$$\langle a, b\rangle = \int_M dm\,(\mathrm{Ad}(m)a, \mathrm{Ad}(m)b)$$

where dm is the normalised Haar measure on M. Then the subspaces $\mathfrak{v}, \mathfrak{k}, \mathfrak{h}_2, \ldots, \mathfrak{h}_r, \mathfrak{m}$ are mutually orthogonal and the operators ad$a$ and $S(v)$ are skew-symmetric for all $a \in \mathfrak{m}$ and $v \in \mathfrak{v}$. We may assume that $b_1, \ldots, b_d$ is an orthonormal basis with respect to $\langle\cdot\,,\cdot\rangle$.

For all $u > 0$ let $\gamma_u \colon \mathfrak{g} \to \mathfrak{g}$ be the linear map such that $\gamma_u(b_i) = u^{w_i} b_i$ for all $i \in \{1, \ldots, d\}$, where $w_i = 0$ if $b_i \in \mathfrak{m}$ and $w_i = k$ if $b_i \in \mathfrak{h}_k$. Next define $[\cdot\,,\cdot]_u \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$[a, b]_u = \gamma_u^{-1}\Big(\big[\gamma_u(a), \gamma_u(b)\big]\Big).$$

Then $\mathfrak{g}_u = \big(\mathfrak{g}, [\cdot\,,\cdot]_u\big)$ is a Lie algebra and $\gamma_u \colon \mathfrak{g}_u \to \mathfrak{g}$ a Lie algebra isomorphism. Define similarly the nilpotent Lie algebra $\mathfrak{q}_{Nu} = \big(\mathfrak{q}, [\cdot\,,\cdot]_{Nu}\big)$ with

$$[a, b]_{Nu} = \gamma_u^{-1}\Big(\big[\gamma_u(a), \gamma_u(b)\big]_N\Big).$$

Then $\mathfrak{q}_{Nu}$ is the nilshadow of $\mathfrak{g}_u$. If $a *_{Nu} b$ denotes the Campbell–Baker–Hausdorff formula in a and b with respect to $[\cdot\,,\cdot]_{Nu}$ on $\mathfrak{q}_{Nu}$ then $Q_{Nu} = (\mathfrak{q}, *_{Nu})$ is the connected simply connected nilpotent Lie group with Lie algebra $\mathfrak{q}_{Nu}$. Set $G_{Nu} = M \times Q_{Nu}$. We denote by $*_{Nu}$ the multiplication on G_{Nu} and by $g^{(-1)_{Nu}}$ the inverse of g. Define $\tau_u \colon \mathfrak{g}_{Nu} \to \mathcal{L}(\mathfrak{g}_{Nu})$ by $\tau_u(a)b = \Big(\mathrm{ad}\,a_\mathfrak{m} + S\big(\gamma_u(a_\mathfrak{v})\big)\Big) b_\mathfrak{q}$, where $a_\mathfrak{m}$ and $a_\mathfrak{v}$ are the components of a in $\mathfrak{m}$ and $\mathfrak{v}$ and $b_\mathfrak{q}$ is the component of b in $\mathfrak{q}$. If $\overline{T_u} \colon \mathfrak{g}_{Nu} \to \mathrm{Aut}(\mathfrak{g}_{Nu})$ is the homomorphism such that $\overline{T_u}(\exp_{G_{Nu}} a) = e^{\tau_u(a)}$ and $T_u \colon G_{Nu} \to \mathrm{Aut}(G_{Nu})$ is the Lie group homomorphism such that

$$T(\exp_{G_{Nu}} a) \exp_{G_{Nu}} b = \exp_{G_{Nu}}\big(\overline{T_u}(\exp_{G_{Nu}} a) b\big)$$

for all $a, b \in \mathfrak{g}_{Nu}$ then $(g, h) \mapsto g \;{}_{T_u}{*}\; h = (T_u(h^{(-1)_{Nu}})g) *_{Nu} h$ defines a Lie group multiplication on the set G_{Nu} of which the Lie algebra is isomorphic to $\mathfrak{g}_u$ (compare [**14**, p. 229]). Here $\exp_{G_{Nu}}$ denotes the usual exponential map on G_{Nu}. We set $G_u = (G_{Nu}, {}_{T_u}{*})$ and $T = T_1$. Then with $u = 1$ the Lie group G is isomorphic to $(G_{Nu}, {}_T{*})$ and from now on we identify G with $(G_{N1}, {}_T{*})$. We also delete the u in a symbol if $u = 1$. As a consequence

$$(2) \qquad \big(dL_{G_u}(a)\varphi\big)(g) = \Big(dL_{G_{Nu}}\big(\overline{T_u}(g^{(-1)_{Nu}})a\big)\varphi\Big)(g)$$

for all $a \in \mathfrak{g}$, $g \in G_u$ and $\varphi \in C_c^\infty(G_u)$. But it follows from the skew-symmetry of ad$\mathfrak{m}$ and $S(\mathfrak{v})$ that $\overline{T_u}$ is a unitary representation of G_{Nu} on $\mathfrak{g}$ equipped with the inner product $\langle\cdot\,,\cdot\rangle$.

We choose and fix a Lebesgue measure on the vector space $\mathfrak{q}$. Then we fix the Haar measure on Q_{Nu} such that $\int_{Q_{Nu}} \varphi = \int_{\mathfrak{q}} \varphi \circ \exp_{Q_{Nu}}$ for all $\varphi \in C_c(Q_{Nu})$. Then the Haar measure on G_{Nu} is the product measure of the normalised Haar measure on the compact group M and the Haar measure on Q_{Nu}. Finally since $\left|\det \overline{T_u}(g)\right| = 1$ for all g it follows that we can choose the Haar measure on G_u such that $\int_{G_u} \varphi = \int_{G_{Nu}} \varphi$ for all $\varphi \in C_c(G_u)$. Note that this fixes the Haar measure on $G = G_1$.

If $a_1, \ldots, a_{d'}$ is the algebraic basis of $\mathfrak{g}$ then $u\gamma_u^{-1}(a_1), \ldots, u\gamma_u^{-1}(a_{d'})$ is an algebraic basis for $\mathfrak{g}_u$. Now set $A_k^{[u]} = dL_{G_u}\big(u\gamma_u^{-1}(a_k)\big)$ for all $k \in \{1, \ldots, d'\}$. Then (2) implies that

$$(A_k^{[u]}\varphi)(g) = \sum_{j=1}^{d} \langle \overline{T_u}(g)b_j, u\gamma_u^{-1}(a_k)\rangle \, (\widetilde{B}_j^{(u)}\varphi)(g) = \sum_{j=1}^{d} u^{1-w_j}\, r_{kj}^{(u)}(g)\, (\widetilde{B}_j^{(u)}\varphi)(g) \tag{3}$$

where

$$\widetilde{B}_i^{(u)} = dL_{G_{Nu}}(b_i),$$

$r_{kj}^{(u)} = r_{kj} \circ \Gamma_u$, $r_{kj}(g) = \langle \overline{T}(g)b_j, a_k\rangle$ and $\Gamma_u \colon G_u \to G$ is the lifting of the isomorphism γ_u.

Next, define the subelliptic operator $H_{[u]}$ on G_u by

$$H_{[u]} = -\sum_{k,l=1}^{d'} c_{kl}\, A_k^{[u]}\, A_l^{[u]} .$$

If $S^{[u]}$ is the semigroup generated by $H_{[u]}$ then by subellipticity there exists a $c_1 > 0$ such that

$$\left\|(\lambda I + H_{[u]})^{-1}\right\|_{2\to 2} \leqslant \lambda^{-1} \tag{4}$$

$$\left\|(\lambda I + H_{[u]})^{-1} A_k^{[u]}\right\|_{2\to 2} \leqslant c_1\, \lambda^{-1/2} \tag{5}$$

uniformly for all $\lambda, u > 0$ and $k \in \{1, \ldots, d'\}$.

Let $d_0 = \dim \mathfrak{v}$ and $d_q = \dim \mathfrak{q}$. Then $\mathfrak{n} = \{b_{d_0+1}, \ldots, b_{d_q}\}$. Moreover, set $J(\mathfrak{n}) = \bigcup_{n=0}^{\infty} \{d_0+1, \ldots, d_q\}^n$. First we prove L_2 bounds on nilpotent derivatives.

Proposition 2.1. *For all $\alpha \in J(\mathfrak{n})$ there exists a $c > 0$ such that*

$$\|B^\alpha S_t\|_{2\to 2} \leqslant c\, t^{-\|\alpha\|/2} \tag{6}$$

for all $t \geqslant 1$.

The proof requires some preparation. For all $u > 0$ consider the unitary representation $U^{(u)}$ of the Lie group N in $L_2(G_{Nu})$ defined by $U^{(u)}(n) = L_{G_{Nu}}(n)$. For $m \in \mathbb{N}_0$ define the space

$$\mathcal{X}_m^{(u)} = \bigcap_{\alpha \in J_m(\mathfrak{n})} D(\widetilde{B}^{(u)\alpha})$$

with norm

$$|||\varphi|||_m^{(u)} = \max_{\alpha\in J_m(\mathfrak{n})} \|\widetilde{B}^{(u)\alpha}\varphi\|_2 ,$$

where $\|\cdot\|_2$ is the L_2-norm on $L_2(G_{Nu})$ and $J_m(\mathfrak{n}) = \{\alpha \in J(\mathfrak{n}) : |\alpha| \leqslant m\}$. So $\mathcal{X}_m^{(u)}$ is the Banach space of m-times differentiable elements with respect to the representation $U^{(u)}$ and the vector space basis $b_{d_1+1}, \ldots, b_{d_q}$ for $\mathfrak{n}$, with the usual norm. Define the seminorm $N_m^{(u)} \colon \mathcal{X}_m^{(u)} \to \mathbf{R}$ by

$$N_m^{(u)}(\varphi) = \max_{\substack{\alpha\in J(\mathfrak{n})\\ |\alpha|=m}} \|\widetilde{B}^{(u)\alpha}\varphi\|_2 .$$

Next we need bounds on $N_m^{(u)}\big((\lambda I + H_{[u]})^{-1}\varphi\big)$.

Lemma 2.2. *For all $m \in \mathbf{N}$ there exist $c, \lambda_0 > 0$ such that*

$$N_m^{(u)}\big((\lambda I + H_{[u]})^{-1}\varphi\big) \leqslant c\,\lambda^{-1} N_m^{(u)}(\varphi)$$

for all $\lambda \geqslant \lambda_0$, $u \geqslant 1$ and $\varphi \in L_{2;\infty}(G_{Nu})$.

Proof: Since $\widetilde{B}_i^{(u)} r_{kj}^{(u)} = 0$ if $b_i \in \mathfrak{n}$ and $\mathfrak{n}$ is an ideal it follows from (3) that there exists a $c > 0$ such that

$$\big\|[\widetilde{B}^{(u)\alpha}, A_k^{[u]}]\varphi\big\|_2 \leqslant c\, N_m^{(u)}(\varphi)$$

for all $k \in \{1, \ldots, d'\}$, $u \geqslant 1$, $\varphi \in L_{2;\infty}(G_{Nu})$ and $\alpha \in J(\mathfrak{n})$ with $|\alpha| = m$. Let $u \geqslant 1$, $\lambda > 0$, $\varphi \in L_{2;\infty}(G_{Nu})$ and write $\psi = (\lambda I + H_{[u]})^{-1}\varphi$. For all $\alpha \in J(\mathfrak{n})$ with $|\alpha| = m$ one has

$$\begin{aligned}
\|\widetilde{B}^{(u)\alpha}\psi\|_2 &\leqslant \big\|(\lambda I + H_{[u]})^{-1}\widetilde{B}^{(u)\alpha}\varphi\big\|_2 + \big\|(\lambda I + H_{[u]})^{-1}[\widetilde{B}^{(u)\alpha}, H_{[u]}]\psi\big\|_2 \\
&\leqslant \lambda^{-1} N_m^{(u)}(\varphi) + \sum_{k,l=1}^{d'} |c_{kl}|\, \big\|(\lambda I + H_{[u]})^{-1}[\widetilde{B}^{(u)\alpha}, A_k^{[u]}]A_l^{[u]}\psi\big\|_2 \\
&\qquad\qquad + \sum_{k,l=1}^{d'} |c_{kl}|\, \big\|(\lambda I + H_{[u]})^{-1}A_k^{[u]}[\widetilde{B}^{(u)\alpha}, A_l^{[u]}]\psi\big\|_2 \\
&\leqslant \lambda^{-1} N_m^{(u)}(\varphi) + c\,\lambda^{-1} \sum_{k,l=1}^{d'} |c_{kl}|\, N_m^{(u)}(A_l^{[u]}\psi) + c\,c_1\,\lambda^{-1/2} \sum_{k,l=1}^{d'} |c_{kl}|\, N_m^{(u)}(\psi) ,
\end{aligned}$$

where we used (4) and (5). So with $c_2 = c \sum\limits_{k,l=1}^{d'} |c_{kl}|$ one deduces that

$$N_m^{(u)}(\psi) \leqslant \lambda^{-1} N_m^{(u)}(\varphi) + c_2\,\lambda^{-1} \max_{k\in\{1,\ldots,d'\}} N_m^{(u)}(A_k^{[u]}\psi) + c_1\,c_2\,\lambda^{-1/2} N_m^{(u)}(\psi) . \tag{7}$$

Next, if $k \in \{1, \ldots, d'\}$ and $\alpha \in J(\mathfrak{n})$ with $|\alpha| = m$ then it follows from subellipticity that

$$\begin{aligned}\|\widetilde{B}^{(u)\alpha}A_k^{[u]}\psi\|_2^2 &\leqslant 2\|A_k^{[u]}\widetilde{B}^{(u)\alpha}\psi\|_2^2 + 2\big\|[\widetilde{B}^{(u)\alpha}, A_k^{[u]}]\psi\big\|_2^2 \\ &\leqslant 2\mu^{-1}\operatorname{Re}(\widetilde{B}^{(u)\alpha}\psi, H_{[u]}\widetilde{B}^{(u)\alpha}\psi) + 2c^2 N_m^{(u)}(\psi)^2 \\ &\leqslant 2\mu^{-1}\operatorname{Re}(\widetilde{B}^{(u)\alpha}\psi, \widetilde{B}^{(u)\alpha}H_{[u]}\psi) \\ &\qquad + 2\mu^{-1}\operatorname{Re}\big(\widetilde{B}^{(u)\alpha}\psi, [H_{[u]}, \widetilde{B}^{(u)\alpha}]\psi\big) + 2c^2 N_m^{(u)}(\psi)^2 .\end{aligned}$$

But $H_{[u]}\psi = \varphi - \lambda\psi$. So

$$\begin{aligned}\operatorname{Re}(\widetilde{B}^{(u)\alpha}\psi, \widetilde{B}^{(u)\alpha}H_{[u]}\psi) &= \operatorname{Re}(\widetilde{B}^{(u)\alpha}\psi, \widetilde{B}^{(u)\alpha}\varphi) - \lambda\operatorname{Re}(\widetilde{B}^{(u)\alpha}\psi, \widetilde{B}^{(u)\alpha}\psi) \\ &\leqslant N_m^{(u)}(\psi)\, N_m^{(u)}(\varphi) \leqslant 2^{-1}(N_m^{(u)}(\psi)^2 + N_m^{(u)}(\varphi)^2)\end{aligned}$$

and

$$\begin{aligned}&\Big|\operatorname{Re}\big(\widetilde{B}^{(u)\alpha}\psi, [H_{[u]}, \widetilde{B}^{(u)\alpha}]\psi\big)\Big| \\ &\quad\leqslant \sum_{i,j=1}^{d'} |c_{ij}|\,\Big|\big(\widetilde{B}^{(u)\alpha}\psi, [A_i^{[u]}, \widetilde{B}^{(u)\alpha}]A_j^{[u]}\psi\big)\Big| + \sum_{i,j=1}^{d'} |c_{ij}|\,\Big|\big(\widetilde{B}^{(u)\alpha}\psi, A_i^{[u]}[A_j^{[u]}, \widetilde{B}^{(u)\alpha}]\psi\big)\Big| \\ &\quad\leqslant c_2\, N_m^{(u)}(\psi) \max_{j\in\{1,\dots,d'\}} N_m^{(u)}(A_j^{[u]}\psi) + c_2 \max_{i\in\{1,\dots,d'\}} N_m^{(u)}(A_i^{[u]}\psi) N_m^{(u)}(\psi) + c\, c_2 N_m^{(u)}(\psi)^2\end{aligned}$$

by anti-symmetry of $A_i^{[u]}$ and an estimate on the commutator $[A_i^{[u]}, \widetilde{B}^{(u)\alpha}]$. Hence

$$\begin{aligned}\max_{k\in\{1,\dots,d'\}} N_m^{(u)}(A_k^{[u]}\psi)^2 &\leqslant \mu^{-1}N_m^{(u)}(\varphi)^2 + 4c_2\,\mu^{-1}N_m^{(u)}(\psi) \max_{j\in\{1,\dots,d'\}} N_m^{(u)}(A_j^{[u]}\psi) \\ &\qquad + 2(\mu^{-1} + c^2 + \mu^{-1}c\, c_2)N_m^{(u)}(\psi)^2 .\end{aligned}$$

Therefore

$$\max_{k\in\{1,\dots,d'\}} N_m^{(u)}(A_k^{[u]}\psi) \leqslant \mu^{-1/2}\, N_m^{(u)}(\varphi) + c_3\, N_m^{(u)}(\psi)$$

where $c_3 = 4c_2\,\mu^{-1} + \big(2(\mu^{-1} + c^2 + \mu^{-1}c\, c_2)\big)^{1/2}$. Together with (7) it follows that

$$N_m^{(u)}(\psi) \leqslant \lambda^{-1}(1 + c_2\,\mu^{-1/2})N_m^{(u)}(\varphi) + c_2(c_3\,\lambda^{-1} + c_1\,\lambda^{-1/2})\, N_m^{(u)}(\psi) .$$

So the lemma follows with $\lambda_0 = 1 + 4c_2^2(c_1 + c_3)^2$. ∎

Now we are able to prove Proposition 2.1.

PROOF OF PROPOSITION 2.1: Let $m \in \mathbf{N}$, $m \geqslant 2$. It follows from the first five steps in the proof of [**9**, Lemma 2.2] that there exists a $\nu \in \langle 0, 1\rangle$ such that $L'_{2;1}(G_u) \subset \big(L_2(G_{Nu}), L'_{2;mr}(G_{Nu})\big)_{\gamma,\infty;\mathrm{K}}$ and the embedding is continuous uniformly for all $u \geqslant 1$, where $L'_{2;1}(G_u)$ is the Sobolev space defined with respect to the group G_u and the algebraic basis $u\gamma_u^{-1}(a_1), \dots, u\gamma_u^{-1}(a_{d'})$ and $L'_{2;mr}(G_{Nu})$ is the Sobolev space defined with respect to the group G_{Nu} and the algebraic basis $b_1, \dots, b_{d_1}, b_{d_q+1}, \dots, b_d$, where $d_1 = \dim \mathfrak{h}_1$. Here we used the real interpolation spaces with respect to the K-method of Peetre. Obviously

$L'_{2;mr}(G_{Nu}) \subset L'^{(Q)}_{2;mr}(G_{Nu})$, the Sobolev space with respect to the set $b_1, \ldots, b_{d_1}$. But $L'^{(Q)}_{2;mr}(G_{N1}) \subseteq L^{(Q)}_{2;m}(G_{N1})$, the Sobolev space with respect to the set $b_1, \ldots, b_{d_q}$ and the embedding is continuous. Therefore, by scaling, the space $L'^{(Q)}_{2;mr}(G_{Nu})$ is embedded in the space $L^{(Q)}_{2;m}(G_{Nu})$ and the embedding is continuous uniformly for all $u \geqslant 1$. Obviously, $L^{(Q)}_{2;m}(G_{Nu}) \subseteq \mathcal{X}^{(u)}_m$ and the norm of the embedding is bounded by 1. Combining these embeddings there exists a $c > 0$ such that

$$\|\varphi\|_{(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{\nu,2,\mathrm{K}}} \leqslant c\Big(\|\varphi\|_2 + \sum_{k=1}^{d'} \|A^{[u]}_k \varphi\|_2\Big)$$

uniformly for all $u \geqslant 1$ and $\varphi \in L'_{2;1}(G_u)$. Hence by (4) and (5) there exists a $c_0 > 0$ such that

$$\big\|(\lambda I + H_{[u]})^{-1}\varphi\big\|_{(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{\nu,2,\mathrm{K}}} \leqslant c_0\, \lambda^{-1/2}\|\varphi\|_2 \tag{8}$$

for all $u \geqslant 1$, $\lambda \geqslant 1$ and $\varphi \in \mathcal{X}^{(u)}_0$.

By Lemma 2.2 there exist $c_1 \geqslant 1$ and $\lambda_0 > 0$ such that such that

$$|||(\lambda I + H_{[u]})^{-1}\varphi|||_m \leqslant c_1\, \lambda^{-1}|||\varphi|||_m$$

for all $\lambda \geqslant \lambda_0$, $u \geqslant 1$ and $\varphi \in L_{2;\infty}(G_{Nu})$. But $L_{2;\infty}(G_{Nu})$ is dense in $\mathcal{X}^{(u)}_m$ by an argument similar to the proof of [5, Lemma 2.4]. Hence if $\lambda = 1 \vee \lambda_0$ then the map $(\lambda I + H_{[u]})^{-1}$ is continuous from $\mathcal{X}^{(u)}_0$ into $\mathcal{X}^{(u)}_0$ with norm bounded by 1 and from $\mathcal{X}^{(u)}_m$ into $\mathcal{X}^{(u)}_m$ with norm bounded by c_1. Therefore, by interpolation, for all $\gamma \in \langle 0, 1\rangle$ the map $(\lambda I + H_{[u]})^{-1}$ is continuous from $(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{\gamma,2,\mathrm{K}}$ into $(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{\gamma,2,\mathrm{K}}$ with norm bounded by c_1. But by (8) the map $(\lambda I + H_{[u]})^{-1}$ is also continuous from $\mathcal{X}^{(u)}_0$ into $(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{\nu,2,\mathrm{K}}$ with norm bounded by c_0. Hence, by interpolation, for all $\gamma \in \langle 0, 1\rangle$ the map $(\lambda I + H_{[u]})^{-1}$ is continuous from $(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{\gamma,2,\mathrm{K}}$ into $(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{\gamma+(1-\gamma)\nu,2,\mathrm{K}}$ with norm bounded by $c_0 + c_1$. Using interpolation once more, it follows that there exists an $N \in \mathbf{N}$ such that the map $(\lambda I + H_{[u]})^{-N}$ is continuous from $\mathcal{X}^{(u)}_0$ into $(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{1-(2m)^{-1},2,\mathrm{K}}$ with norm bounded by $(1 + c_0 + c_1)^N$. By (11) in [6] one has $(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{1-(2m)^{-1},2,\mathrm{K}} \subseteq \mathcal{X}^{(u)}_{m-1}$ and the embedding is continuous, uniformly for all $u > 0$. Hence there exists a $c_2 > 0$ such that

$$\|\widetilde{B}^{(u)\alpha}\varphi\|_2 \leqslant |||\varphi|||^{(u)}_{m-1} \leqslant c_2\, \|\varphi\|_{(\mathcal{X}^{(u)}_0, \mathcal{X}^{(u)}_m)_{1-(2m)^{-1},2,\mathrm{K}}}$$

uniformly for all $u \geqslant 1$, $\varphi \in L_{2;\infty}(G_{Nu})$ and $\alpha \in J(\mathfrak{n})$ with $|\alpha| = m - 1$. Then

$$\|\widetilde{B}^{(u)\alpha}\varphi\|_2 \leqslant c_2(1 + c_0 + c_1)^N \big\|(\lambda I + H_{[u]})^N\varphi\big\|_2$$

for all $\varphi \in L_{2;\infty}(G_u)$. In particular,

$$\begin{aligned}\|\widetilde{B}^{(u)\alpha} S^{[u]}_1\|_{2\to2} &\leqslant c_2(1 + c_0 + c_1)^N \sum_{n=0}^{N} \binom{N}{n} \lambda^{N-n}\, \|H^n_{[u]} S^{[u]}_1\|_{2\to2} \\ &\leqslant c_2(1 + c_0 + c_1)^N \Big(\lambda^N + \sum_{n=1}^{N} \binom{N}{n} \lambda^N\, \|H_{[u]} S^{[u]}_{1/n}\|^n_{2\to2}\Big)\,.\end{aligned}$$

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But there exists a $c_3 \geqslant 1$ such that $\|HS_t\|_{2\to 2} \leqslant c_3\, t^{-1}$ for all $t > 0$. Then $\|H_{[u]}S^{[u]}_{1/n}\|_{2\to 2} = u^2\|HS_{n^{-1}u^2}\|_{2\to 2} \leqslant c_3 n$ uniformly for all $u > 0$. Then

$$\|\widetilde{B}^{(u)\alpha}S^{[u]}_1\|_{2\to 2} \leqslant c_4$$

for all $u \geqslant 1$, where $c_4 = c_2c_3(2(1+c_0+c_1)\lambda)^N N$. Finally, it follows from (2) that $B^{(u)}_i = \sum_{j=1}^{d}(\rho_{ij}\circ\Gamma_u)\,\widetilde{B}^{(u)}_j$ for all $i \in \{1,\dots,d\}$ and $u > 0$, where $\rho_{ij}(g) = \langle\overline{T}(g)b_j, b_i\rangle$. But if $b_i \in \mathfrak{n}$ then for a non-vanishing term in the sum one must have $b_j \in \mathfrak{n}$. In addition, $\widetilde{B}^{(u)}_k\rho_{ij} = 0$ if $b_k \in \mathfrak{n}$ and ρ_{ij} is bounded. So for all $\alpha \in J(\mathfrak{n})$ there exist bounded continuous functions ρ_β such that $B^{(u)\alpha} = \sum_{\beta\in J(\mathfrak{n});\,|\beta|=|\alpha|}(\rho_\beta\circ\Gamma_u)\widetilde{B}^{(u)\beta}$. Then for all $\alpha \in J(\mathfrak{n})$ there exists a $c_5 > 0$ such that $\|B^{(u)\alpha}S^{[u]}_1\|_{2\to 2} \leqslant c_5$, uniformly for all $u \geqslant 1$. Then

$$\|B^\alpha S_t\|_{2\to 2} = t^{-\|\alpha\|/2}\|B^{(t^{1/2})\alpha}S^{[t^{1/2}]}_1\|_{2\to 2} \leqslant c_5\, t^{-\|\alpha\|/2}$$

for all $t \geqslant 1$ and Proposition 2.1 follows. □

COROLLARY 2.3. *For all $\alpha \in J(\mathfrak{n})$ there exists a $c > 0$ such that*

$$\|B^\alpha K_t\|_\infty \leqslant c\, V'(t)^{-1/2}t^{-\|\alpha\|/2}$$

for all $t \geqslant 1$.

PROOF: For all $t \geqslant 1$ and $\alpha \in J(\mathfrak{n})$ one has

$$\begin{aligned}\|B^\alpha K_{3t}\|_\infty = \|B^\alpha S_{3t}\|_{1\to\infty} &\leqslant \|B^\alpha S_{2t}\|_{2\to\infty}\|S_t\|_{1\to 2}\\ &= \|B^\alpha S_t\|_{2\to 2}\|S_t\|_{2\to\infty}\|S^*_t\|_{2\to\infty} = \|B^\alpha S_t\|_{2\to 2}\|K_t\|_2\|K^\dagger_t\|_2\end{aligned}$$

where $K^\dagger$ is the kernel of S^*. Then the corollary follows from the bounds (6), the Gaussian bounds on K and $K^\dagger$ and the doubling property. □

The next lemma is the key in the induction step to turn L_∞-bound on derivatives and Gaussian bounds on K into Gaussian bounds on derivatives of K.

LEMMA 2.4. *Suppose $a \in \mathfrak{g}$, $t \geqslant 1$, $b, c_0, c_2, c, \delta > 0$, $\Phi \in C^\infty(G)$ and suppose that*

$$|\Phi(g)| \leqslant c_0\, G_{b,t}(g)$$

for all $g \in G$,

$$\left\|dL(a)^2\Phi\right\|_\infty \leqslant c_2\, V'(t)^{-1/2}$$

and $|\exp ua|' \leqslant c\left(1 \vee |u|^{1/\delta}\right)$ for all $u \in \mathbf{R}$. Then

$$\left|(dL(a)\Phi)(g)\right| \leqslant (2e^{bc^2}\tau^{-1}c_0 + 2^{-1}c_2\,\tau)\, G_{4^{-1}b,t}(g)$$

for all $\tau \in \langle 0, t^{\delta/2}]$ and $g \in G$.

PROOF: If $u \in \mathbf{R}$ then

$$\left(|g|'\right)^2 \leqslant 2\Big(\big|\exp(-ua)g\big|'\Big)^2 + 2(|\exp ua|')^2 \leqslant 2\Big(\big|\exp(-ua)g\big|'\Big)^2 + 2c^2\left(1 \vee |u|^{2/\delta}\right)$$

and hence

$$-\Big(\big|\exp(-ua)g\big|'\Big)^2 \leqslant -2^{-1}(|g|')^2 + c^2\left(1 \vee |u|^{2/\delta}\right)$$

for all $g \in G$. Therefore

$$\Big|\Big(\big(I - L(\exp ua)\big)\Phi\Big)(g)\Big| \leqslant 2e^{bc^2} c_0\, V'(t)^{-1/2}\, e^{-2^{-1}b(|g|')^2 t^{-1}}$$

for all $g \in G$ and $u \in \langle 0, t^{\delta/2}]$.

Using the identity

$$f'(0) = u^{-1}\big(f(u) - f(0)\big) + u^{-1}\int_0^u dv\,(u-v)\,f''(v)$$

with $f(u) = \big(L(\exp ua)\Phi\big)(g)$ one deduces that

$$\begin{aligned}\Big|\big(dL(a)\Phi\big)(g)\Big| &\leqslant u^{-1}\Big|\Big(\big(I - L(\exp ua)\big)\Phi\Big)(g)\Big| + u^{-1}\int_0^u dv\,(u-v)\,\big\|dL(a)^2\Phi\big\|_\infty \\ &\leqslant 2e^{bc^2}\, u^{-1} c_0\, V'(t)^{-1/2}\, e^{-2^{-1}b(|g|')^2 t^{-1}} + 2^{-1} u\, c_2\, V'(t)^{-1/2}\end{aligned}$$

for all $g \in G$ and $u \in \langle 0, t^{\delta/2}]$. Then the lemma follows by setting $u = \tau\, e^{-4^{-1}b(|g|')^2 t^{-1}}$. ∎

Next we prove the Gaussian bounds for the nilpotent derivatives of the kernel. If $|\cdot|$ and $|\cdot|_{G_N}$ are moduli on G and G_N with respect to the basis $b_1, \ldots, b_d$, respectively, then it follows from (2) and the orthogonality of the $\overline{T}(g)$ that $|g| = |g|_{G_N}$ for all $g \in G_N$. Then by [**15**, Proposition III.42], for any neighbourhood Ω of the identity element there exists a $c > 0$ such that

$$c^{-1}\,|g|_{G_N} \leqslant |g|' \leqslant c\,|g|_{G_N} \tag{9}$$

or all $g \in G\backslash\Omega$.

PROPOSITION 2.5. *For all $\alpha \in J(\mathfrak{n})$ there are $b, c > 0$ such that*

$$|B^\alpha K_t| \leqslant c\, t^{-\|\alpha\|/2}\, G_{b,t} \tag{10}$$

for all $t \geqslant 1$.

PROOF: If $|\alpha| = 0$ then the bounds (10) equal the bounds (1). The proof is by induction. Let $k \in \{d_0 + 1, \ldots, d_q\}$, $\alpha \in J(\mathfrak{n})$ and suppose that the bounds (10) are valid for $B^\alpha K_t$. Then by (9) and the inclusion "$\phi\big(R(t)\big) \subset B(ct)$" in the proof of [**15**, Proposition IV.5.6], applied to the simply connected nilpotent group Q_N it follows that there exists a $c > 0$ such that $|\exp ub_k|' \leqslant c\,|u|^{1/\omega_k}$ for all $|u| \geqslant 1$. Then the bounds (10) follow for $B_k B^\alpha K_t$ from Lemma 2.4 by taking $\Phi = B^\alpha K_t$, $a = b_k$, $\delta = \omega_k$ and $\tau = t^{\omega_k/2}$, using the induction hypothesis and the L_∞-bounds of Corollary 2.3 on $B_k^2 B^\alpha K_t$. Hence by induction the bounds (10) are valid for all $\alpha \in J(\mathfrak{n})$. The proof is complete. ∎

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3. Derivative bounds

The derivative bounds on the kernel in a general direction follow from the derivative bounds in a nilpotent direction and an estimate on $\langle b_j, \mathrm{Ad}(g)b_i\rangle$. We need two lemmas before we prove the required bounds on $\mathrm{Ad}(g)$.

Lemma 3.1. *If $v \in \mathfrak{v}$ then $K(v)\mathfrak{q}_{N;k} \subseteq \mathfrak{q}_{N;k+1}$ for all $k \in \{1, \ldots, r\}$.*

Proof: It is straightforward to prove that $\mathrm{ad}v$ and $K(v)$ are derivations on $\mathfrak{g}_N$. If $v' \in \mathfrak{v}$ then $(\mathrm{ad}v)v' = [v, v']_N$ and since $K(v)$ is a polynomial in $\mathrm{ad}v$ without a constant term and $\mathrm{ad}v$ is a derivation of $\mathfrak{q}_N$ it follows that $K(v)\mathfrak{v} \subseteq \mathfrak{q}_{N;2}$. Clearly $K(v)\mathfrak{n} = [v, \mathfrak{n}]_N \subseteq \mathfrak{q}_{N;2}$. So $K(v)\mathfrak{q} \subseteq \mathfrak{q}_{N;2}$. Since $K(v)$ is a derivation of $\mathfrak{q}_N$ the corollary follows. ∎

For all $\xi \in \mathbf{R}^{d_q}$ define $\|\xi\| = \sum_{i=1}^{d_q} |\xi_i|^{1/w_i}$. Then it follows from the Campbell–Baker–Hausdorff formula and the first displayed formula on page 55 of [**15**] that there exists a $c > 0$ such that

$$\|\xi\| \leqslant c\,|\exp_{G_N} \xi_1 b_1 *_{G_N} \ldots *_{G_N} \exp_{G_N} \xi_{d_q} b_{d_q}|_{G_N} \tag{11}$$

for all $\xi \in \mathbf{R}^{d_q}$ with $\|\xi\| \geqslant 1$.

Lemma 3.2. *For all $i, j \in \{1, \ldots, d\}$ and $m_1, \ldots, m_{d_q} \in \mathbf{N}_0$ there exists a $c > 0$ such that*

$$\begin{aligned} &\left|\left\langle b_j, (\mathrm{ad}\xi_{d_q} b_{d_q})^{m_{d_q}} \ldots (\mathrm{ad}\xi_{d_0+1} b_{d_0+1})^{m_{d_0+1}} K(\xi_{d_0} b_{d_0})^{m_{d_0}} \ldots K(\xi_1 b_1)^{m_1} b_i\right\rangle\right| \\ &\qquad \leqslant \begin{cases} c\,\|\xi\|^{\omega_j - \omega_i} & \text{if } \omega_j \geqslant \omega_i \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{12}$$

for all $\xi \in \mathbf{R}^{d_q}$ with $\|\xi\| \geqslant 1$.

Proof: By composition it suffices to prove the statement for $m_1 + \ldots + m_{d_q} = 1$. Let $l \in \{1, \ldots, d_q\}$ be such that $m_l = 1$. Suppose the left hand side of (12) is not zero.

If $l \leqslant d_0$ then $v = \xi_l b_l \in \mathfrak{v}$ and one has $K(v)\mathfrak{m} = \{0\}$ and $K(v)\mathfrak{q}_{N;k} \subseteq \mathfrak{q}_{N;k+1}$ for all $k \in \{1, \ldots, r\}$ by Lemma 3.1. Hence $i \leqslant d_q$ and $\omega_j = w_j > w_i \geqslant \omega_i$. Therefore with $c = \left|\langle b_j, K(b_l)b_i\rangle\right|$ one has

$$\left|\langle b_j, K(\xi_l b_l)b_i\rangle\right| \leqslant c\,|\xi_l| \leqslant c\,\|\xi\| \leqslant c\,\|\xi\|^{\omega_j - \omega_i}$$

since $\|\xi\| \geqslant 1$.

If $l > d_0$ then with $n = \xi_l b_l \in \mathfrak{n}$ one has $(\mathrm{ad}n)b_i = -[b_i, n] \in \mathfrak{h}_{w_l} \cap \mathfrak{n}$ if $b_i \in \mathfrak{m}$. Hence $b_j \in \mathfrak{n}$ and $\omega_j - \omega_i = w_j = w_l$. Then $\left|\langle b_j, (\mathrm{ad}\xi_l b_l)b_i\rangle\right| \leqslant \left|\langle b_j, [b_l, b_i]\rangle\right| \|\xi\|^{\omega_j - \omega_i}$ since $|\xi_l| \leqslant \|\xi\|^{w_l}$. Alternatively, if $b_i \in \mathfrak{v}$ then $(\mathrm{ad}n)b_i = -(\mathrm{ad}b_i)n \in \mathfrak{q}_{N;w_l} \cap \mathfrak{n}$. So $b_j \in \mathfrak{n}$ and $\omega_j - \omega_i = w_j \geqslant w_l$. Then $\left|\langle b_j, (\mathrm{ad}\xi_l b_l)b_i\rangle\right| \leqslant \left|\langle b_j, [b_l, b_i]\rangle\right| \|\xi\|^{\omega_j - \omega_i}$ since $|\xi_l| \leqslant \|\xi\|^{w_l} \leqslant \|\xi\|^{w_j}$. Finally, if $b_i \in \mathfrak{n}$ then $(\mathrm{ad}n)b_i \in \mathfrak{q}_{N;w_i + w_l}$. So $w_j \geqslant w_i + w_l$ and one can argue as before. ∎

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Lemma 3.3. *For all $i,j \in \{1,\ldots,d\}$ there exists a $c>0$ such that*

$$\left|\left\langle b_j, \mathrm{Ad}(g^{-1})b_i\right\rangle\right| \leqslant \begin{cases} c\left(1+\left(|g|'\right)^{\omega_j-\omega_i}\right) & \text{if } \omega_j \geqslant \omega_i \\ 0 & \text{otherwise}\end{cases}$$

for all $g \in G$.

Proof: Since $G = MQ$ and M is compact it follows that $\mathrm{Ad}(M)b_i$ is a bounded subset of $\mathfrak{m}\oplus\mathfrak{v}$ or $\mathfrak{h}_{w_i}\cap\mathfrak{n}$ if $b_i \in \mathfrak{m}\oplus\mathfrak{v}$ or $b_i \in \mathfrak{n}$, respectively. So it suffices to consider the case $g \in Q$. By [**14**, Theorem 3.18.11], there exists a $\xi \in \mathbf{R}^{d_q}$ such that $g = \exp\xi_1 b_1 \ldots \exp\xi_{d_q} b_{d_q}$. Then

$$\mathrm{Ad}(g^{-1}) = e^{-\mathrm{ad}\xi_{d_q}b_{d_q}} \ldots e^{-\mathrm{ad}\xi_{d_0+1}b_{d_0+1}} e^{-K(\xi_{d_0}b_{d_0})} \ldots e^{-K(\xi_1 b_1)} e^{-S(\xi_{d_0}b_{d_0})} \ldots e^{-S(\xi_1 b_1)} .$$

But $S(v)$ is an orthogonal transformation, leaving $\mathfrak{m}\oplus\mathfrak{v}$, $\mathfrak{k}$, $\mathfrak{h}_2,\ldots,\mathfrak{h}_r$ invariant. Moreover, $\exp_{G_N}\xi_1 b_1 *_{G_N} \cdots *_{G_N} \exp_{G_N}\xi_{d_q}b_{d_q} = \exp\xi_1 b_1 \ldots \exp\xi_{d_q}b_{d_q} = g$. Therefore by (11) and (9) it suffices to show that there exists a $c>0$ such that

$$\left|\left\langle b_j, e^{-\mathrm{ad}\xi_{d_q}b_{d_q}} \ldots e^{-\mathrm{ad}\xi_{d_0+1}b_{d_0+1}} e^{-K(\xi_{d_0}b_{d_0})} \ldots e^{-K(\xi_1 b_1)} b_i\right\rangle\right| \leqslant \begin{cases} c\,\|\xi\|^{\omega_j-\omega_i} & \text{if } \omega_j \geqslant \omega_i \\ 0 & \text{otherwise}\end{cases}$$

uniformly for all $\xi \in \mathbf{R}^{d_q}$ with $\|\xi\| \geqslant 1$. But this follows from Lemma 3.2 by expanding the (terminating) power series of the exponentials of the nilpotent endomorphisms. ∎

Next we derive a global a priori Gaussian bound which will be improved by subsequent additional arguments.

Lemma 3.4. *For all $\alpha \in J(d)$ there exist $b,c>0$ such that*

$$|B^\alpha K_t| \leqslant c\,G_{b,t} \tag{13}$$

for all $t \geqslant 1$.

Proof: Let $\alpha \in J(d)$. It follows from the semigroup property and the bounds (1) that there exists a $c>0$ such that

$$\|B^\alpha S_t\|_\infty = \|B^\alpha S_t\|_{1\to\infty} \leqslant \|B^\alpha S_{1/2}\|_{\infty\to\infty}\,\|S_{t-1/2}\|_{1\to\infty} \leqslant c\,V'(t-1/2)^{-1/2}$$

for all $t \geqslant 1$. Hence there exists a $c>0$ such that $\|B^\alpha S_t\|_\infty \leqslant c\,V'(t)^{-1/2}$ for all $t \geqslant 1$.

Now we prove the lemma by induction to $|\alpha|$. The bounds (13) are valid if $|\alpha| = 0$. Let $k \in \{1,\ldots,d\}$, $\alpha \in J(d)$ and suppose the bounds (13) are valid for $B^\alpha K_t$. Then the bounds (13) for $B_k B^\alpha K_t$ follow from Lemma 2.4 by taking $\Phi = B^\alpha K_t$, $a = b_k$, $\tau = 1$ and $\delta = 1$, using the induction hypothesis and the above proved L_∞-bounds on $B_k^2 B^\alpha K_t$. Hence the lemma follows by induction. ∎

The proof of Theorem 1.1 is an easy consequence of the next lemma.

LEMMA 3.5. *Let R be a right invariant differential operator, $b, c > 0$, $\delta \geqslant 0$ and $i \in \{1, \ldots, d\}$. Suppose that $|RK_t| \leqslant c\, t^{-\delta} G_{b,t}$ for all $t \geqslant 1$. Then there exist $b', c' > 0$ such that $|B_i RK_t| \leqslant c'\, t^{-\delta} t^{-\omega_i/2} G_{b',t}$ for all $t \geqslant 1$.*

PROOF: For all $j \in \{1, \ldots, d\}$ define $\psi_{ij} \colon G \to \mathbf{R}$ by

$$\psi_{ij}(g) = \langle b_j, \mathrm{Ad}(g^{-1}) b_i \rangle.$$

Then $B_i\, L_G(g) = \sum_{j=1}^{d} L_G(g)\, \psi_{ij}(g)\, B_j$ for all $g \in G$. Hence

$$B_i(\psi * \varphi) = \sum_{j=1}^{d} (\psi \psi_{ij}) * B_j \varphi$$

for all $i \in \{1, \ldots, d\}$ and $\varphi, \psi \in L_{1;\infty}(G)$. In particular,

$$B_i RK_{2t} = B_i((RK_t) * K_t) = \sum_{j=1}^{d} (\psi_{ij}\, RK_t) * B_j K_t \tag{14}$$

for all $t > 0$. By Proposition 2.5 and Lemma 3.4 there exist $b_1, c_1 > 0$ such that $|B_j K_t| \leqslant c_1\, t^{-\omega_j/2} G_{b_1,t}$ for all $t \geqslant 1$ and $j \in \{1, \ldots, d\}$. Moreover, by Lemma 3.3, one can restrict the sum in (14) to $j \in \{1, \ldots, d\}$ with $\omega_j \geqslant \omega_i$. So suppose $\omega_j \geqslant \omega_i$. Then there exists a $c_2 > 0$ such that $\left|\psi_{ij}(g)\right| \leqslant c_2 \Bigl(1 + \bigl(|g|'\bigr)^{\omega_j - \omega_i}\Bigr)$ for all $g \in G$. Therefore by the assumption on RK_t one has

$$\begin{aligned}
\left|\psi_{ij}(g)\, (RK_t)(g)\right| &\leqslant c\, c_2 \Bigl(1 + \bigl(|g|' t^{-1/2}\bigr)^{\omega_j - \omega_i}\, t^{(\omega_j - \omega_i)/2}\Bigr)\, t^{-\delta}\, G_{b,t}(g) \\
&\leqslant c\, c_2 (1 + c_3)\, t^{-\delta}\, t^{(\omega_j - \omega_i)/2}\, G_{2^{-1}b,t}(g)
\end{aligned}$$

for all $g \in G$ and $t \geqslant 1$, where $c_3 = \sup_{x > 0} x^{\omega_j - \omega_i} e^{-2^{-1} b x^2}$. Then

$$\begin{aligned}
\left|(\psi_{ij}\, RK_t) * B_j K_t\right| &\leqslant c c_1 c_2 (1 + c_3)\, t^{-\delta}\, t^{-\omega_i/2}\, G_{2^{-1}b,t} * G_{b_1,t} \\
&\leqslant c'\, t^{-\delta}\, t^{-\omega_i/2}\, G_{b',t}
\end{aligned}$$

for some $b', c' > 0$, uniformly for all $t \geqslant 1$, since the convolution of two Gaussians is bounded by a Gaussian. ∎

PROOF OF THEOREM 1.1: The first estimate for $|\alpha| = 0$ is given by (1) and it follows by induction to $|\alpha|$ from Lemma 3.5. The second estimate follows similarly once one can prove that $B_i K_t$ has bounds $|B_i K_t| \leqslant c\, t^{-1/2} G_{b,t}$ for all $t \geqslant 1$. Since $a_1, \ldots, a_{d'}$ is an algebraic basis it suffices to prove that for all $\alpha \in J(d')$ with $|\alpha| \geqslant 1$ there exist $b, c > 0$ such that

$$|A^\alpha K_t| \leqslant c\, t^{-1/2}\, G_{b,t} \tag{15}$$

for all $t \geqslant 1$. The proof is again by induction.

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If $|\alpha| = 1$ then (15) has been proved in [**13**, Proposition 1], for a sublaplacian and the proof for a complex subelliptic operator is the same, once one has the Gaussian bounds (1). Let $\alpha \in J(d')$ with $|\alpha| \geqslant 1$, $k \in \{1, \dots, d'\}$ and suppose the bounds (15) for $A^\alpha K_t$ are valid. Then the bounds (15) follow for $A_k A^\alpha K_t$ from Lemma 3.5 by writing A_k as a linear combination of $B_1, \dots, B_d$. Hence by induction the bounds (15) are valid for all $\alpha \in J(d')$ with $|\alpha| \geqslant 1$ and the proof of the theorem is complete. $\square$

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