CORRESPONDENCE

(*To the Editors of the* Journal of the Institute of Actuaries)

He looked again, and found it was
A Double Rule of Three:
'And all its mystery', he said,
'Is clear as day to me!'

Sylvie and Bruno, LEWIS CARROLL.

DEAR SIRS,

It would be accounted a miracle if an author were able to accept pointed criticism of his work at its face value. Indeed, more often than not he can demonstrate—to his own satisfaction at least—that such criticism is unjustified and based on misunderstanding.

Messrs Beard and Perks (*J.I.A.* LXXV, 75–86) devote more than passing attention to a paper I contributed to the *Skandinavisk Aktuarietidskrift* in 1947. They severely criticize:

(i) the sampling procedure which is stated to be 'by no means...obvious or...realistic';

(ii) the use of the Poisson law for the probability distribution of deaths which they could not 'accept as theoretically appropriate';

(iii) the supposed 'inconsistencies' in the asymptotic formulas I derived which led to the 'extraordinary' result that the Central Limit Theorem did not apply unless q, the rate of mortality, tended to zero;

(iv) the numerical examples of probability distributions of policy-claims used as illustrations since these 'would be materially affected by a refitting of the data [of duplicate policies] to provide a distribution with a less pronounced tail'.

These criticisms appear so damning that your readers can scarcely have been encouraged to refer to my paper, in a periodical of which but few copies exist in England, to check their validity.

The problem posed in my 1947 paper is a familiar one: the observation of E policies on lives aged x has resulted in y policy-claims—what is the probability of such an event? Now although statisticians have recently been asking themselves whether, in problems of sampling, there is complete justification in the assumption that the number of objects sampled is to be held fixed, it is difficult to propose an alternative which is not subject to even more weighty disadvantages. If N *lives* had been observed to result in d deaths, few of us would think in terms of anything but sampling with fixed N.

On the assumption, then, that E is a fixed datum and that y is a random variable, two hypotheses were made:

(*a*) that q is the probability of death of any one individual within the observational period independently of the deaths of other individuals, and

(*b*) that π_j ($j=1, 2, 3, \ldots, \mathrm{E}$) is the probability that any individual observed has j policies included in the observations.

The mathematical formula employed was effectively an extension of Bayes's

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rule (cp. Problem 2, Ch. IV, Uspensky, *Introduction to Mathematical Probability*), viz.

$$\Pr.(C \mid A) = \frac{\sum_{i=1}^{n} \Pr.(B_i)\Pr.(A \mid B_i)\Pr.(C \mid AB_i)}{\sum_{i=1}^{n} \Pr.(B_i)\Pr.(A \mid B_i)}$$

$$= \frac{\sum_{i=1}^{n} \Pr.(B_i) \sum_{j=1}^{m} \Pr.(A_j \mid B_i)\Pr.(C \mid A_j B_i)}{\sum_{i=1}^{n} \Pr.(B_i) \sum_{j=1}^{m} \Pr.(A_j \mid B_i)}, \qquad (1)$$

where A consists of m mutually exclusive events A_j ($j=1, 2, \ldots, m$). In its application to the problem under discussion C represents the y policy-claims and A stands for the E policies which form the observations. The 'causes' B_i ($i=1, 2, 3, \ldots, n$) are represented by the L lives supposed to have been observed ($L=1, 2, 3, \ldots, E$). The m mutually exclusive events

$$A_j \; (j=1, 2, 3, \ldots, m)$$

are given by the m partitions of L satisfying the pair of relations

$$\sum_k l_k^{(j)} = L \quad \text{and} \quad \sum_k k l_k^{(j)} = E,$$

where $l_k^{(j)}$ represents the number of lives in the jth partition with k policies assured. The probability of this jth partition is then

$$\Pr.(A_j \mid B_i) = \frac{L!}{l_1^{(j)}!\, l_2^{(j)}!\, l_3^{(j)}! \ldots} \pi_1^{l_1^{(j)}} \pi_2^{l_2^{(j)}} \pi_3^{l_3^{(j)}} \ldots .$$

Now for a specified set of lives $l_k^{(j)}$ ($k=1, 2, 3, \ldots$) the corresponding deaths may be written $d_k^{(js)}$($k=1, 2, 3, \ldots$), where s assumes as many values as there are sets satisfying the relation $y = \sum_k k d_k^{(js)}$. Hence

$$\Pr.(C \mid A_j B_i) = \sum_s \Pr.(d_1^{(js)}, d_2^{(js)}, d_3^{(js)}, \ldots \mid l_1^{(j)}, l_2^{(j)}, l_3^{(j)}, \ldots),$$

where the probability appearing in the terms on the right is the product of the probabilities of the independent events appearing therein and was, in my paper, evaluated on the assumption of the Poisson law.

In order to complete the calculations required by relation (1) it is necessary to set values on the prior probabilities $\Pr.(B_i)$ ($i=1, 2, \ldots, n$). Adopting

$$\Pr.(B_i) \equiv \Pr.(L) = 1/E, \qquad (2)$$

and making the appropriate substitutions indicated above in relation (1), the formula of my 1947 paper results.

It will be noticed that the problem was posed in terms of E policies exposed to risk and y policy-claims and that L, the number of lives involved, does not appear in the solution or in the sampling process implied.

Although the Poisson law was used in my paper for the probability distribution of deaths, this was purely for mathematical convenience and no theoretical justification was attempted. The fact of the matter is that, in general, neither binomial nor Poisson law is correct, either of them being but more or less crude approximations to the rather complex distribution law implied by actuarial practice in the collection of mortality observations (*Skand. Act.* 1948, pp. 14–45).

With regard to the asymptotic formulae derived in my 1947 paper I refer interested readers to chap. 11 of Feller's *An Introduction to Probability Theory and its Applications* (reviewed in *J.I.A.* LXXVII, 324) where the leading term method is developed and illustrated. Note also that in the derivation of the Poisson law as the limit of the binomial, q is assumed to tend to zero and thus the further terms of an asymptotic expansion involving increasingly higher powers of q are *a fortiori* of lower order than the leading term.

Finally, it may be mentioned that it is possible to re-graduate the distributions of duplicate policies given in my paper to provide a much improved fit. These re-graduations necessitate only small changes in the parameters of the discrete Pareto laws and do not affect by one iota the numerical examples of claim distributions I chose to illustrate the theory.

Yours faithfully,

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