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A counter-example to a C^2 closing lemma

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Abstract Let M be a compact manifold that contains a two-dimensional punctured torus Given $p \in M$ and an integer $r \geq 2$, there exists $X \in \mathfrak{X}^\infty(M)$ having non-trivial recurrent trajectories and such that, for some neighbourhood $\mathcal{U}$ of $X|_{(M-\{p\})}$ in $\mathfrak{X}^r(M-\{p\})$, no $Y \in \mathcal{U}$ has closed orbits

1 *Introduction*

The C^r-Closing Lemma Problem is very important because it is strongly related to the extension, to classes of differentiability $r \geq 2$, of the C^1-General Density Theorem and the Stability Conjecture We shall explain this

Let N be a manifold, $X \in \mathfrak{X}^1(N)$, $p \in N$ and γ_p be the trajectory passing through the point p, we say that $x \in \{p, \gamma_p\}$ is *non-wandering* if there exists a sequence of real numbers $t_n \to \infty$ and a sequence of points of the manifold $p_n \to p$ such that either $X_{t_n}(p_n) \to p$ or $X_{(-t_n)}(p_n) \to p$, where $X_t(t \in \mathbb{R})$ is the flow induced by X If $p_n = p$ in this definition and moreover $X_t(p) \neq p$, for all $t \in \mathbb{R} - \{0\}$, the non-wandering $x \in \{p, \gamma_p\}$ will be called *non-trivial recurrent*

The statement of the C^r-Closing Lemma Problem is as follows

Let M be a smooth compact manifold, $r \geq 2$ an integer, $f \in \mathrm{Diff}^r(M)$ (resp $X \in \mathfrak{X}^r(M)$) and p be a non-wandering point of f (resp of X) There is $g \in \mathrm{Diff}^r(M)$ (resp $Y \in \mathfrak{X}^r(M)$) arbitrarily close to f (resp to X) in the C^r-topology so that p is a periodic point of g (resp of Y)

C Pugh proved not only the C^1-Closing Lemma but also that if, for a given $r \geq 1$, the C^r-Closing Lemma had a positive answer, then *generically*, in the C^r-topology, vector fields would have the property that the union of their closed orbits and singularities is dense in their non-wandering set **[Pg.2]** Relevant to this is a partial claim of the Stability Conjecture This generic property must be satisfied for vector fields whose topological orbit structure is preserved by small C^r-perturbations **[Pa-Sm]**

Nevertheless, C Pugh has already put in doubt the validity of the C^2-Closing Lemma **[Pg.1]** Our result strengthens these doubts We give a negative answer to this problem when a point is removed from the manifold The considered C^r-topology is the Whitney one which is defined as follows

Given a compact manifold M and a closed subset Λ of it, let $\| \ \|_r$ be a norm on $\mathfrak{X}^r(M)$ compatible with its C^r-topology and let $\{V_i | i \in \mathbb{N}\}$ be a locally finite open

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covering of $M-\Lambda$ When $\{\varepsilon_i\}$ varies among all possible sequences of positive real numbers, a fundamental system of neighbourhoods, of any given C^r vector field X on $M-\Lambda$, in the Whitney C^r-topology is the one formed by the open sets

$$\mathcal{U}(\{\varepsilon_i\})=\{Y\in\mathfrak{X}^r(M-\Lambda) \mid \|Y|_{\bar{V}_i}-X|_{\bar{V}_i}\|_r<\varepsilon_i\}$$

$\mathfrak{X}^r(M-\Lambda)$ will be the space of C^r vector fields on $M-\Lambda$ with the Whitney C^r-topology

The only result known about the C^r-Closing Lemma Problem, when $r\geq 2$, is that it is true for diffeomorphisms of the circle and for a large class of flows on the torus with non-trivial recurrence [**Gu.2**] Now we state our result

THEOREM A *Let M be a compact manifold that contains a two-dimensional punctured torus Given $p\in M$ and an integer $r\geq 2$, there exists $X\in\mathfrak{X}^\infty(M)$ having non-trivial recurrent trajectories and such that, for some neighbourhood $\mathcal{U}$ of $X|_{(M-\{p\})}$ in $\mathfrak{X}^r(M-\{p\})$, no $Y\in\mathcal{U}$ has closed orbits*

Given $X\in\mathfrak{X}^1(M)$ and a non-wandering point q, Pugh's C^1-Closing Lemma states that, for any neighbourhood V of q and for any neighbourhood $\mathcal{U}$ of X in $\mathfrak{X}^1(M)$ there exists $Y\in\mathcal{U}$ having a closed trajectory through q and such that $X|_{(M-V)}=Y|_{(M-V)}$ Therefore, for vector fields on compact manifolds, as opposed to the case of the C^1-Closing Lemma, a positive answer to the C^2-Closing Lemma is not always possible by local perturbations around the non-wandering point

We wish to mention some C^r-Closing Lemma type results One is the Peixoto's C^r-Connecting Lemma which was used to characterize structurally stable vector fields on two-manifolds [**Pe**] Another is Mañé's C^1-Ergodic Closing Lemma that was utilized to characterize structurally stable diffeomorphisms of two-manifolds [**Ma**] Moreover, we have the Takens C^1-Connecting result which was used to prove generic properties in conservative systems [**Ta**] Finally, the Pixton–Robinson C^r-Connecting result for diffeomorphisms of the sphere S^2 which is a positive answer in the direction of the C^r-Closing Lemma [**Px**]

In the context of Bifurcation Theory [**So**], [**N-P-T**], the following question has been asked 'Given a hyperbolic saddle point, of a smooth flow (or diffeomorphism), such that its unstable manifold accumulates on itself, is it always possible to produce, via a small C^r-perturbation of the flow, $r\geq 2$, a homoclinic orbit of this saddle point?'

A negative answer to this question (on punctured manifolds) can be obtained by modifying, locally around one of its hyperbolic saddle points, the example of Theorem A In the example of Theorem A, no unstable manifold of any given hyperbolic point accumulates on itself See also [**Pg.3**]

Definitions used in this work can be found in [**Me-Pa**]

It will be seen that it is enough to prove Theorem A when the manifold is the bidimensional torus § 2 is devoted to providing a general idea of the paper, relevant to this we prove Theorem 2 1 which deals with a simpler example (on an infinitely punctured torus T^2) but contains the main ideas of the proof of Theorem A §§ 3, 4 provide the proof of two results used in § 2 Theorem A is proved in § 5

2 *The idea of the proof of Theorem* A

It will be convenient to introduce the following notation

σ will be the *golden mean* which is the positive root σ of the equation

$$\sigma^2+\sigma-1=0$$

$\{q_n/q_{n+1}\}$ will be the sequence of *principal convergents* of σ, that is $q_0=q_1=1$ and, for $n\geq 1$, $q_{n+1}=q_n+q_{n-1}$

Real numbers will be used to also *denote elements* of $\mathbb{R}/\mathbb{Z}$

$\{\tau(i)\}$ will be the sequence defined as follows $\tau(0)=\tau(1)=0$ and, for $i>1$, $\tau(i)=m$ provided that $q_m<i\leq q_{m+1}$

$\theta=\theta(\alpha)=\sum_i \alpha^{\tau(|i|)}$ It is observed that if $\alpha\in(0,\frac{1}{2})$, then $3<\theta(\alpha)<4$ (Lemma 4 1)

h $C\to\mathbb{R}/\mathbb{Z}$ will be the monotone continuous map of degree one such that $h(0)=0$ and

(1) For all $i\in\mathbb{Z}$, $h^{-1}(i\sigma)$ is an interval of length $\alpha^{\tau(|i|)}$ Here the point $i\sigma$ is to be taken modulo 1

(2) The union of all of the intervals $h^{-1}(i\sigma)$, with $i\in\mathbb{Z}$, is a full Lebesgue measure subset of the circle $C=\mathbb{R}/\theta\mathbb{Z}$

Υ will be the homeomorphism of the circle $C=\mathbb{R}/\theta\mathbb{Z}$ such that $\theta=\theta(0\ 01)$ and

(1) If R_σ $\mathbb{R}/\mathbb{Z}\to\mathbb{R}/\mathbb{Z}$ is the geometric rotation $x\to x+\sigma$ then, for all $y\in C$, $R_\sigma\circ h(y)=h\circ\Upsilon(y)$

(2) Υ is smooth away from $\{x|x$ or $\Upsilon(x)$ is an endpoint of $h^{-1}(0)\}$ and moreover, for all $i\in\mathbb{Z}$ such that $h^{-1}(i\sigma)$ and $h^{-1}((i+1)\sigma)$ have the same length, $\Upsilon'|_{h^{-1}(i\sigma)}\equiv 1$

$\mathscr{F}$ will be a continuous oriented foliation on the torus T^2 which is a suspension of the map Υ The foliation $\mathscr{F}$ is topologically equivalent to the example of class C^1 constructed by Denjoy [**De**] The arguments of [**Gu.1**, Smoothing Theorem] can be used to prove that $\mathscr{F}$ can be constructed to be smooth when restricted to $T^2-\{$endpoints of $h^{-1}(0)\}$

$\mathcal{N}$ will be the union of all intervals $h^{-1}(i\sigma)$ such that $i\in\mathbb{Z}$ and σi, considered as a point of $\mathbb{R}/\mathbb{Z}$, belongs to the closed subinterval of $\mathbb{R}/\mathbb{Z}-\{\frac{1}{2}\}$ with endpoints $-(q_{\tau(\tau(|i|))})\sigma$ and $(q_{\tau(\tau(|i|))})\sigma$ Observe that $\mathcal{N}$ is a closed subset of $\mathbb{R}/\mathbb{Z}$

X will be a smooth vector field, without singularities, tangent to $\mathscr{F}$ and defined on the manifold $M=T^2-\mathcal{N}$ The arguments of [**Gu.1**, Smoothing Theorem] imply the existence of such an X which has the additional property that can be extended to a smooth vector field on T^2 whose set of singularities is precisely $\mathcal{N}$

Dom (f) will be the domain of definition of a function f

The main result of this section is the following

2 1 THEOREM *If $\mathcal{U}$ is a neighbourhood of X, in $\mathfrak{X}^2(T^2-\mathcal{N})$, small enough, no vector field $Y\in\mathcal{U}$ has closed orbits*

To state Proposition 2 2, which is a fundamental step in the proof of Theorem 2 1, we shall use the following definitions

Let A and B be the transversal edges of a flow box of $\mathscr{F}$ Suppose that the foliation goes from A to B We say that $Y\in\mathfrak{X}^2(T^2-\mathcal{N})$ *properly connects A with B* if there is a homotopy of open segments $\lambda(t)$, $t\in[0,1]$, contained in $T^2-(A\cup B)$ such that

For all $t \in [0, 1]$, $\lambda(t)$ connects A and B Moreover, $\lambda(0)$ (resp $\lambda(1)$) is an arc of trajectory of X (resp of Y) going from A to B

The order of the integers determines in a natural way the *order '<' of the connected components* of $\mathcal{N}$ Given $n \in \mathbb{N}$ and a neighbourhood $\mathcal{U}$ of Y in $\mathfrak{X}^2(T^2 - \mathcal{N})$, we *say that Y is in $\mathcal{H}_n(\mathcal{U})$*, if $Y \in \mathcal{U}$ and any pair of consecutive connected components $h^{-1}(\sigma i) < h^{-1}(\sigma j)$ of $\mathcal{N}$, such that $-q_n \le i < j \le q_n$, are properly connected by Y

2 2 PROPOSITION *There exists a natural number n such that if $\mathcal{U}$ is a neighbourhood of X in $\mathfrak{X}^2(T^2 - \mathcal{N})$, small enough, and $Y \in \mathcal{U}$, then $Y \in \mathcal{H}_n(\mathcal{U})$ Moreover for all $k \ge n$, any $Z \in \mathcal{H}_k(\mathcal{U}) - \mathcal{H}_{k+1}(\mathcal{U})$ has no closed orbits*

Using this proposition, the proof of which is outlined in 2 8 but done in § 4, we shall proceed to give the proof of Theorem 2 1 The following will be needed
$\|x\| = \mathrm{Inf}_{p \in \mathbb{Z}} |x + p|$, where $x \in \mathbb{R}$ In this way $\|x\|$ defines a *metric on* $\mathbb{R}/\mathbb{Z}$
R_σ $\mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ will be the *geometrical rotation* $x \to x + \sigma$
$I(x, y)$ and $I[x, y]$ will be the *open and closed subintervals* of $\mathbb{R}/\mathbb{Z} - \{\frac{1}{2}\}$, respectively, with endpoints x and y
$\mathcal{U}$ will be a neighbourhood of X in the Whitney C^2-topology
$Q(i)$ will be $h^{-1}((|i|/i)(q_{|i|})\sigma)$, when $i \ne 0$ and $Q(0) = h^{-1}(0)$
$\mathcal{B}(A, B)$ will be a flow box of $\mathcal{F}$ having A and B as transversal edges and such that the foliation goes from A to B

2 3 LEMMA [**La**] *If $|q| > 0$ is an integer such that $|q| < q_{n+1}$, then $\|q\sigma\| \ge \|q_n\sigma\|$ Conversely, if $n \ge 1$, q_{n+1} is the smallest positive integer such that $\|q_{n+1}\sigma\| < \|q_n\sigma\|$*

Using this lemma and the fact that Y is semiconjugate to R_σ we conclude that

2 4 COROLLARY *Given a natural number $n > 3$, the union of $h^{-1}(I[0, q_n\sigma])$ and the arc of trajectory of $\mathcal{F}$ joining $0 \in Q(0)$ and $Q(n)$ contains a simple closed curve Γ_n which is uniquely determined Also Γ_n is non-null homotopic and so $T^2 - \Gamma_n$ is an open annulus*

We can introduce the following notation
κ_n $\mathbb{R}^2 - \{0\} \to T^2$ will be the covering map such that Γ_n, as in corollary above, is covered by closed curves going around the origin and the circle C transversal to $\mathcal{F}$ is covered by 'radial' curves starting at the origin See figure 2 1

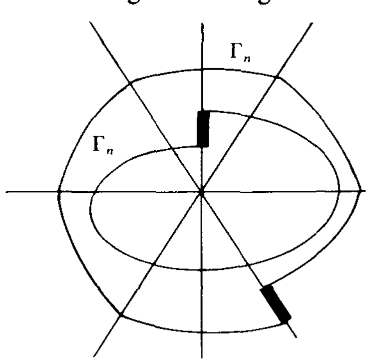

FIGURE 2 1

$\mathscr{C}_n$ will be the union of c_n and the two connected components of $\kappa_n^{-1}(\mathscr{B}(Q(0), Q(n))$ that meet c_n, where c_n is the closure of a connected component of $\kappa_n^{-1}(T^2-\Gamma_n)$ and Γ_n is as in the corollary above Observe that c_n is a fundamental domain See, in figure 2 2 n, the compact annulus $\mathscr{C}_n$

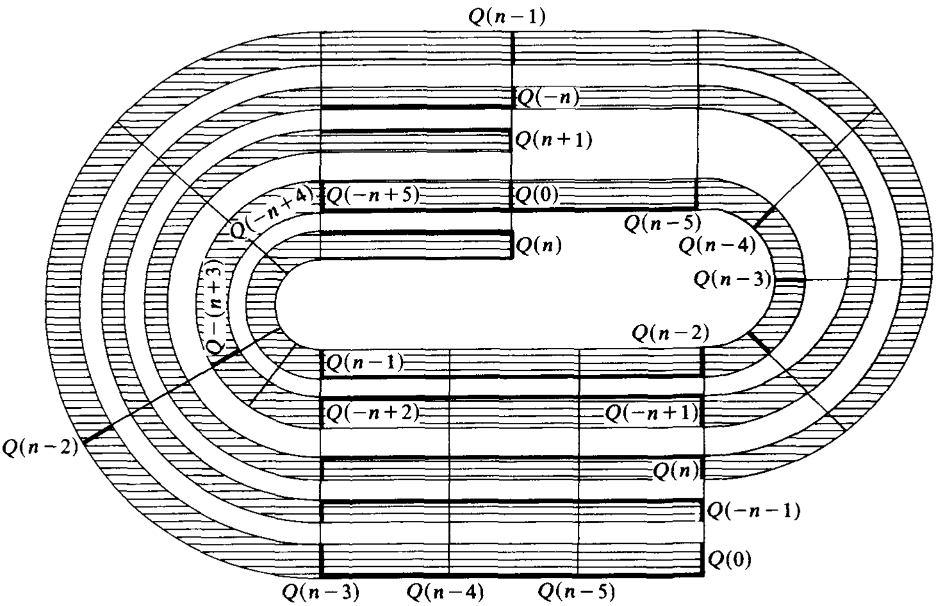

FIGURE 2 2 n

The proof of the following lemma will be given in § 3

2 5 LEMMA *The annulus $\mathscr{C}_n$ is – up to a homeomorphism – that of figure* 2 2 n *In this figure, the shaded strips are formed by $\mathscr{C}_n \cap \kappa_n^{-1}(\mathscr{B}(Q(-n-1), Q(n+1)))$, also, the segments that meet transversally these shaded strips are contained in $\kappa_n^{-1}(C)$*

2 6 *Remarks* If $\mathscr{U}$ is small enough and $Y \in \mathscr{U}$, then

(1) It will be seen in Lemma 4 2 that there exists an oriented continuous foliation $\mathscr{F}_Y$ defined *everywhere* on T^2, tangent to Y and of class C^2 when restricted to $T^2-\{\text{endpoints of } Q(0)\}$ Moreover, the forward Poincaré map $U = U_Y \; C \to C$ induced by $\mathscr{F}_Y$ is defined everywhere

(2) If $Y \in \mathscr{H}_k(\mathscr{U})$, for some k, there is a fundamental domain D of $\kappa_k \; \mathbb{R}^2-\{0\} \to T^2$ whose boundary is determined by the union of $h^{-1}(I[0, q_k\sigma])$, $\mathscr{N} \cap \mathscr{B}(Q(0), Q(k))$ and a selection of the minimal number of arcs of trajectory that are needed to connect all consecutive connected components of $\mathscr{N} \cap \mathscr{B}(Q(0), Q(k))$ See figure 2 3 k

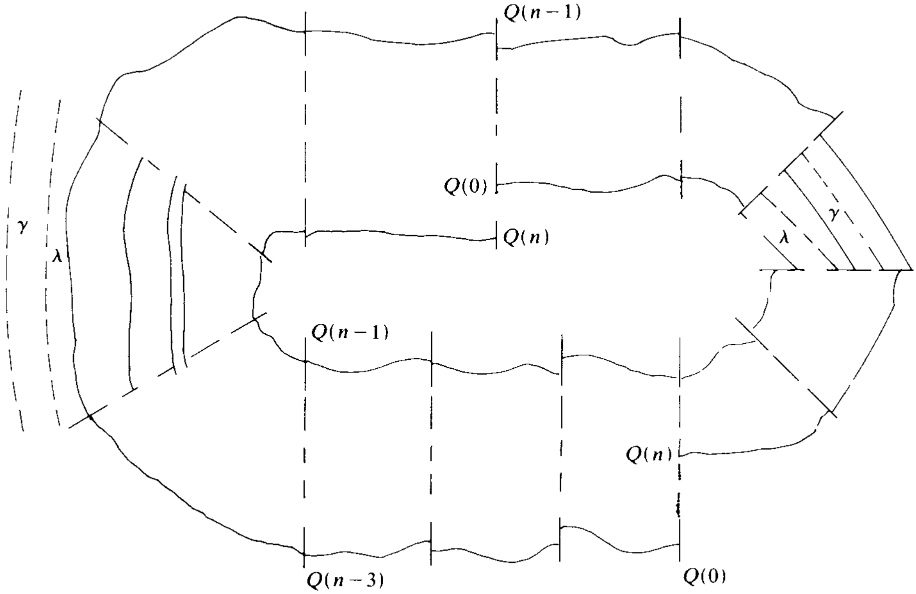

FIGURE 2 3 n

Now we can show the following

2 7 *Proof of Theorem* 2 1 It follows from Proposition 2 2, that for $Y \in \mathscr{U}$ to have closed orbits it will be necessary that, for all $k > n$, $Y \in \mathscr{H}_k(\mathscr{U})$ However, we claim that

(1) If $Y \in \mathscr{H}_k(\mathscr{U})$, for some k, any closed orbit of Y meets C at least q_k times In fact, $\kappa_k^{-1}(C) \cap \mathscr{C}_k$ has q_k connected components because, by Lemma 2 3, $\mathscr{B}(Q(0), Q(k)) \cap C$ does so By the remarks in 2 6 and since $Y \in \mathscr{H}_k(\mathscr{U})$, any closed trajectory of Y has to meet these q_k connected components

Therefore, it follows from the claim (1) above that Y cannot have closed orbits

We observe that the tables above contain the notation needed for the precise statements of the results of the next sections that are used in 2 8 below

2 8 *Outline of the proof of Proposition* 2 2 Certainly, given $n \in \mathbb{N}$, if $\mathscr{U}$ is small enough, $Y \in \mathscr{H}_n(\mathscr{U})$ Moreover, the example has been constructed so that the proof of Lemma 2 2 is the same for all given k. The required topological and metric symmetries are provided by Lemmas 2 5 and 4 3, respectively Roughly speaking Lemma 4 3 says that

(1) When k is large, $\mathscr{C}_k$ is very thin Also, the shaded strips of figure 2 2 k are much thicker than the strips in between them

The C^2-topology is only needed in Lemma 4 2 This lemma is put together with (1), and the remarks of 2 6 so that somewhere, as suggested in figure 2 3 k, there must be trajectories of $Y \in \mathscr{U}$ – the dotted curves γ and λ in that figure – that block any possibility of having closed orbits These trajectories γ and λ appear near to and as a consequence of the consecutive connected components of

$$(\mathscr{B}(Q(-k-1), Q(k)) \cup \mathscr{B}(Q(k), Q(k+1))) \cap \mathscr{N}$$

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that (by the assumption that $Y \in \mathcal{H}_k(\mathcal{U}) - \mathcal{H}_{k+1}(\mathcal{U})$) are not properly connected by Y

2 9 *Outline of the proof of Theorem* A We construct a vector field $\tilde{X}$ on $T^2 - Q(0)$ so that if B is a small neighbourhood of a connected component of $\mathcal{N}$, then the phase portrait of $\tilde{X}|_B$ is that of figure 5 1 Away from a small neighbourhood of $\mathcal{N} - Q(0)$ in $T^2 - Q(0)$ the phase portraits of X and $\tilde{X}$ are the same Moreover all singularities of $\tilde{X}$ are hyperbolic

Following basically the same argument as that of the proof of Theorem 2 1, we can find a neighbourhood of $\tilde{X}$, in $T^2 - Q(0)$, made up of vector fields (with singularities but) without closed orbits From this point it is not difficult to prove Theorem A when the manifold is the torus T^2

3 *The golden mean*

We shall recall some properties of the golden mean σ The proofs of these facts can be found in **[Cx]** or, in the more general context of continued fractions, in **[Her**, Ch V], **[La]** and **[Sl]**

3 1 LEMMA *Let $\mathbb{R}/\mathbb{Z} - \{\frac{1}{2}\}$ be provided with the orientation induced by that of $\mathbb{R}$ For all $n > 2$, the elements of the sequence $\{q_i\sigma\}$ are ordered in the oriented interval $\mathbb{R}/\mathbb{Z} - \{\frac{1}{2}\}$ as in figure* 3 1

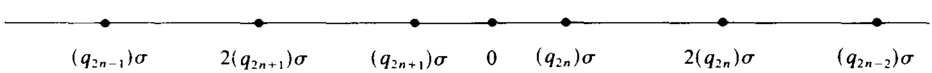

FIGURE 3 1

3 2 LEMMA *For all $n > 1$, the intervals $\{(R_\sigma)^j(I[0, q_n\sigma])|0 \le j < q_{n+1}\}$ and $\{(R_\sigma)^k(I[0, (q_{n+1})\sigma])|0 \le k < q_n\}$ cover $\mathbb{R}/\mathbb{Z}$, and, moreover, their interiors are pairwise disjoint*

3 3 LEMMA *The number σ and the sequence $\{q_n\}$ satisfy the following properties*

(i) *For all $n > 1$, $\|q_n\sigma\|/\|q_{n-1}\sigma\| = \sigma$*

(ii) *For all $n > 6$, $q_n > n + 7$*

As a direct consequence of this lemma, we have that

3 4 LEMMA *Let $R_{\sigma n}$ $I(-q_n\sigma, q_n\sigma) \to I(-q_n\sigma, q_n\sigma)$ be the map induced by R_σ, that is, $R_{\sigma,n}(x) = y$ if, for some positive integer j, $(R_\sigma)^j(x) = y$ and $(R_\sigma)^k(x)$ does not belong to $I(-q_n\sigma, q_n\sigma)$, for $1 \le k < j$*

Then $R_{\sigma,n}$ is a piecewise orientation preserving isometry that satisfies the following

(i) *Its graph is that of figure 3 2, in particular,*

$$R_{\sigma n}(I(0, -q_n\sigma)) = I(0, q_n\sigma), \qquad R_{\sigma n}(I[0, -q_{n+1}\sigma]) = I[0, q_{n+1}\sigma]$$

and

$$R_{\sigma n}(I(-q_{n+1}\sigma, q_n\sigma)) = (q_n\sigma, q_{n+1}\sigma)$$

(ii) *Up to re-scaling, via the linear map $x \to -\sigma x$, the maps $R_{\sigma,n}$ and $R_{\sigma,n+1}$ are the same, for all $n > 1$*

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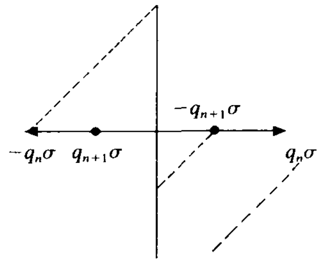

FIGURE 3 2

3 5 *Proof of Lemma* 2 5 It follows from Lemmas 2 3 and 3 1 that the two connected components of $\kappa_n^{-1}(C)\cap\mathscr{C}_n$ meeting $Q(0)$ are precisely $h^{-1}(I[q_n\sigma, q_{n-1}\sigma])$ and $h^{-1}(I[0, q_{n-2}\sigma])$ and, moreover, that these components meet $\mathscr{B}(Q(-n-1), Q(n+1))$ as in figure 2 2 *n* Since $q_{n-2}+q_{n-1}=q_n$, $\mathscr{B}(Q(-n+1), Q(0))$ meets these components at $Q(0)$ and $Q(-n+1)$ only and so it is localized as in figure 2 2 *n* With these type of arguments, the proof of the lemma can easily be completed

4 *Proof of Proposition* 2 2

To prove this proposition we shall need some lemmas

4 1 LEMMA *Let* $\alpha\in(0,\frac{1}{2})$ *Then*

$$\theta=\theta(\alpha)=3+\frac{2\alpha}{1-\alpha-\alpha^2}$$

Proof First we shall prove that

(1) $S=S(\alpha)=\sum q_n\alpha^n=1/(1-\alpha-\alpha^2)$

In fact, let $u=\sum\alpha^n=1/(1-\alpha)$ By induction we may easily see that, for all $n\in\mathbb{N}$, $q_n<2^n$ Therefore S is absolutely convergent and it can be re-written as

$$\begin{aligned} S &= u+\alpha^2u+\alpha^3u+\quad +(q_n-q_{n-1})\alpha^nu+(q_{n+1}-q_n)\alpha^{n+1}u+ \\ &= u(1+\alpha^2+\alpha^3+\quad +q_{n-2}\alpha^n+q_{n-1}\alpha^{n+1}+\quad) \\ &= u+u\alpha^2S \end{aligned}$$

This implies that $S=1/(1-\alpha-\alpha^2)$ The lemma follows immediately because

$$\theta-1=2(1+\alpha+\alpha^2+q_2\alpha^3+\quad +q_{n-1}\alpha^n+\quad)$$

Although the proof of next lemma is contained in the proofs of Lemmas 5 2 and 5 3, part (i) will be proved now

4 2 LEMMA *Let* $\varepsilon=0\ 1$ *If* $\mathscr{U}$ *is small enough and* $Y\in\mathscr{U}$, *then*

(i) *there exists an oriented continuous foliation* $\mathscr{F}_Y$ *defined everywhere on* T^2, *tangent to* Y *and of class* C^2 *when restricted to* $T^2-\{$*endpoints of* $h^{-1}(0)\}$

(ii) *Let* $U=U_Y$ $C\to C$ *be the forward Poincaré map induced by* $\mathscr{F}_Y$ and $n\in\mathbb{N}$ *For all points p, q, s, t belonging to a connected component of* Dom (U_n) *of length less than* 2^{-n}, *the following is satisfied*

$$\left|\frac{\|(U_n)(p)-(U_n)(q)\|}{\|p-q\|}-\frac{\|(U_n)(s)-(U_n)(t)\|}{\|s-t\|}\right|<\left(\frac{(1+\varepsilon)^2}{2}\right)^n$$

where $\delta\in\{-1,1\}$, $\hat{D}=C-(\mathrm{Sp}(D_\delta)\cup\mathrm{Sp}(D_{-2\delta}))$, U_n *is either* $(U^\delta|_{\hat{D}})^n$ *or* $(\Upsilon^{-\delta})\circ U^\delta\circ(U^\delta|_{\hat{D}})^{n-1}$, *and* $\mathrm{Sp}(D_{\delta k})$, *with* $k\in\{1,2\}$, *is the union of* $\bigcup\{h^{-1}(\delta(k-1+q_n)\sigma)|n\in\mathbb{N}-\{0,1,2\}\}$ *and* $\{x|\Upsilon^{k-1}(x)$ *is an endpoint of* $h^{-1}(0)\}$

Proof of (i) Let L_X be the unit tangent vector field to $T^2-\{\text{endpoints of } h^{-1}(0)\}$ induced by $\mathcal{F}$ Let $\{V_i\}_{i\geq 1}$ be a locally finite open covering of $T^2-\mathcal{N}$ and let $\{\psi_i\}$ be a smooth partition of unity strictly subordinate to $\{V_i\}$ The neighbourhood $\mathcal{U}$ has the form

$$\mathcal{U}=\{Y\in\mathfrak{X}^2(T^2-\mathcal{N})\,|\,\|Y|_{V_i}-X|_{V_i}\|_2\leq\varepsilon_i\},$$

where $\|\ \|_2$ is the uniform C^2-norm on $\mathfrak{X}^2(T^2)$ and $\{\varepsilon_i\}$ is a sequence of positive real numbers

Given $Y\in\mathcal{U}$, we may define on $T^2-\{\text{endpoints of } h^{-1}(0)\}$

$$L_0=L_X\quad\text{and}\quad L_i=\sum_{k=1}^{i}\psi_k(\tilde{L}_Y-L_X)+L_X,$$

where $\tilde{L}_Y$ is the unit tangent vector field induced by Y and defined on $T^2-\mathcal{N}$

Let W be an arbitrary open neighbourhood of $\{\text{endpoints of } h^{-1}(0)\}$ Since the support of ψ_k is contained in V_k, we may take the terms of the sequence $\{\varepsilon_i\}$ so small that

$$\left\|\sum_{k=1}^{\infty}\psi_k(\tilde{L}_Y-L_X)|_{(T^2-W)}\right\|_2\leq\sum_{k=1}^{\infty}\|\psi_k\|_2\|\tilde{L}_{Y|_{V_k}}-L_{X|_{V_k}}\|_2<\infty,$$

where $\|\ \|_2$ is also the uniform C^2-norm on the space of real valued C^2-maps on T^2

This implies that $\{L_i\}$ converges to a vector field L_Y of class C^2 which, as each L_i, is defined on $T^2-\{\text{endpoints of } h^{-1}(0)\}$ Certainly, L_Y restricted to $T^2-\mathcal{N}$ is precisely $\tilde{L}_Y$ Under these conditions, it is easy to see that L_Y induces a foliation $\mathcal{F}_Y$ as required to prove part (i) of this lemma

4 3 Lemma *Given* $n>0$, *let* $\mu_n=\|h^{-1}(I(0,q_n\sigma))\|$ *Then, for all* $n>1$, $\mu_n=\alpha^{n-1}\mu_1$, *and therefore,*

$$\|h^{-1}(I(q_{n-1}\sigma,q_n\sigma))\|=\mu_{n-1}-\mu_n-\alpha^{n-1}=\alpha^{n-2}(\mu_1-\mu_2-\alpha)$$

Proof By the way that the length of the intervals $h^{-1}(i\sigma)$, $i\in\mathbb{Z}$, have been chosen and the fact that Υ is semi-conjugate to R_σ, the lemma follows immediately from Lemma 4 1 and the re-scaling property of Lemma 3 4

4 4 Lemma *If* $\mathcal{U}$ *is small enough, then for all* $n>5$ *such that* $\alpha^{n-6}<2^{-n}$ *and* $((1+\varepsilon)^2/2)^n<\alpha^6$, *where* $\varepsilon=0\ 1$ *and* $\alpha=0\ 01$, *any* $Y\in\mathcal{H}_n(\mathcal{U})-\mathcal{H}_{n+1}(\mathcal{U})$ *has no orbits*

Proof We shall use the following notation Given $x,y\in C$ such that $\|x-y\|<\theta/2$, the subinterval of C having x and y as endpoints and such that its length is less than $\theta/2$, *will be denoted by* xy Let

$$\Lambda_{n1}=\mathcal{B}(h^{-1}(I[0,-(q_{n-1})\sigma]),h^{-1}(I[(q_{n-1})\sigma,0]))$$

$$\Lambda_{n2}=\mathcal{B}(h^{-1}(I[0,-q_n\sigma]),h^{-1}(I[0,q_n\sigma]))$$

$$\Lambda_{n3}=\mathcal{B}(h^{-1}(I[(q_{n-1})\sigma,-q_n\sigma]),h^{-1}(I[q_n\sigma,-(q_{n-1})\sigma]))$$

See these flow boxes in figure 4 1

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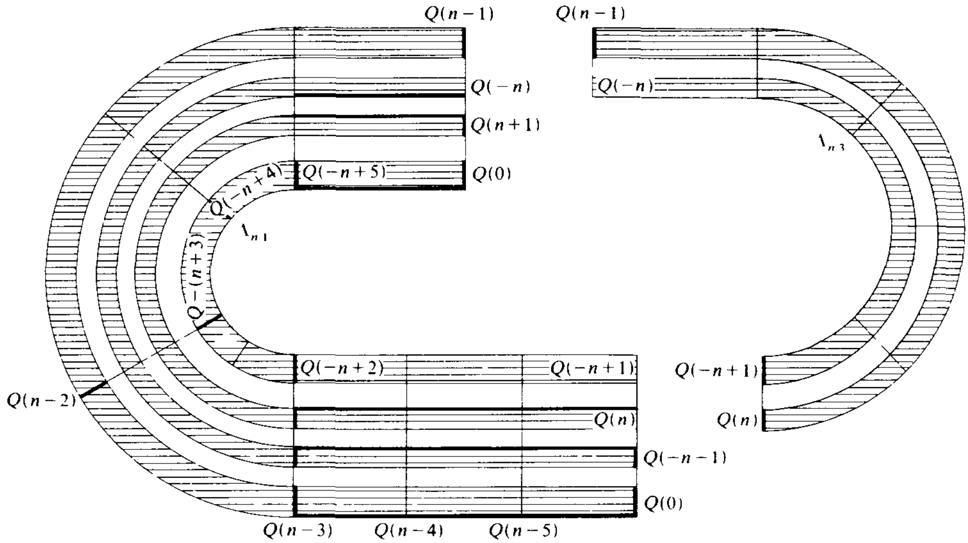

FIGURE 4 1

Since $Y \in \mathscr{H}_n(\mathscr{U})$ it is easy to check that when either of the following two conditions is satisfied the lemma is true

(1 1) $Y|_{T^2-\mathcal{N}}$ does not properly connect the transversal edges of $\Lambda_{n,1}$, or

(1 2) $Y|_{T^2-\mathcal{N}}$ does not properly connect either the transversal edges of $\Lambda_{n\,2}$ or the transversal edges of $\Lambda_{n\,3}$

Since $Y \in \mathscr{H}_n(\mathscr{U}) - \mathscr{H}_{n+1}(\mathscr{U})$,

(2) There are two consecutive connected components, $\bar{a}_j\bar{b}_j$ and a_jb_j, of $\mathcal{N}$ which belong to $\mathscr{B}(Q(n),\ Q(n+1)) \cup \mathscr{B}(Q(-n-1),\ Q(-n))$ that are not properly connected by Y (At the moment, the index 'j' is unnecessary but it will change with the considerations just before (5), below)

This gives rise to the following three alternatives

(3) $\bar{a}_j\bar{b}_j$ and a_jb_j are necessarily contained in one of the following flow boxes whose union covers T^2

(3 1) $\mathscr{B}(h^{-1}(I(-(q_{n-4})\sigma,\ -2(q_{n-4})\sigma),\ h^{-1}(I(0,\ -(q_{n-4})\sigma)))$

(3 2) $\mathscr{B}(h^{-1}(I(0,\ (q_{n-4})\sigma)),\ h^{-1}(I((q_{n-4})\sigma,\ 2(q_{n-4})\sigma))$

(3 3) $\mathscr{B}(h^{-1}(I(0,\ -(q_{n-4})\sigma)),\ h^{-1}(I(0,\ (q_{n-4})\sigma)))$

If *alternative* (3 1) occurs, that is if $\bar{a}_j\bar{b}_j$, and a_jb_j are contained in the flow box

$$\mathscr{B}(h^{-1}(I(-(q_{n-4})\sigma,\ -2(q_{n-4})\sigma),\ h^{-1}(I(0,\ -(q_{n-4})\sigma)))$$

of figure 4 2, then, by definition of $\mathcal{N}$

(4) $\bar{a}_j\bar{b}_j$, and a_jb_j are also contained in the subinterval $h^{-1}(I[-(q_{\tau(\tau(|i|))})\sigma,\ (q_{\tau(\tau(|i|))})\sigma])$ of C, which crosses the flow box

$$\mathscr{B}(h^{-1}(I(-(q_{n-4})\sigma,\ -2(q_{n-4})\sigma),\ h^{-1}(I(0,\ -(q_{n-4})\sigma))),$$

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along global cross sections, determining a partition of it into sub-flow boxes. The flow box $\mathscr{B}(\bar{a}_1\bar{b}_{11}, a_1b_{11})$ of figure 4.2 is a typical sample of such sub-flow boxes, moreover, by definition of $\mathcal{N}$, for all $i \in \{1, 2, \ldots, 11\}$, the segments a_ib_i and $\bar{a}_i\bar{b}_i$ of figure 4.2 are consecutive connected components of $\mathcal{N}$ (i.e. holes of the punctured torus).

Therefore, we may assume that $\bar{a}_j\bar{b}_j$ and a_jb_j, considered in (2), are contained in the transversal edges of the flow box $\mathscr{B}(\bar{a}_1\bar{b}_{11}, a_1b_{11})$ that was described in (4). By letting j vary in the set $\{2, 3, 6, 7\}$, we obtain all possibilities. See figure 4.2.

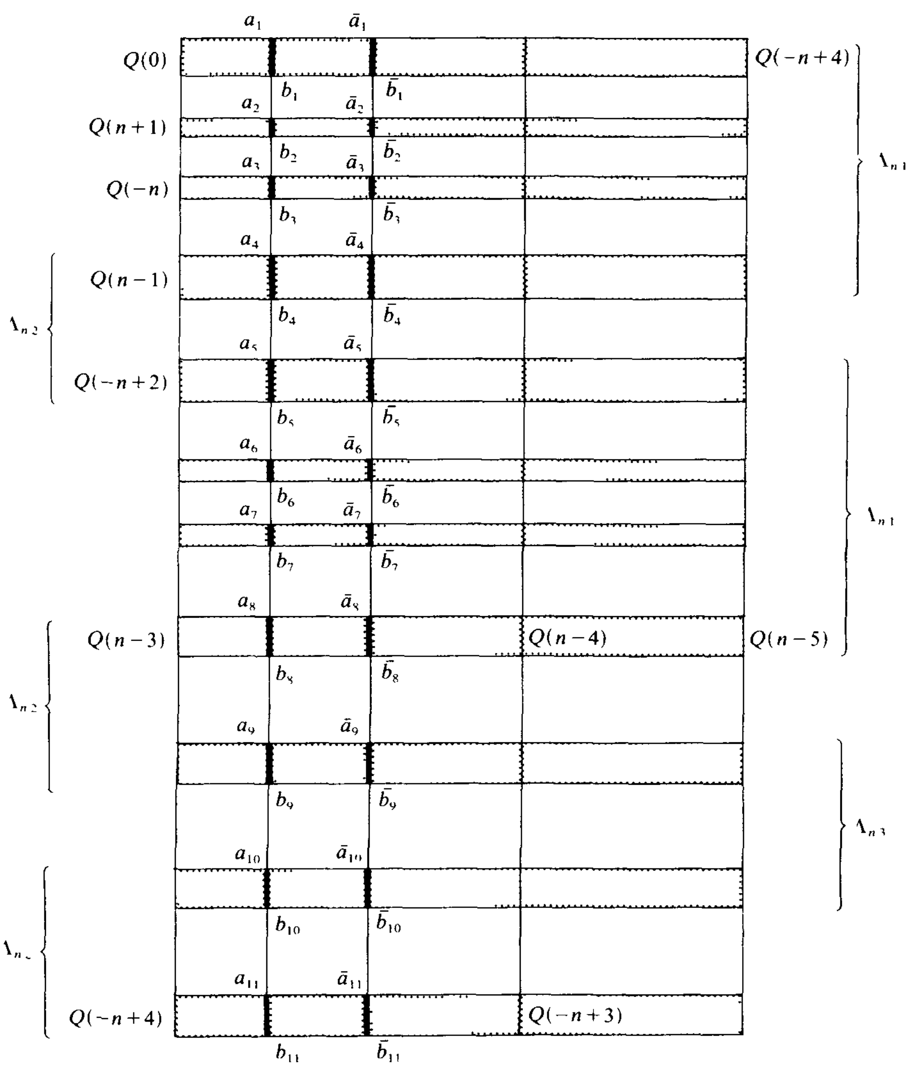

FIGURE 4.2. See in figure 4.3 how this flow box is situated in $\mathscr{C}_n$.

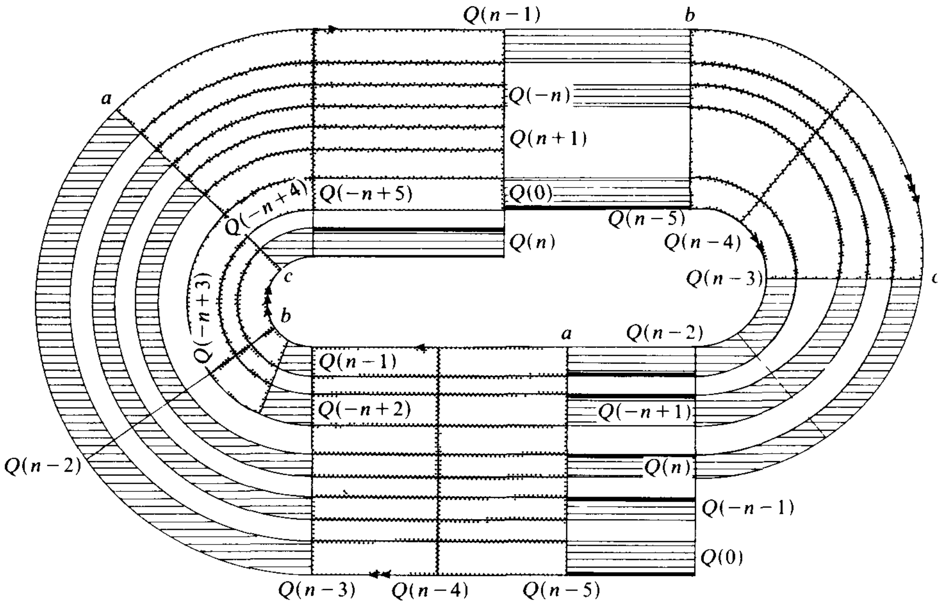

FIGURE 4 3

Using the properties of X and Lemma 4 3, it is easy to check (5)-(7) below

(5) $\|a_1b_1\| \geq \alpha^{n-5}$, $\|\bar{a}_1\bar{b}_1\| \geq \alpha^{n-5}$,

$\|a_{11}b_{11}\| \geq \alpha^{n-4}$, $\|\bar{a}_{11}\bar{b}_{11}\| \geq \alpha^{n-4}$,

$\|a_8b_8\| = \|\bar{a}_8\bar{b}_8\| = \alpha^{n-4}$ if $\bar{a}_8\bar{b}_8 \subset \mathscr{B}(Q(n-4), Q(n-3)) - Q(n-4)$,

$\|a_8b_8\| = \|\bar{a}_8\bar{b}_8\| = \alpha^{n-5}$

if $\bar{a}_8\bar{b}_8 \subset \mathscr{B}(Q(n-5), Q(n-4)) - (Q(n-5) \cup Q(n-4))$,

$\|a_2b_2\| = \|\bar{a}_2\bar{b}_2\| = \|a_3b_3\| = \|\bar{a}_3\bar{b}_3\| = \|a_6b_6\| = \|\bar{a}_6\bar{b}_6\| = \|a_7b_7\| = \|\bar{a}_7\bar{b}_7\| = \alpha^n$,

$\|a_4b_4\| = \|\bar{a}_4\bar{b}_4\| = \alpha^{n-2}$, $\|a_5b_5\| = \|\bar{a}_5\bar{b}_5\| = \alpha^{n-3}$,

$\|a_9b_9\| = \|\bar{a}_9\bar{b}_9\| = \alpha^{n-1}$, $\|a_{10}b_{10}\| = \|\bar{a}_{10}\bar{b}_{10}\| = \alpha^{n-2}$

(6) For all $i \in \{1, 2, \quad, 10\}$, $\|b_ia_{i+1}\| = \|\bar{b}_i\bar{a}_{i+1}\|$

(7) $\|b_4a_5\| < 3\alpha^{n+1}$, $\sum_{i\neq 4} \|b_ia_{i+1}\| < \alpha^{n+1}$

It follows from Lemma 3 2 and the definition of $\tau(n)$ that

(8) Any arc of trajectory of $\mathscr{B}(\bar{a}_1\bar{b}_{11}, a_1b_{11})$ meets C at most n times

Now we shall prove that

(9) If (1 1) is not satisfied, then (1 2) is true In other words, when alternative (3 1) occurs, the lemma is true

Since $Y \in \mathscr{H}_n(\mathscr{U})$, (9) implies that for Y to have closed orbits it is necessary that Y properly connect both $\bar{b}_1\bar{a}_4$ with b_1a_4 and $\bar{b}_5\bar{a}_8$ with b_5a_8 This leads to some possibilities all of which are similar to (10) below The argument is essentially the same in all cases We shall only prove

(10) The assumption (2) is satisfied for $j=2$, also, $Y \in \mathscr{U}$ properly connects $\bar{b}_1\bar{a}_4$ with b_1a_4 by an arc of trajectory that joins $\bar{b}_2\bar{a}_3$ with b_1a_2, and moreover, Y properly connects $\bar{b}_6\bar{a}_7$ with b_6a_7

First, we shall also assume that

(11) $\mathscr{B}(\bar{a}_1\bar{b}_{11}, a_1b_{11})$ does not meet either $Q(n-4)$ or the transversal edges of $\Lambda_{n\,1}$

Under these conditions, we claim that

(12) If $\mathscr{U}$ is as in Lemma 4 2 then Y does not properly connect either $\bar{b}_9\bar{a}_{10}$ with b_9a_{10} or $\bar{b}_{10}\bar{a}_{11}$ with $b_{10}a_{11}$

In fact, Let $U = U_Y$ $C \to C$ and $\pi = \pi_Y$ $\bar{a}_1\bar{b}_{11} \to a_1b_{11}$ be the forward Poincaré maps induced by Y Suppose that $U^k = \pi$ Then there are points $p_1 \in \bar{b}_2\bar{a}_3$ and $p_2 \in \bar{b}_6\bar{a}_7$ belonging to Dom (π) and such that $\pi(p_1) \in b_1a_2$ and $\pi(p_2) \in b_6a_7$ If we assume that there exists $p_3 \in \bar{b}_9\bar{a}_{10}$ belonging to Dom (π), then $\pi(p_3)$ belongs to b_9a_{10} because, as $Y \in \mathscr{H}_n(\mathscr{U})$, any two consecutive connected components of $\mathscr{N}$ belonging to either $\mathscr{B}(\bar{a}_9\bar{b}_9, a_9b_9)$ or $\mathscr{B}(\bar{a}_{10}\bar{b}_{10}, a_{10}b_{10})$ are properly connected by Y

Before proceeding, observe that we may apply Lemma 4 2 to points of the interval p_1p_3 In fact, $\|p_1p_3\| < \alpha^{n-6} < 2^{-n}$ and, since $Y \in \mathscr{H}_n(\mathscr{U})$ and $\bar{b}_1\bar{a}_{11}$ is contained in $(\pi_X)^{-1}(C - ((\mathrm{Sp}\,(D_1) \cup \mathrm{Sp}\,(D_{-2}))))$, we have that p_1p_3 is contained in the domain of definition of

$$(U|_{U^{-1}(C-((\mathrm{Sp}\,(D_1)\cup \mathrm{Sp}\,(D_{-2}))))})^k$$

Using (6), (7) and (8) and assuming that $\bar{a}_8\bar{b}_8 \subset \mathscr{B}(Q(n-4), Q(n-3)) - Q(n-4)$, it can be seen that

$$\|\pi(p_1)\pi(p_2)\| > 3\alpha^n + \alpha^{n-2} + \alpha^{n-3},$$
$$\|p_1p_2\| < 4\alpha^{n+1} + 2\alpha^n + \alpha^{n-2} + \alpha^{n-3},$$
$$\|\pi(p_2)\pi(p_3)\| < 4\alpha^{n+1} + \alpha^n + \alpha^{n-4} + \alpha^{n-1},$$
$$\|p_2p_3\| > \alpha^n + \alpha^{n-4} + \alpha^{n-1}$$

Therefore, as $\alpha = 0\,01$,

$$\frac{\|\pi(p_1)\pi(p_2)\|}{\|p_1p_2\|} > 1 + \frac{\alpha^3}{4},$$
$$\frac{\|\pi(p_2)\pi(p_3)\|}{\|p_2p_3\|} < 1 + 2\alpha^5,$$
$$\frac{\|\pi(p_1)\pi(p_2)\|}{\|p_1p_2\|} - \frac{\|\pi(p_2)\pi(p_3)\|}{\|p_2p_3\|} > \frac{\alpha^3}{8} > \alpha^5$$

However, this last inequality is not possible by (8), by Lemma 4 2 and the assumptions of this lemma Similarly, Y does not properly connect $\bar{b}_{10}\bar{a}_{11}$ with $b_{10}a_{11}$ This proves (12)

The assumption (11) is unnecessary For instance if $\bar{b}_1\bar{a}_{11}$ meets $Q(n-4)$, we use Lemma 4 2 in the case that $U_n = U_n(Y) = Y \circ \pi^{-1}$ This proves the lemma when alternative 3 1 is satisfied Up to orientation of $\mathscr{F}$, alternatives 3 1 and 3 2 are the same (See figure 4 4) The proof of the lemma following alternative 3 3 is similar to the case considered See figures 4 5 and 4 6

4 5 *Proof of Proposition* 2 2 It follows directly from Lemma 4 4

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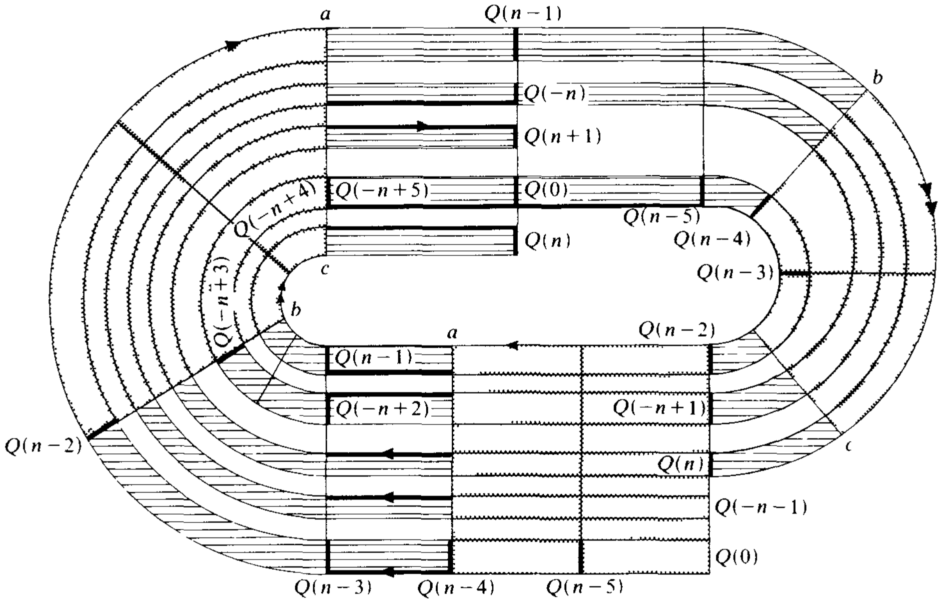

FIGURE 4 4

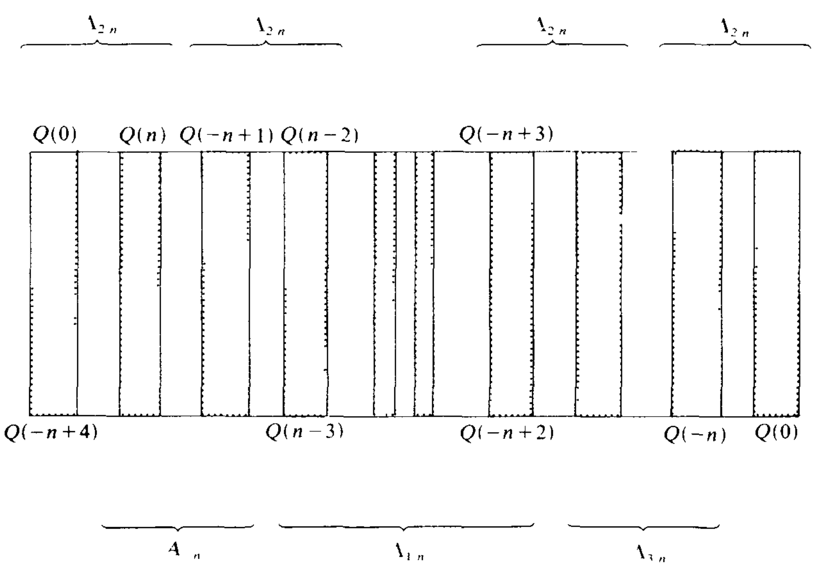

FIGURE 4 5 See in figure 4 6 how this canonical rectangle is situated in $\mathscr{C}_n$

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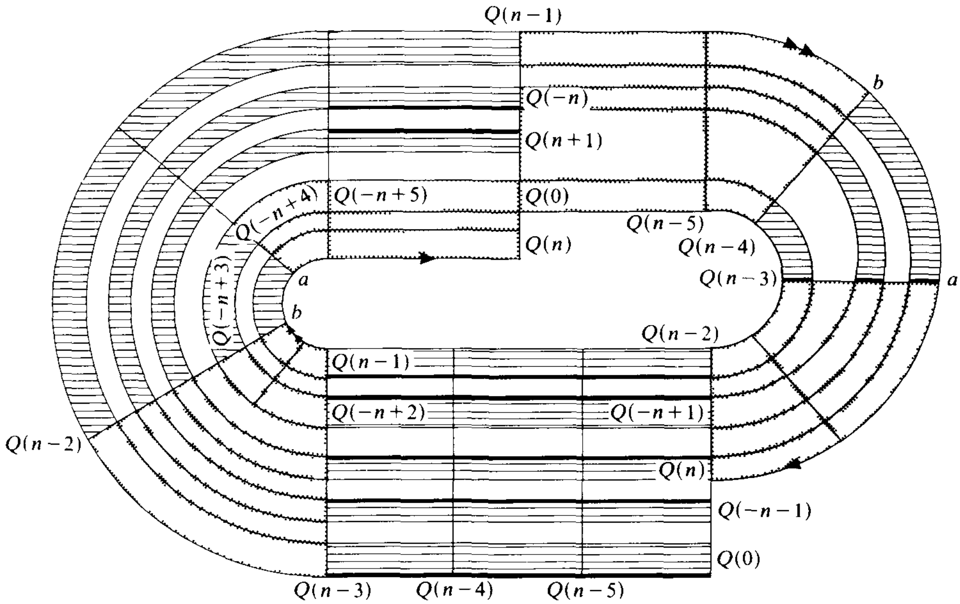

FIGURE 4 6

5 *Proof of Theorem* A

This section will be devoted to the proof of our main result which will be obtained as a corollary of Theorem 5 1 below

A *canonical rectangle of a flow* Y on T^2 will be a rectangle (closed disk) R such that two of its opposite edges are arcs of trajectory of Y and its other two opposite edges, transversal to Y, are one of *entrance* to R and the second of *exit* from R

To define the vector field X that appears in the statement of Theorem 5 1 and for later use, we shall introduce the following notation

Y will be as in § 2 with the following additional property For $\delta \in \{1, -1\}$, $(\mathrm{Y}^{\delta})'|_{(C-(D_{\delta}\cup D_{(-2\delta)}))} \equiv 1$, where

$D_{\delta k}$, with $k \in \{1, 2\}$ is the union of $\bigcup\{\Sigma(\delta(k-1+q_n)\sigma) | n \in \mathbb{N}-\{0,1,2\}\}$ and $\{x | \mathrm{Y}^{k-1}(x)$ is an endpoint of $h^{-1}(0)\}$, and

$\Sigma(i)$, with $i \in \mathbb{Z}$, is an open subinterval of C whose closure is contained in $h^{-1}(i\sigma) -$ {endpoints of $h^{-1}(i\sigma)$} and $\|h^{-1}(i\sigma)\| - \|\Sigma(i)\| < \|h^{-1}(i\sigma)\|^{30}$

$\mathcal{N}_{\delta k}$, with $\delta \in \{-1, 1\}$ and $k \in \{1, 2\}$, will be the union of all intervals $\Sigma(\delta(k-1+i))$, when $i > 2$ and $\Sigma(\delta i) \subset \mathcal{N}$ Here δk is the product of δ times k

$\mathrm{Sp}\,(D_{\delta k})$, with $\delta \in \{-1, 1\}$ and $k \in \{1, 2\}$, will be the union of all intervals $h^{-1}(i\sigma)$ such that $\Sigma(i)$ is contained in $D_{\delta k}$

$\mathrm{Sp}\,(\mathcal{N}_{\delta k})$, with $\delta \in \{-1, 1\}$ and $k \in \{1, 2\}$, will be the union of all intervals $h^{-1}(i\sigma)$ such that $\Sigma(i)$ is contained in $\mathcal{N}_{\delta k}$

X will be a smooth vector field on T^2 such that

(1) X is transversal to $C -$ {endpoints of $h^{-1}(0)$}

(2) The forward Poincaré map $C \to C$ induced by X is a restriction of Y and its domain of definition contains $C - \mathrm{Sp}\,(\mathcal{N}_1 \cup \mathcal{N}_{(-2)})$

(3) Given a connected component $\Sigma(i)$ of $\mathcal{N}_1 \cup \mathcal{N}_{(-2)}$, there is a canonical rectangle B_i of X such that its transversal edges are the intervals $h^{-1}(i\sigma)$ and $h^{-1}((i+1)\sigma)$, and the phase portrait of $X|_{B_i}$ is that of figure 5 1, where the four singularities are hyperbolic, $\Sigma(i)$ is contained in the stable manifold of the sink of $X|_{B_i}$, $\Sigma(i+1)$ is contained in the unstable manifold of the source of $X|_{B_i}$ Observe that one of the transversal edges of B_i is a connected component of Sp $(\mathcal{N}_1 \cup \mathcal{N}_{(-1)})$

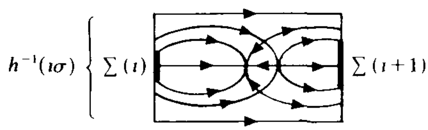

FIGURE 5 1

(4) Each B_i contains a canonical rectangle $\tilde{B}_i$ such that the set of singularities of $X|_{B_i}$ and the transversal edge of B_i meeting $\mathcal{N}_1 \cup \mathcal{N}_{(-1)}$ are contained in $\tilde{B}_i$ Moreover, the complement in T^2 of any neighbourhood of {endpoints of $h^{-1}(0)$} meets only finitely many $\tilde{B}_i$'s

(5) The only singularities of X that are not contained in the rectangles B_i as above are the endpoints of $h^{-1}(0)$ See figure 5 2

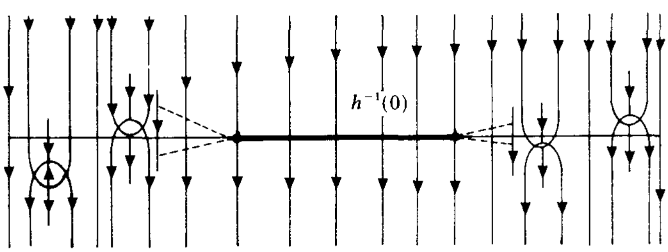

FIGURE 5 2

The arguments of [**Gu.1**, Smoothing Theorem] imply the existence of such an X Observe that (4) is possible because the connected components of $\mathcal{N}_1 \cup \mathcal{N}_{(-1)}$ accumulate at {endpoints of $h^{-1}(0)$} only

5 1 THEOREM *Let $\mathcal{U}$ be a neighbourhood of X in the Whitney C^2-topology The vector field X has non-trivial recurrent points Moreover, if $\mathcal{U}$ is small enough, no vector field belonging to $\mathcal{U}$ has closed orbits*

To prove Theorem 5 1, we shall need some lemmas and the following notation $\Xi(\varepsilon, Y)$, with $\varepsilon > 0$, will be the set of homeomorphisms U $C \to C$ such that, for $\delta \in \{-1, 1\}$,

(1) U^δ restricted to $C - (U^{-\delta}(\Lambda) \cup \Lambda)$ is of class C^2, where $\Lambda =$ {endpoints of $Q(0)$}

(2) $(Y^{-\delta}) \circ U^\delta$ is of class C^2 in the whole of C

(3) $\|Y^{-d} \circ U^\delta - \mathrm{Id}\|_2 < \varepsilon$, where Id is the identity map of C and $\| \;\; \|_2$ is the C^2-uniform norm

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5 2 LEMMA *Let $\varepsilon = 0$ 1 and $n \in \mathbb{N} - \{0\}$ Given $U \in \Xi(\varepsilon, \Upsilon)$, for all points p, q, s, t belonging to a connected component of* Dom (U_n) *of length less than 2^{-n}, the following is satisfied*

$$\left|\frac{\|(U_n)(p)-(U_n)(q)\|}{\|p-q\|}-\frac{\|(U_n)(s)-(U_n)(t)\|}{\|s-t\|}\right|<\left(\frac{(1+\varepsilon)^2}{2}\right)^n$$

where $\hat{D} = C - (D_\delta \cup D_{-2\delta})$ and U_n is either $(U^\delta|_{\hat{D}})^n$ or $(\Upsilon^{-\delta}) \circ U^\delta \circ (U^\delta|_{\hat{D}})^{n-1}$

Proof We shall prove this lemma only when $\delta = 1$ Since $(\Upsilon^{-1}|_{(C-(D_{-1}\cup D_2))})' \equiv 1$ and $U \in \Xi(\varepsilon, \Upsilon)$ we have that

(1) $\|(U|_{\hat{D}}) - \mathrm{Id}\|_2 < \varepsilon$

Moreover, we shall show, by induction on n that if $x \in$ Dom $(U|_{\hat{D}})^n$, then

(a n) $$|(U^n)''(x)| < (1+\varepsilon)^{n-1}((1+\varepsilon)^n - 1)$$

In fact, (a 1) follows directly from (1) Suppose that (a n) is valid Then, using (1)

$$\begin{aligned}|(U^{n+1})''(x)| &\le |(U^n)''(U(x))| \ |(U'(x))^2| + |(U^n)'(U(x))| \ |U''(x)| \\ &< (1+\varepsilon)^{n-1}((1+\varepsilon)^n-1)(1+\varepsilon)^2 + (1+\varepsilon)^n\varepsilon \\ &= (1+\varepsilon)^n((1+\varepsilon)^{n+1}) - (1+\varepsilon)^{n+1} + (1+\varepsilon)^n\varepsilon \\ &= (1+\varepsilon)^n((1+\varepsilon)^{n+1}) + (1+\varepsilon)^n(\varepsilon - 1 - \varepsilon) \\ &= (1+\varepsilon)^n((1+\varepsilon)^{n+1} - 1)\end{aligned}$$

This finishes the proof of (a n), for all n

By the Mean Value Theorem and (a n), it follows that, given $x, y \in$ Dom $[U|_{\hat{D}}]^n$,

$$|(U^n)'(x) - (U^n)'(y)| \le ((1+\varepsilon)^2)^n \ \|x-y\| < \left(\frac{(1+\varepsilon)^2}{2}\right)^n$$

The lemma follows immediately from this and the Mean Value Theorem

5 3 LEMMA *Let $\varepsilon = 0$ 1 and $\Lambda = \{$endpoints of $h^{-1}(0)\}$ If $\mathcal{U}$ is small enough and $Y \in \mathcal{U}$, then*

(C 1) *Y is transversal to $C - \Lambda$ and $Y|_{T^2-C}$ is topologically equivalent to $X|_{T^2-C}$ Also $\bigcup\{\Sigma(\imath) \mid |\imath| \le 2\}$ is contained in the stable manifold of the attractor of Y originating from that of X having the same property*

(C 2) Dom $((\Upsilon_Y)^\delta) \cap \mathcal{N}_\delta$ *is empty, where Υ_Y $C \to C$ is the forward Poincaré map induced by Y*

Moreover, given an arbitrary sequence $\{x_k\}$ of points of Sp $(\mathcal{N}_1) \cup$ Sp $(\mathcal{N}_{(-1)}) - (\mathcal{N}_1 \cup \mathcal{N}_{(-1)})$ *which meet, at most once, any given connected component of* Sp $(\mathcal{N}_1) \cup$ Sp $(\mathcal{N}_{(-1)}) - (\mathcal{N}_1 \cup \mathcal{N}_{(-1)})$, *there exists $U \in \Xi(\varepsilon, \Upsilon)$ such that*

(C 3) *For all $x \in \{x_k\} \cap$* Dom $((\Upsilon_Y)^\delta) \cap$ Sp $(\mathcal{N}_\delta)$, $U^\delta(x) = ((\Upsilon_Y)^\delta)(x)$

(C 4) $(\Upsilon_Y)^\delta$ *and U^δ are equal when restricted to the set* Dom $((\Upsilon_Y)^\delta) -$ (Sp $(\mathcal{N}_\delta) \cup ((\Upsilon_Y)^{-\delta}$(Sp $(\mathcal{N}_{-\delta}))))$

Proof Given $\imath \in \mathbb{Z}$, write $\hat{\imath} = 2\imath$ if $\imath \ge 0$, and $\hat{\imath} = -2\imath - 1$ if $\imath < 0$ Let $\{B_\imath\}$ and $\{\tilde{B}_\imath\}$ be the sequences of canonical rectangles considered in the definition of X Let $\{V_\imath\}_{\imath \ge 1}$ be a locally finite open covering of $T^2 - \Lambda$ such that

(1 1) Each $\tilde{B}_\imath$ is contained in $V_{\hat{\imath}}$ and each $B_\imath$ is contained in $M_{\hat{\imath}} = \bigcup\{V_\jmath \mid \jmath \le \hat{\imath}\}$

(1 2) Each $B_\imath$ is disjoint from $\bigcup\{V_\jmath \mid \jmath > \hat{\imath}\}$ and each $\tilde{B}_\imath$ is disjoint from $\bigcup\{V_\jmath \mid \jmath \ne \hat{\imath}\}$

(1 3) For all $\iota \in \mathbb{N}-\{0,1\}$, $V_\iota \subset M_\iota - M_{\iota-2}$

(1 4) For all $\iota \in \mathbb{N}$, there exist smooth closed curves λ_ι and ρ_ι such that $\lambda_\iota \cap \rho_\iota = C \cap \bar{M}_\iota$, $T^2 - M_\iota$ is the union of two disks each of which contains an endpoint of $h^{-1}\{0\}$, the boundary $\partial(T^2 - M_\iota)$, of $T^2 - M_\iota$, is equal to the closure of $\lambda_\iota \cup \rho_\iota - \lambda_\iota \cap \rho_\iota$, and the flow enters $T^2 - M_\iota$ through λ_ι See figure 5 3

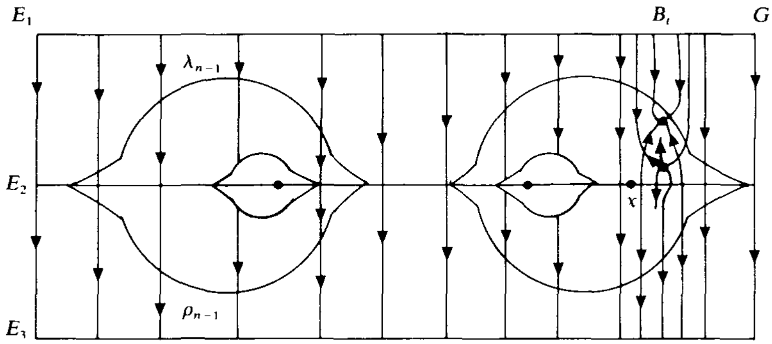

FIGURE 5 3

Let $\| \quad \|_2$ be a norm on $\mathfrak{X}^2(T^2)$ compatible with its C^2-topology When $\{\varepsilon_\iota\}$ varies among all possible sequences of positive real numbers, a fundamental system of neighbourhoods of $X|_{T^2-\Lambda}$ in the Whitney C^2-topology is the one formed by the open sets

$$\mathcal{U}(\{\varepsilon_\iota\}) = \{Y \in \mathfrak{X}^2(T^2 - \Lambda) \mid \| Y|_{\bar{V}_\iota} - X|_{\bar{V}_\iota} \|_2 < \varepsilon_\iota\}$$

Let $\{\psi_\iota \; T^2 - \Lambda \to [0,1]\}$ be a smooth partition of unity strictly subordinate to the locally finite covering $\{V_\iota\}$ Given $Y \in \mathfrak{X}^2(T^2 - \Lambda)$, we define

$$Y_0 = X \quad \text{and} \quad Y_\iota = \sum_{k=1}^{\iota} \psi_k(Y - X) + X$$

Let $\mathcal{U}_0 = \mathcal{U}(\{1, 1, \quad, 1, \quad\})$ Suppose inductively on n, that for $\iota \in \{0, 1, 2, \quad, n\}$, there is a positive real number ε_ι, with $\varepsilon_0 = 1$, such that if $Y \in \mathcal{U}_n = \mathcal{U}(\{\varepsilon_1, \varepsilon_2, \quad, \varepsilon_n, 1, 1, 1, \quad\})$ and $T_\iota \; C \to C$ is the forward Poincaré map induced by Y_ι, then there exists a map $U_\iota \; C \to C$, with $U_0 = \Upsilon$, defined everywhere and such that

(2 1 ι) Items (C 1) and (C 2) of this lemma are true for $Y = Y_\iota$ and $\Upsilon_Y = T_\iota$ Also $U_\iota \in \Xi(\varepsilon, \Upsilon)$

(2 2 ι) If $\hat{j} \leq \iota$ and $x \in \{x_k\} \cap \mathrm{Dom}\,((\Upsilon_Y)^\delta) \cap \mathrm{Sp}\,(\mathcal{N}_\delta) \cap B_j$ Then $(U_\iota)^\delta(x) = ((T_\iota)^\delta)(x)$

(2 3 ι) If $\hat{j} > \iota$, then $(U_\iota)^\delta$ and $(T_\iota)^\delta$ are equal when restricted to $\mathrm{Dom}\,((T_\iota)^\delta) \cap \mathrm{Sp}\,(\mathcal{N}_\delta) \cap B_j$

(2 4 ι) $(T_\iota)^\delta$ and $(U_\iota)^\delta$ are equal when restricted to the set

$$\mathrm{Dom}\,((T_\iota)^\delta) - (\mathrm{Sp}\,(\mathcal{N}_\delta) \cup ((T_\iota)^{-\delta}(\mathrm{Sp}\,(\mathcal{N}_{-\delta}))))$$

(2 5 ι) $\|\Upsilon^{-\delta} \circ (U_\iota)^\delta - \Upsilon^{-\delta} \circ (U_{\iota-1})^\delta\|_2 < \frac{\varepsilon}{2^{\iota+1}}$, where $U_{-1} = U_0 = \Upsilon$

Let G be the canonical rectangle of Y_n containing $M_{n+1}-M_{n-1}$ such that its transversal edges E_1 (of entrance) and E_3 (of exit) are contained in C, and each of its edges, which is an arc of trajectory of Y_n, meets ∂M_{n-1} exactly once See figure 5 3 Let $E_2 = C \cap (T^2 - M_{n-1})$

Let us prove (2 1 $n+1$)-(2 5 $n+1$) Since the support of Y_{n+1} is contained in V_{n+1},

(3) Y_{n+1} is a perturbation of Y_n localized in $V_{n+1} \subset M_{n+1} - M_{n-1}$

It may or may not be that there exists $\tilde{B}_t$ contained in V_{n+1} If so, it may or may not happen that $\{x_k\}$ meets $\tilde{B}_t$ Since, by (1 2), V_{n+1} meets at most one term of $\{\tilde{B}_k\}$, this gives rise to a couple of cases, similar to (4) below, to be considered We shall proceed studying only

(4) For some t, $\tilde{B}_t$ is contained in V_{n+1}, the transversal edges $h^{-1}(t\sigma)$ and $h^{-1}((t+1)\sigma)$ of B_t are contained in E_1 and Sp ($\mathcal{N}_{-1}$), respectively, and $\{x_k\}$ meets $h^{-1}((t+1)\sigma) - \Sigma(t+1)$ at x

It follows from (3) that ε_{n+1} can be taken so small that items (C 1) and (C 4) of this lemma are true when $Y = Y_{n+1}$ and $\Upsilon_Y = T_{n+1}$ Also, U_{n+1} can be defined to be equal to U_n in $C - (E_1 \cup E_2)$ We shall only need to define both U_{n+1} and $(U_{n+1})^{-1}$ at E_2

It follows from (2 3 n) and (2 4 n) that $(U_n)^{-1}$ and $(T_n)^{-1}$ are equal in an open neighbourhood of $h^{-1}((t+1)\sigma) \cap \mathrm{Dom}((T_n)^{-1})$ Therefore, using (1 1)-(1 3) and (3), $\varepsilon_{n+1} > 0$ can be chosen so small that $(U_{n+1})^{-1}$ can be defined in a neighbourhood $F_{(-1)}$ of $h^{-1}((t+1)\sigma) \subset E_2$ so that

(5) The statements (2 2 $n+1$)-(2 5 $n+1$) are true when $(U_{n+1})^{\delta}$ and $(T_{n+1})^{\delta}$ are restricted to F_δ where $F_1 = (U_n)^{-1}(F_{-1}) \subset E_1$

Let f_n $\lambda_{n+1} - h^{-1}((t+1)\sigma) \to E_1$ (resp g_n $\lambda_{n+1} - h^{-1}((t+1)\sigma) \to E_1$) be the backward Poincaré map induced by $Y_n|_G$ (resp $Y_{n+1}|_G$) Certainly if ε_{n+1} is small enough, g_n is so close to f_n that by defining

$$H_1 = E_1 - U_n(h^{-1}((t+1)\sigma)), \qquad H_{-1} = E_2 - h^{-1}((t+1)\sigma)$$

and

$$U_{n+1}|_{H_1} = U_n \circ (f_n) \circ (g_n)^{-1},$$

we have that

(6) The statements (2 2 $n+1$)-(2 5 $n+1$) are true when $(U_{n+1})^{\delta}$ and $(T_{n+1})^{\delta}$ are restricted to H_δ

With the same procedure, U_{n+1} is defined in E_2 In this way we have defined U_{n+1} in the whole of C Conditions (2 3 $n+1$) and (2 4 $n+1$) ensure that U_{n+1} is well defined

Under these conditions, it is easy to finish the proof of (2 1 $n+1$)

This implies that $\{\Upsilon^{-1} \circ U_i\}$ converges to a C^2 map that can be written as $\Upsilon^{-1} \circ U$ and such that

(7) $\|\Upsilon^{-\delta} \circ U^{\delta} - \mathrm{Id}\|_2 < \varepsilon$,

where $\mathrm{Id} = \Upsilon^{-1} \circ U_0$ is the identity map of C and $\| \ \|_2$ is the C^2 uniform norm

The lemma is proved

The order of the integers determines in a natural way the *order* '<' *of the intervals* of the set $\{\Sigma(i) | i \in \mathbb{Z}\}$ and, therefore, of the connected components of $\mathcal{N}_1 \cup \mathcal{N}_{-1}$ Let

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$\Sigma(i) < \Sigma(j)$ be a pair of consecutive connected components of $\mathcal{N}_1 \cup \mathcal{N}_{-1}$ Let $\tilde{B}_{i_1}$ and $\tilde{B}_{j_1}$ be the elements of $\{\tilde{B}_k\}$ containing $\Sigma(i)$ and $\Sigma(j)$, respectively We say that $Y \in \mathcal{U}$ *weakly connects* $\Sigma(i)$ *with* $\Sigma(j)$ if there is a homotopy of open segments $\lambda(t)$, $t \in [0, 1]$, such that

For all $t \in [0, 1]$, $\lambda(t)$ has both endpoints contained in $\tilde{B}_{i_1} \cup \tilde{B}_{j_1}$ Moreover, $\lambda(0)$(resp $\lambda(1)$) is an arc of trajectory of X(resp of Y) going from its repelling singularity contained in $\tilde{B}_{i_1}$ to its attracting singularity contained in $\tilde{B}_{j_1}$

Given $n \in \mathbb{N}$ and $Y \in \mathcal{U}$, we *say that* Y *is in* $\Delta(\mathcal{U})_n$, if any pair $\Sigma(i) < \Sigma(j)$ of consecutive connected components of $\mathcal{N}_1 \cup \mathcal{N}_{-1}$, with $-q_n \le i < j \le q_n$, are weakly connected by Y

5 5 Lemma *If* $\mathcal{U}$ *is small enough then for any natural number* $n > 5$ *such that* $\alpha^{n-6} < 2^{-n}$ *and* $((1+\varepsilon)^2/2)^n < \alpha^6$, *where* $\varepsilon = 0\,1$ *and* $\alpha = 0\,01$, *any* $Y \in \Delta(\mathcal{U})_n - \Delta(\mathcal{U})_{n+1}$ *has no closed orbits*

Proof Except for (12) of the proof of Lemma 4 4 – whose corresponding version in this new case will be proved below – the same proof applies to this similar case

Here $\mathcal{B}(\tilde{\theta}, \theta)$ will be the canonical rectangle of X with transversal edges $\tilde{\theta} = \bar{a}_1\bar{b}_{11}$ and $\theta = a_1 b_1$ We shall suppose the following

(1) The connected components of $\mathcal{N}_1 \cup \mathcal{N}_{-1}$ contained in θ (resp $\tilde{\theta}$) are the closed intervals A_iB_i (resp $\bar{A}_i\bar{B}_i$), with $i \in \{1, 2, \quad , 11\}$, that are distributed, along θ (resp $\tilde{\theta}$), according to the order $<$, as $a_i < A_i < B_i < b_i$ (resp $\bar{a}_i < \bar{A}_i < \bar{B}_i < \bar{b}_i$)

(2) If $Y \in \mathcal{U}$, then $\bar{A}_2\bar{B}_2$ and A_2B_2 are not weakly connected by Y Also Y, when restricted to $\mathcal{B}(\tilde{\theta}, \theta)$, connects $\bar{B}_2\bar{A}_3$ with B_1A_2 and $\bar{B}_6\bar{A}_7$ with B_6A_7

(3) $\mathcal{B}(\tilde{\theta}, \theta)$ does not meet either $Q(n-4)$ or the transversal edges of $\Lambda_{n\,1}$

Under these conditions, we claim that

(4) If $\mathcal{U}$ is small enough then $Y|_{\mathcal{B}(\tilde{\theta}\,\theta)}$ does not connect either $\bar{B}_9\bar{A}_{10}$ with B_9B_{10} or $\bar{B}_{10}\bar{A}_{11}$ with $B_{10}A_{11}$

In fact, suppose that $(Y_Y)^k \equiv \pi \quad \tilde{\theta} \to \theta$ is the forward Poincaré map induced by Y Then there are points $p_1 \in \bar{B}_2\bar{A}_3$ and $p_2 \in \bar{B}_6\bar{A}_7$ belonging to Dom (π) and such that $\pi(p_1) \in B_1A_2$ and $\pi(p_2) \in B_6A_7$ If we assume that there exists $p_3 \in \bar{B}_9\bar{A}_{10}$ belonging to Dom (π), then $\pi(p_3)$ belongs to B_9A_{10} because, as $Y \in \Delta(\mathcal{U})_n$, any two consecutive connected components of $\mathcal{N}_1 \cup \mathcal{N}_{-1}$ belonging to either $\mathcal{B}(\bar{a}_9\bar{b}_9,\ a_9b_9)$ or $\mathcal{B}(\bar{a}_{10}\bar{b}_{10},\ a_{10}b_{10})$ are weakly connected by Y

Let $\{x_k\}$ be the sequence formed by the points of the intersection of Sp $(\mathcal{N}_1) \cup$ Sp $(\mathcal{N}_{(-1)}) - (\mathcal{N}_1 \cup \mathcal{N}_{(-1)})$ and $\bigcup\{$arcs of trajectory of Y connecting p_i with $\pi(p_i) | i = 1, 2, 3\}$ Certainly, $\{x_k\}$ meets, at most once, any given connected component of Sp $(\mathcal{N}_1) \cup$ Sp $(\mathcal{N}_{(-1)}) - (\mathcal{N}_1 \cup \mathcal{N}_{(-1)})$ Therefore, corresponding to $\{x_k\}$, we may consider the map $U \in \Xi(\varepsilon, Y)$ of Lemma 5 3, that satisfies $U^k(p_i) = \pi(p_i)$, with $i = 1, 2, 3$ By (2) and the fact that $Y \in \Delta(\mathcal{U})_n$, $\tilde{\theta}$ is contained in the domain of definition of $(U|_{U^{-1}(C-(D_1 \cup D_{-1}))})^k$ Under these circumstances and because $\|p_1p_3\| < \alpha^{n-6} < 2^{-n}$, we may apply Lemma 5 2 to points of the interval p_1p_3 so as to obtain a contradiction and prove (4) As we said, we carry out this last argument as in the proof of Lemma 4 4 Observe that, for all i, $\|h^{-1}(\sigma i)\| - \|\Sigma(i)\|$ is negligible

5.6 *Proof of Theorem* 5.1. Let $\mathscr{U}$ and n be as in Lemma 5.5. Let $Y\in \Delta(\mathscr{U})_n-\Delta(\mathscr{U})_{n+1}$. It follows from the last lemma, that for $Y\in\mathscr{U}$ to have closed orbits it will be necessary that, for all $k>n$, $Y\in\Delta(\mathscr{U})_k$. However if $Y\in\Delta(\mathscr{U})_k$, for some k, any closed orbit of Y meets C at least q_k times. Therefore, Y cannot have closed orbits.

5.7 *Proof of Theorem* A. In the following sequence of statements, we will use the same pattern of arguments of the proof of Lemma 5.3, corresponding to the use of a partition of unity.

Let $\Psi: T^2\to[0,1]$ be a smooth function which vanishes exactly at $h^{-1}(0)$. Theorem 5.1 is also true for the vector field $Z=\Psi X|_{T^2-h^{-1}(0)}$. Let $p\in T^2$ and $H: (T^2-h^{-1}(0))\to (T^2-\{p\})$ be a smooth diffeomorphism. The vector field ΨX can be constructed so flat at the set of its singularities that the vector field H_*Z extends smoothly to the whole T^2 and also, for some neighbourhood $\mathscr{V}$ of H_*Z, $H^{-1}(\mathscr{V})\subset\mathscr{U}$, where $\mathscr{U}$ is as in Theorem 5.1. This proves Theorem A when the manifold is the torus.

Certainly the example can be embedded in any higher genus two-manifold. Also, the proof of the general case follows by taking the torus as a normally hyperbolic attractor [**H-P-S**] of a smooth flow (on a higher dimensional manifold) that is a gradient-like Morse–Smale vector field [**Pa-Sm**] away from this attractor. This proves Theorem A.

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