# AN ABSTRACT FORMULATION OF THE LEBESGUE DECOMPOSITION THEOREM

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Some concepts in measure theory can be generalized by means of classes of null sets. When measures are considered then the classes of all sets of measure zero play the role of classes of null sets. The purpose of this paper is to give an abstract formulation and proof of the Lebesgue decomposition theorem.

## 1

Throughout this paper $(X, \mathscr{S})$ is a measurable space and $\mathscr{S}$ is a $\sigma$-algebra of sets. The notation $E^c$ is used for the complement of a set $E$ and $E \Delta F$ for the symmetric difference of two sets $E$ and $F$. A nonempty class $\mathscr{N}$ of sets such that $\mathscr{N} \subset \mathscr{S}$, $\mathscr{N}$ is closed under countable unions of sets, and $E \cap F \in \mathscr{N}$ whenever $E \in \mathscr{N}$ and $F \in \mathscr{S}$ is called a class of null sets of $\mathscr{S}$. Evidently $\phi \in \mathscr{N}$. The notation $(\mathscr{S} - \mathscr{N})C$ indicates that each subclass of disjoint sets of $\mathscr{S} - \mathscr{N}$ is countable. If $\mathscr{M}$ and $\mathscr{N}$ are two classes of null sets of $\mathscr{S}$ and $(\mathscr{S} - \mathscr{N})C$ holds, then there exists a measurable set $Z_{mn}$ (to $\mathscr{M}, \mathscr{N}$ are assigned indices $m$, $n$) with the following properties: (i) $Z_{mn} \in \mathscr{M}$, (ii) $E \in \mathscr{M}$, $E \subset Z_{mn}^c$ imply $E \in \mathscr{N}$. The set $Z_{mn}$ is determined uniquely in the following sense. A measurable set $B$ has the properties (i) and (ii) if and only if $Z_{mn} \Delta B \in \mathscr{M} \cap \mathscr{N}$ (cf. [1]). Let $\mathscr{M}$ and $\mathscr{N}$ be two classes of null sets of $\mathscr{S}$. We say that $\mathscr{N}$ is absolutely continuous with respect to $\mathscr{M}$, denoted $\mathscr{N} \ll \mathscr{M}$, if $\mathscr{M} \subset \mathscr{N}$. We say that $\mathscr{M}$ and $\mathscr{N}$ are equivalent, denoted $\mathscr{M} \equiv \mathscr{N}$, if both $\mathscr{N} \ll \mathscr{M}$ and $\mathscr{M} \ll \mathscr{N}$. We say that $\mathscr{N}$ is singular with respect to $\mathscr{M}$, denoted $\mathscr{N} \perp \mathscr{M}$, if there exists a measurable set $A$ such that $E \cap A \in \mathscr{M}$ and $E \cap A^c \in \mathscr{N}$, whenever $E \in \mathscr{S}$. We note that the relation $\mathscr{N} \perp \mathscr{M}$ is symmetric. We say that $\mathscr{N}$ is $s$-singular with respect to $\mathscr{M}$, denoted $\mathscr{N}\, s\, \mathscr{M}$, if given $E \in \mathscr{S}$ there exists a measurable set $F \subset E$ such that $F \in \mathscr{M}$ and both sets $E$ and $F$ belong to $\mathscr{N}$, or both sets $E$ and $F$ belong to $\mathscr{S} - \mathscr{N}$ (cf. [4]). $\mathscr{N}\, s\, \mathscr{M}$ need not be symmetric.

EXAMPLE. Let $X = [0, 1]$ and $\mathscr{S}$ be the class of all subsets of $X$. Let $\mathscr{M} = \{E : E \in \mathscr{S}$ and $E$ countable$\}$ and $\mathscr{N} = \{\phi\}$. We show that $\mathscr{N}\, s\, \mathscr{M}$. Given $E \in \mathscr{S}$, then: if $E = \phi$, put $F = \phi$ then $F \in \mathscr{M}$, $F \subset E$ and both $E$ and $F \in \mathscr{N}$. If $E$ countable, put $F = E$, then $F \in \mathscr{M}$, $F \subset E$ and both $E$ and $F \in \mathscr{S} - \mathscr{N}$. If $E$



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uncountable, put $F$ = countable subset of $E$ such that $F \neq \phi$, then $F \subset E$, $F \in \mathscr{M}$ and both $E$ and $F \in \mathscr{S} - \mathscr{N}$. To show that $\mathscr{M}\, s \mathscr{N}$ does not hold consider $E \in \mathscr{S}$, $E$ uncountable then the only set $F \subset E$, $F \in \mathscr{N}$ is $F = \phi$ and $F \in \mathscr{M}$ but $E \in \mathscr{S} - \mathscr{M}$.

## 2

Let $(X, \mathscr{S})$ be a measurable space and $\mathscr{M}, \mathscr{N}$ be two classes of null sets of $\mathscr{S}$. Let $\mathscr{N}_M = \{E : E \in \mathscr{S} \text{ and } E \cap M \in \mathscr{N}\}$ for each $M \in \mathscr{M}$. Then $\mathscr{N}_M$ is a class of null sets of $\mathscr{S}$ for each $M \in \mathscr{M}$. If $\mathscr{L} = \bigcap_{M \in \mathscr{M}} \mathscr{N}_M$, then $\mathscr{L}$ is a class of null sets of $\mathscr{S}$.

LEMMA 1. *Let $(X, \mathscr{S})$ be a measurable space and $\mathscr{M}, \mathscr{N}$ be two classes of null sets of $\mathscr{S}$. Let $\mathscr{L} = \bigcap_{M \in \mathscr{M}} \mathscr{N}_M$. If $E \in \mathscr{S}$ there exists a measurable set $F$ such that $F \subset E$, $F \in \mathscr{M}$ and both sets $E$ and $F$ belong to $\mathscr{L}$ or both belong to $\mathscr{S} - \mathscr{L}$.*

PROOF. If $E \in \mathscr{L}$, then each measurable subset of $E$ also belongs to $\mathscr{L}$. It is sufficient to put $F = E \cap M$ for an arbitrary set $M \in \mathscr{M}$.

If $E \in \mathscr{S} - \mathscr{L}$ then there exists at least one set $M_0 \in \mathscr{M}$ such that $E \cap M_0 \notin \mathscr{N}$. Put $F = E \cap M_0$. We show that $F \notin \mathscr{L}$. On the contrary, suppose that $F \in \mathscr{L}$, hence for each $M \in \mathscr{M}$, $F \cap M \in \mathscr{N}$, but if $M = M_0$ we have $E \cap M_0 \notin \mathscr{N}$. This is a contradiction. Thus $F \in \mathscr{S} - \mathscr{L}$.

LEMMA 2. *Let $(X, \mathscr{S})$ be a measurable space and $\mathscr{M}, \mathscr{N}$ be two classes of null sets of $\mathscr{S}$. Let $\mathscr{L}$ be as above. Then $\mathscr{L}\, s\, \mathscr{M}$, $\mathscr{L} \ll \mathscr{N}$ and $\mathscr{N} \cap \mathscr{M} \equiv \mathscr{L} \cap \mathscr{M}$.*

PROOF. From Lemma 1 it immediately follows that $\mathscr{L}\, s \mathscr{M}$. Now we prove $\mathscr{L} \ll \mathscr{N}$. Let $E \in \mathscr{N}$ then also $E \cap M \in \mathscr{N}$ for each $M \in \mathscr{M}$, hence $E \in \mathscr{N}_M$ for each $M \in \mathscr{M}$. Thus $E \in \mathscr{L}$.

To prove $\mathscr{N} \cap \mathscr{M} \equiv \mathscr{L} \cap \mathscr{M}$ it is sufficient to show that $\mathscr{L} \cap \mathscr{M} \subset \mathscr{N} \cap \mathscr{M}$ since the reverse inclusion follows at once from $\mathscr{L} \ll \mathscr{N}$. Let $E \in \mathscr{L} \cap \mathscr{M}$, then $E \in \mathscr{N}_M$ for each $M \in \mathscr{M}$ and $E \in \mathscr{M}$, hence $E \in \mathscr{N}_E$ that is $E \in \mathscr{N}$. Thus $E \in \mathscr{N} \cap \mathscr{M}$.

THEOREM 1. *Let $(X, \mathscr{S})$ be a measurable space and $\mathscr{M}, \mathscr{N}$ be two classes of null sets of $\mathscr{S}$. Then there exist two classes of null sets $\mathscr{N}_0$ and $\mathscr{N}_1$ such that $\mathscr{N}_1\, s \mathscr{M}$, $\mathscr{N}_0 \ll \mathscr{M}$* and *$\mathscr{N} = \mathscr{N}_0 \cap \mathscr{N}_1$. The class $\mathscr{N}_1$ is always unique. If $(\mathscr{S} - \mathscr{N})C$ then $\mathscr{N}_0$ is unique also.*

PROOF. Put $\mathscr{N}_1 = \mathscr{L}$, $\mathscr{L}$ as in Lemma 1 then we have $\mathscr{N}_1\, s \mathscr{M}$. Let $\mathscr{N}_0 = \bigcap_{N \in \mathscr{N}_1} \mathscr{L}_N$, where $\mathscr{L}_N = \{E : E \in \mathscr{S} \text{ and } E \cap N \in \mathscr{N}\}$ for each $N \in \mathscr{N}_1$, then $\mathscr{N}_0$ and $\mathscr{L}_N$ for each $N \in \mathscr{N}_1$ are classes of null sets of $\mathscr{S}$.

Now we prove $\mathscr{N}_0 \ll \mathscr{M}$. From Lemma 1 we have, if $E$ is any measurable set there exists a measurable set $L$ such that $L \subset E$, $L \in \mathscr{N}_1$ and both sets $E$ and $L$ belong to $\mathscr{N}_0$ or both sets belong to $\mathscr{S} - \mathscr{N}_0$. If moreover $E \in \mathscr{M}$ then from the

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equation (a) $E = L \cup (E-L)$ follows that also $L$ and $E-L$ belong to $\mathscr{M}$. From Lemma 2 we have $\mathscr{N}_1 \cap \mathscr{M} = \mathscr{N} \cap \mathscr{M}$ and $\mathscr{N}_0 \cap \mathscr{N}_1 = \mathscr{N} \cap \mathscr{N}_1$ hence $\mathscr{N}_1 \cap \mathscr{M} = \mathscr{N}_0 \cap \mathscr{N}_1 \cap \mathscr{M}$ and from the latter equation it follows that $L \in \mathscr{N}_0$. Now it is sufficient to show that $E-L \in \mathscr{N}_0$. There exists a set $G$ such that $G \subset E-L$, $G \in \mathscr{N}_1$ and both $G$ and $E-L$ belong to $\mathscr{N}_0$ or both $G$ and $E-L$ belong to $\mathscr{S}-\mathscr{N}_0$. But $G = (E-L) \cap G$ and so $G \in \mathscr{N}_1 \cap \mathscr{M}$, hence $G \in \mathscr{N}_0$ and also $E-L \in \mathscr{N}_0$. Now from (a) we have $E \in \mathscr{N}_0$, hence $\mathscr{M} \subset \mathscr{N}_0$. Thus $\mathscr{N}_0 \ll \mathscr{M}$.

To prove $\mathscr{N} = \mathscr{N}_0 \cap \mathscr{N}_1$ it is sufficient to show that $\mathscr{N}_0 \cap \mathscr{N}_1 \subset \mathscr{N}$ since the reverse inclusion follows from the definitions of $\mathscr{N}_0$ and $\mathscr{N}_1$. From Lemma 2 we have $\mathscr{N}_0 \cap \mathscr{N}_1 = \mathscr{N} \cap \mathscr{N}_1$, hence $\mathscr{N}_0 \cap \mathscr{N}_1 \subset \mathscr{N}$.

To prove the uniqueness of the class $\mathscr{N}_1$ suppose that $\mathscr{N}_0, \mathscr{N}_1, \mathscr{R}_0$ and $\mathscr{R}_1$ are classes of null sets of $\mathscr{S}$ such that $\mathscr{N}_0 \cap \mathscr{N}_1 = \mathscr{N} = \mathscr{R}_0 \cap \mathscr{R}_1, \mathscr{N}_0 \ll \mathscr{M}$, $\mathscr{R}_0 \ll \mathscr{M}, \mathscr{N}_1 \, s \, \mathscr{M}$ and $\mathscr{R}_1 \, s \, \mathscr{M}$. We wish to prove $\mathscr{N}_1 = \mathscr{R}_1$. Let $E \in \mathscr{S}$ from $\mathscr{N}_1 \, s \, \mathscr{M}$ follows, there exists a measurable set $F_1$ such that $F_1 \subset E$, $F_1 \in \mathscr{M}$ and both sets $E$ and $F_1$ belong to $\mathscr{N}_1$ or both belong to $\mathscr{S}-\mathscr{N}_1$. Since $\mathscr{R}_1 \, s \, \mathscr{M}$ there exists a measurable set $F_2$ such that $F_2 \subset E$, $F_2 \in \mathscr{M}$ and both sets $E$ and $F_2$ belong to $\mathscr{R}_1$ or both belong to $\mathscr{S}-\mathscr{R}_1$. Put $F = F_1 \cup F_2$ then $F \subset E$ and $F \in \mathscr{M}$. Then $F \in \mathscr{N}_0$ and $F \in \mathscr{R}_0$ since $\mathscr{N}_0 \ll \mathscr{M}$ and $\mathscr{R}_0 \ll \mathscr{M}$. If moreover $E \in \mathscr{N}_1$ then also $F \in \mathscr{N}_1$ and from the equations $\mathscr{N}_0 \cap \mathscr{N}_1 = \mathscr{N} = \mathscr{R}_0 \cap \mathscr{R}_1$ it follows that $F \in \mathscr{R}_1$ hence $F_2 \in \mathscr{R}_1$, then $E \in \mathscr{R}_1$. Thus $\mathscr{N}_1 \subset \mathscr{R}_1$. Similarly we prove $\mathscr{R}_1 \subset \mathscr{N}_1$.

Now suppose $(\mathscr{S}-\mathscr{N})C$ then there exists a measurable set $Z_{mn}$ with the properties (i) and (ii). Let $\mathscr{N}_{mn} = \{E : E \in \mathscr{S}$ and $E \cap Z_{mn} \in \mathscr{N}\}$. We prove $\mathscr{N}_1 = \mathscr{N}_{mn}$. The inclusion $\mathscr{N}_1 \subset \mathscr{N}_{mn}$ follows at once from the definition of the class $\mathscr{N}_1$. On the other hand let $E \in \mathscr{N}_{mn}$ and let $M \in \mathscr{M}$. From the equation $E \cap M = (E \cap M \cap Z_{mn}) \cup [(E \cap M)-Z_{mn}]$ follows $E \cap M \in \mathscr{N}$ for each $M \in \mathscr{M}$, since both sets on the right belong to $\mathscr{N}$. The first set because it is a subset of $E \cap Z_{mn}$, the second because of the property (ii). We have $E \in \mathscr{N}_M$ for each $M \in \mathscr{M}$, hence $E \in \mathscr{N}_1$. Thus $\mathscr{N}_1 = \mathscr{N}_{mn}$. We note that now $\mathscr{N}_0 = \bigcap_{N \in \mathscr{N}_{mn}} \mathscr{L}_N$.

Let $\mathscr{K} = \{E : E \in \mathscr{S}$ and $E-Z_{mn} \in \mathscr{N}\}$. We show that $\mathscr{N}_0 = \mathscr{K}$. First we prove $E-Z_{mn} \in \mathscr{N}_1$ for each $E \in \mathscr{S}$. This follows from the relations $(E-Z_{mn}) \cap Z_{mn} = \phi \in \mathscr{N}$. If moreover $E \in \mathscr{N}_0$ then also $E-Z_{mn} \in \mathscr{N}_0$, hence $E-Z_{mn} \in \mathscr{N}$, since $\mathscr{N}_0 \cap \mathscr{N}_1 = \mathscr{N}$. Thus $E \in \mathscr{K}$. On the other hand, let $E \in \mathscr{K}$ and let $N \in \mathscr{N}_{mn}$ then from the equation $E \cap N = (E \cap N \cap Z_{mn}) \cup [(E \cap N)-Z_{mn}]$ follows $E \cap N \in \mathscr{N}$ for each $N \in \mathscr{N}_{mn}$, hence $E \in \mathscr{L}_N$ for each $N \in \mathscr{N}_{mn}$. Thus $E \in \mathscr{N}_0$.

Now we wish to prove $\mathscr{N}_0 = \mathscr{R}_0$, where $\mathscr{N}_0 \ll \mathscr{M}$, $\mathscr{R}_0 \ll \mathscr{M}$ and $\mathscr{N}_0 \cap \mathscr{N}_1 = \mathscr{N} = \mathscr{R}_0 \cap \mathscr{N}_1$. We recall that $(E-Z_{mn}) \in \mathscr{N}_1$, whenever $E \in \mathscr{S}$. If moreover $E \in \mathscr{N}_0$ then from the equation (b) $E = (E \cap Z_{mn}) \cup (E-Z_{mn})$ it follows that $E \in \mathscr{R}_0$, since the first set on the right belongs to $\mathscr{M}$ and the second belongs to $\mathscr{N}_0 \cap \mathscr{N}_1$. Thus $\mathscr{N}_0 \subset \mathscr{R}_0$. On the other hand, suppose $E \in \mathscr{R}_0$ then again from (b) it follows that $E \in \mathscr{N}_0$, since the first set on the right belongs to $\mathscr{M}$ and

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the second belongs to $\mathscr{R}_0 \cap \mathscr{N}_1$. Thus $\mathscr{R}_0 \subset \mathscr{N}_0$. This completes the proof.

LEMMA 3. *Let $(X, \mathscr{S})$ be a measurable space and $\mathscr{M}$, $\mathscr{N}$ be two classes of null sets of $\mathscr{S}$. Let $\mathscr{N}\ s\ \mathscr{M}$ and $(\mathscr{S}-\mathscr{N})C$, then $\mathscr{N} \perp \mathscr{M}$. Hence $\mathscr{M}\ s\ \mathscr{N}$.*

PROOF. From $(\mathscr{S}-\mathscr{N})C$ follows the existence of the set $Z_{mn}$ with the properties (i) and (ii). To prove $\mathscr{N} \perp \mathscr{M}$ it is sufficient to put $A = Z_{mn}$, since $E \cap A = E \cap Z_{mn} \in \mathscr{M}$, whenever $E \in \mathscr{S}$. Now it is sufficient to show that $E \cap A^c \in \mathscr{N}$, whenever $E \in \mathscr{S}$. If given $E \in \mathscr{S}$ there exists a measurable set $K$ such that $K \subset E$, $K \in \mathscr{M}$ and both sets $E$ and $K$ belong to $\mathscr{N}$ or both belong to $\mathscr{S}-\mathscr{N}$, since $\mathscr{N}\ s\,\mathscr{M}$. The set $K - Z_{mn}$ belongs to $\mathscr{N}$, this follows from the property (ii) of $Z_{mn}$. Now consider the set $G = E-(K \cup Z_{mn})$, $G$ is a measurable set and from $\mathscr{N}\ s\ \mathscr{M}$ follows that there exists a measurable set $F \subset G$, $F \in \mathscr{M}$ and both sets $F$ and $G$ belong to $\mathscr{N}$ or both belong to $\mathscr{S}-\mathscr{N}$. But $F \in \mathscr{N}$, since $F \subset Z_{mn}^c$ and $F \in \mathscr{M}$, hence $G$ belongs to $\mathscr{N}$. Finally from the equation $E-Z_{mn} = [E-(K \cup Z_{mn})] \cup (K-Z_{mn})$ we have $E \cap A^c = E - Z_{mn} \in \mathscr{N}$ for each $E \in \mathscr{S}$. Thus $\mathscr{N} \perp \mathscr{M}$. Evidently also $\mathscr{M} \perp \mathscr{N}$ and hence $\mathscr{M}\ s\,\mathscr{N}$.

Now we give the abstract formulation of the Lebesgue decomposition theorem.

THEOREM 2. *Let $(X, \mathscr{S})$ be a measurable space and $\mathscr{M}$, $\mathscr{N}$ be two classes of null sets of $\mathscr{S}$. Let $(\mathscr{S}-\mathscr{N})C$. Then there exist two uniquely determined classes of null sets $\mathscr{N}_0$ and $\mathscr{N}_1$ such that $\mathscr{N}_0 \ll \mathscr{M}$, $\mathscr{N}_1 \perp \mathscr{M}$ and $\mathscr{N} = \mathscr{N}_0 \cap \mathscr{N}_1$.*

PROOF. From Theorem 1 we have that there exist two classes of null sets $\mathscr{N}_0$ and $\mathscr{N}_1$ such that $\mathscr{N} = \mathscr{N}_0 \cap \mathscr{N}_1$, $\mathscr{N}_0 \ll \mathscr{M}$ and $\mathscr{N}_1\ s\,\mathscr{M}$. Since $\mathscr{S}-\mathscr{N} = (\mathscr{S}-\mathscr{N}_0) \cup (\mathscr{S}-\mathscr{N}_1) \supset \mathscr{S}-\mathscr{N}_1$, we have also $(\mathscr{S}-\mathscr{N}_1)C$. Now applying Lemma 3 from $\mathscr{N}_1\ s\,\mathscr{M}$ it follows that $\mathscr{N}_1 \perp \mathscr{M}$.

## 3

Let $(X, \mathscr{S})$ be a measurable space and $\mathscr{L}, \mathscr{M}, \mathscr{N}$ be classes of null sets of $\mathscr{S}$. We express some of the properties of $s$-singularity which are listed in [4 p. 629] by means of classes of null sets. For illustration we give the proof of *d*.

a. $\mathscr{N}\ s\ \mathscr{N}$ if and only if $\mathscr{N} = \mathscr{S}$.

*b*. If $\mathscr{N}\ s\,\mathscr{M}$ and $\mathscr{L} \ll \mathscr{M}$, then $\mathscr{N}\ s\ \mathscr{L}$.

c. If $\mathscr{N}\ s\ \mathscr{M}$ and $\mathscr{N} \ll \mathscr{M}$, then $\mathscr{N} = \mathscr{S}$.

d. If $\mathscr{N}\ s\,\mathscr{M}$ and $\mathscr{L}\ s\ \mathscr{M}$, then $(\mathscr{N} \cap \mathscr{L})\, s\, \mathscr{M}$.

e. If $\mathscr{N}\ s\,\mathscr{M}$ and $\mathscr{N}\ s\ \mathscr{L}$, then $\mathscr{N}\ s\ (\mathscr{M} \cap \mathscr{L})$.

f. If $\mathscr{N} \ll \mathscr{M} \cap \mathscr{L}$ and $\mathscr{N}\ s\ \mathscr{L}$, then $\mathscr{N} \ll \mathscr{M}$.

d. PROOF. If given $E \in \mathscr{S}$ there exists a measurable set $F_1 \subset E$, $F_1 \in \mathscr{M}$ and both sets $E$ and $F_1$ belong to $\mathscr{N}$ or both belong to $\mathscr{S}-\mathscr{N}$. Then there exists a

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measurable set $F_2 \subset E$, $F_2 \in \mathscr{M}$ and both sets $E$ and $F_2$ belong to $\mathscr{L}$ or both belong to $\mathscr{S} - \mathscr{L}$. Put $F = F_1 \cup F_2$ then $F \in \mathscr{M}$ and $F \subset E$. Now it is sufficient to show that both sets $E$ and $F$ belong to $\mathscr{N} \cap \mathscr{L}$ or to $\mathscr{S} - (\mathscr{N} \cap \mathscr{L}) = (\mathscr{S} - \mathscr{N}) \cup (\mathscr{S} - \mathscr{L})$. We note that $F_1 \subset F \subset E$ and $F_2 \subset F \subset E$. If $E, F_1 \in \mathscr{N}$ and $E, F_2 \in \mathscr{L}$, then $E$ and $F$ belong to $\mathscr{N} \cap \mathscr{L}$. In all other cases $E$ belongs at least once to $\mathscr{S} - \mathscr{N}$ or $\mathscr{S} - \mathscr{L}$ and so does $F$, hence both $E$ and $F$ belong to $\mathscr{S} - (\mathscr{N} \cap \mathscr{L})$.

## 4

We note, if $(X, \mathscr{S}, \mu)$ is a measure space, where $\mathscr{S}$ is a $\sigma$-algebra of sets, then to $\mu$ we assign the class $\mathscr{M} = \{E : E \in \mathscr{S} \text{ and } \mu(E) = 0\}$. Evidently $\mathscr{M}$ is a class of null sets of $\mathscr{S}$.

If $(X, \mathscr{S}, \mu)$ is a measure space and $\{\mu_n\}_{n=1}^{\infty}$ is a sequence of totally finite measures on $\mathscr{S}$ we say that $\mu$ has the property $\sigma$, if $\mu(E) = \sum_{n=1}^{\infty} \mu_n(E)$ for each $E \in \mathscr{S}$.

We note that the condition '$\mu$ has the property $\sigma$' is weaker than total $\sigma$-finiteness of measures and it can be shown that the property $(\mathscr{S} - \mathscr{M})C$ in a measure space is equivalent to '$\mu$ has the property $\sigma$' as follows.

THEOREM 3. *Let $(X, \mathscr{S}, \mu)$ be a measure space and $\mathscr{M} = \{E : E \in \mathscr{S}$ and $\mu(E) = 0\}$, where $\mathscr{S}$ is a $\sigma$-algebra of sets. Then $(\mathscr{S} - \mathscr{M})C$ if and only if $\mu$ has the property $\sigma$.*

For the proof see [2].

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