A COMBINATORIAL PROOF OF A CONJECTURE OF GOLDBERG AND MOON

Brian Alspach

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1. Introduction. Let T_n denote a tournament of order n, let $G(T_n)$ denote the automorphism group of T_n, let $|G|$ denote the order of the group G, and let $g(n)$ denote the maximum of $|G(T_n)|$ taken over all tournaments T_n of order n. Goldberg and Moon conjectured [2] that $g(n) \leq \sqrt{3}^{n-1}$ for all $n \geq 1$ with equality holding if and only if n is a power of 3. In an addendum to [2] it was pointed out that their conjecture is equivalent to the conjecture that if G is any odd order subgroup of S_n, the symmetric group of degree n, then $|G| \leq \sqrt{3}^{n-1}$ with equality possible if and only if n is a power of 3. The latter conjecture was proved in [1] by John D. Dixon who made use of the Feit-Thompson theorem in his proof. In this paper we avoid use of the Feit-Thompson result and give a combinatorial proof of the Goldberg-Moon conjecture.

2. Preliminary results. If in a tournament T_n there is an arc from a vertex u to a vertex v we say u dominates v and use the notation $(u,v) \in T_n$. Let $\mathcal{O}(u) = \{v \in T_n : (u,v) \in T_n\}$ and $\mathcal{I}(v) = \{u \in T_n : (u,v) \in T_n\}$. If A is a subset of the vertex set of T_n, then $\langle A \rangle$ denotes the subtournament of T_n with vertex set A.

Let A be a finite set of n elements. Arbitrarily label the elements of A with integers chosen from $1, \ldots, r$, $1 \leq r \leq n$, such that each integer between 1 and r inclusive is used as the label for at least one element of A. Pick subsets $A_{p,q} \neq \emptyset$, $1 \leq p \leq r$, $1 \leq q \leq n_p$, such that the following hold:

(i) $\bigcup_{p=1}^{r} \bigcup_{q=1}^{n_p} A_{p,q} = A$ and the $A_{p,q}$ are mutually disjoint;

(ii) each element of $A_{p,q}$ is labelled p;

(iii) $A_{1,q}$ is a singleton for $q = 1, 2, \ldots, n_1$;

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(iv) $|A_{p,q}| \leq 2 \left| \bigcup_{i=1}^{p-1} \bigcup_{j=1}^{n_i} A_{i,j} \right| + 1$ for each p, q satisfying

$2 \leq p \leq r$ and $1 \leq q \leq n_p$.

A collection $\mathfrak{A} = \{A_{p,q} : 1 \leq p \leq r, \ 1 \leq q \leq n_p\}$ satisfying conditions (i) - (iv) will be called a partition of A with r labels and $\sum_{p=1}^{r} n_p$ partition sets.

LEMMA. Let A be a finite set with n elements, $n \geq 2$, and let k be the unique positive integer such that $1 + 3 + \ldots + 3^{k-1} < n \leq 1 + 3 + \ldots + 3^k$. Then for any partition $\mathfrak{A}$ of A with $k-r$ labels, $0 \leq r \leq k-1$, one of the following is true:

(1) $\mathfrak{A}$ has at least $k + r + 1$ partition sets if

$$n > 1 + 3 + \ldots + 3^{k-1}$$

(2) $\mathfrak{A}$ has at least $k + r + 2$ partition sets if

$$n > 1 + 3 + \ldots + 3^{k-1} + 3^{k-1}.$$

Proof. Since the result is true for $n = 2$ we assume $n \geq 3$. Let $\mathfrak{A}$ be a partition of A with k labels. If $n > 1 + 3 + \ldots + 3^{k-1}$, condition (iv) implies $\mathfrak{A}$ has at least $k + 1$ partition sets, and if $n > 1 + 3 + \ldots + 3^{k-1} + 3^{k-1}$, condition (iv) implies $\mathfrak{A}$ has at least $k + 2$ partition sets. Thus (1) and (2) hold for $r = 0$. Let $\mathfrak{A}$ be a partition of A with $k-r$ labels, $1 \leq r \leq k-1$, and s partition sets. It suffices to show that there exists a partition $\mathfrak{A}'$ of A with with $k-r+1$ labels and $s-1$ partition sets.

If there exists an i, $1 \leq i \leq k-r$, such that $n_i \geq 3$, we may assume without loss of generality that $|A_{i,1}| \leq |A_{i,2}| \leq \ldots \leq |A_{i,n_i}|$. The sets $A'_{p,q}$ of $\mathfrak{A}'$ are defined as follows: $A'_{p,q} = A_{p,q}$ for $p = 1, \ldots, i-1$ and $q = 1, \ldots, n_p$. $A'_{i,q} = A_{i,q+2}$ for $q = 1, \ldots, n_i - 2$, $A'_{i+1,1} = A_{i,1} \cup A_{i,2}$, and $A'_{p+1,q} = A_{p,q}$ for $p = i + 1, \ldots, k-r$ and $q = 1, \ldots, n_p$. Label the elements of $A'_{p,q}$ with p for $p = 1, \ldots, k-r+1$ and $q = 1, \ldots, n_p$. It is easy to verify $\mathfrak{A}'$ is a partition of A with $k-r+1$ labels and $s-1$ partition sets.

If there is no i, $1 \leq i \leq k-r$, such that $n_i \geq 3$, then there

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exist i and j, $1 \le i < j \le k-r$, such that $n_i = n_j = 2$. For if there were not there would be at most $k-r+1$ partition sets and by condition (iv) there would be at most

$$1 + 3 + \ldots + 3^{k-r} \le 1 + 3 + \ldots + 3^{k-1} < n$$

elements of A. Pick the smallest i and j for which $n_i = n_j = 2$ with $i < j$. Without loss of generality we may assume $|A_{i,1}| \le |A_{i,2}| \le |A_{j,2}| \le |A_{j,1}|$. Define the sets of $\mathfrak{a}'$ as follows. Pick an element of $A_{j,2}$ and define $A'_{1,1}$ to be this singleton. Distribute the remaining elements of $A_{j,2}$ in the following way in lexicographic order with respect to the indices of $A_{p,q}$: for $p < j$, $p \neq i$, from the elements of $A_{j,2}$ remaining after the previous step (the first step is the definition of $A'_{1,1}$) pick a subset X such that

$$(3) \quad |A_{p,1} \cup X| = 2 \left| \bigcup_{k=1}^{p} A'_{k,1} \right| + 1$$

and define $A'_{p+1,1} = A_{p,1} \cup X$. If $p = i$ pick X such that

$$(4) \quad |A_{i,1} \cup A_{i,2} \cup X| = 2 \left| \bigcup_{k=1}^{i} A'_{k,1} \right| + 1$$

and define $A'_{i+1} = A_{i,1} \cup A_{i,2} \cup X$. If at some stage there are not enough elements of $A_{j,2}$ remaining to achieve equality in (3) or (4), then use all the remaining elements and the process terminates. In any case, the process terminates before or with $A_{j,1}$ where we define $A'_{j+1,1}$ to be $A_{j,1}$ together with all elements of $A_{j,2}$ remaining after the previous stages. If the process terminates with $A_{p,1}$ where $p < i$, then let $A'_{i+1,1} = A_{i,1} \cup A_{i,2}$. If $A_{p,q}$ is any set that receives no element of $A_{j,2}$, define $A'_{p+1,q} = A_{p,q}$. Label any element of $A'_{p,q}$ with p for $p = 1, \ldots, k-r+1$.

$\mathfrak{a}'$ certainly satisfies conditions (i), (ii), and (iii). Condition (iv) is satisfied by $A'_{p,q}$ for $p \neq j+1$. If the process terminates before reaching $A_{j,1}$, that is, if $A'_{j+1,1} = A_{j,1}$, then $A'_{j+1,1}$ satisfies

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condition (iv). Assume $A_{j,2}$ contributes elements to $A'_{j+1,1}$. Then

$$(5) \qquad \left| \bigcup_{p=1}^{j} A'_{p,1} \right| = 1 + 3 + \ldots + 3^{j-1}.$$

Because of the choice of i and j and condition (iv)

$$|A_{p,1}| \leq 3^{p-1} \quad \text{for } p = 1, \ldots, i-1$$

$$|A_{i,q}| \leq 3^{i-1} \quad \text{for } q = 1, 2$$

$$|A_{p,q}| \leq 3^{p-1} + 2 \cdot 3^{p-2} \quad \text{for } p = i+1, \ldots, j.$$

Then $\left| \bigcup_{p=1}^{j} \bigcup_{q=1}^{n_p} A_{p,q} \right| \leq 1 + 3 + \ldots + 3^{j-1} + 2 \cdot 3^{j-1} + 2 \cdot 3^{j-2}$.
Therefore, by (5) $A'_{j+1,1}$ contains at most $2 \cdot 3^{j-1} + 2 \cdot 3^{j-2} < 3^j$ $= 2 \left| \bigcup_{p=1}^{j} A'_{p,1} \right| + 1$ elements. Thus $\mathfrak{a}'$ is a partition of A with k-r+1 labels and s-1 partition sets. The proof of the lemma is complete.

3. Main result. It is easy to verify that $g(1) = 1$, $g(2) = 1$, and $g(3) = 3$. We are now in a position to prove the following:

THEOREM. If $g(n)$ is the maximum value of $|G(T_n)|$ taken over all tournaments T_n of order n, then $g(n) \leq \sqrt{3}^{n-1}$, $n = 1, 2, \ldots$, with equality holding if and only if $n = 3^k$ for some non-negative integer k.

Proof. We proceed by induction after observing the result is true for $n = 1, 2$, and 3. Let T_n be a tournament of order $n \geq 4$. Pick a vertex $u \in T_n$ and let $D = \{v \in T_n : \alpha(u) = v$ for some $\alpha \in G(T_n)\}$. Let $T_d = \langle D \rangle$ and $T_{n-d} = \langle T_n - D \rangle$. If $d < n$, then by the induction hypothesis we have

$$|G(T_n)| \leq |G(T_d)| \, |G(T_{n-d})| \leq \sqrt{3}^{n-2}.$$

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If $T_d = T_n$, then it is clear n must be odd. Letting $G_u = \{\alpha \in G(T_n) : \alpha(u) = u\}$ and writing $n = 2m + 1$ we have $|G(T_n)| = (2m+1)|G_u|$ by [3, Theorem 3.2, p. 5]. Any $\gamma \in G_u$ maps $\mathcal{J}(u)$ onto itself and $\mathcal{O}(u)$ onto itself. Thus, we can view G_u as a subgroup of the direct product of $G(\langle \mathcal{O}(u) \rangle)$ and $G(\langle \mathcal{J}(u) \rangle)$. Therefore, $|G_u| \leq |H_e||G(\langle \mathcal{O}(u) \rangle)|$ where H_e is viewed as the group of automorphisms of T_n leaving u and each vertex of $\mathcal{O}(u)$ fixed and, $|G_u| \leq |H'_e||G(\langle \mathcal{J}(u) \rangle)|$ where H'_e is viewed as the group of automorphisms of T_n leaving u and each vertex of $\mathcal{J}(u)$ fixed.

For $v_1, v_2 \in \mathcal{J}(u)$ let $v_1 \sim v_2$ if $v_1 = v_2$ or if v_1 and v_2 have the same score in $\langle \mathcal{J}(u) \rangle$ and dominate exactly the same vertices of $\mathcal{O}(u)$. It is clear that $\sim$ is an equivalence relation in $\mathcal{J}(u)$. Moreover, for any two distinct vertices $v_1, v_2 \in \mathcal{J}(v)$ there exists an $\alpha \in H_e$ such that $\alpha(v_1) = v_2$ only if $v_1 \sim v_2$. Hence, by the induction hypothesis, if s is the number of equivalence classes in $\mathcal{J}(u)$, then $|H_e| \leq \sqrt{3}^{m-s}$.

Let $s_1 < s_2 < \ldots < s_r$ be the distinct scores that occur in $\langle \mathcal{J}(u) \rangle$. If we assign the label i to the vertices with score s_i, then we claim that the equivalence classes relative to $\sim$ form a partition of $\mathcal{J}(u)$ with r labels. Conditions (i) and (ii) obviously are satisfied. If t is the number of vertices in some equivalence class C whose vertices are labelled i, then there is some vertex $v \in C$ which is dominated by at least $q = \left[{}^{t}/_{2}\right]$ other vertices of C, say $v_1, v_2, \ldots, v_q$, where $[\,]$ denotes the greatest integer function. Clearly $v_1, v_2, \ldots, v_q \in \mathcal{J}(v)$ and any vertex of $\mathcal{O}(u)$ dominated by v is in $\mathcal{O}(v)$ along with u itself. Thus, the score of v_i in $\langle \mathcal{J}(v) \rangle$, $i = 1, \ldots, q$, is strictly less than its score in $\langle \mathcal{J}(u) \rangle$. Since $\langle \mathcal{J}(u) \rangle$ and $\langle \mathcal{J}(v) \rangle$ are isomorphic, there are at least q vertices of $\mathcal{J}(u)$ with score strictly less than s_i. Therefore, conditions (iii) and (iv) are satisfied.

For any two vertices $w_1, w_2 \in \mathcal{O}(u)$, let $w_1 \approx w_2$ if $w_1 = w_2$ or if w_1 and w_2 have the same score in $\langle \mathcal{O}(u) \rangle$ and are dominated by exactly the same vertices of $\mathcal{J}(u)$. Again, $|H'_e| \leq \sqrt{3}^{m-s}$ where s is the number of equivalence classes relative to $\approx$. If

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$s_1 > s_2 > \ldots > s_r$ are the distinct scores in $\langle \mathcal{O}(u) \rangle$ and vertices with with score s_i are given the label i, then the equivalence classes relative to $\approx$ form a partition of $\mathcal{O}(u)$ with r labels.

Let k be the unique positive integer such that $\sqrt{3}^{2k} < 2m + 1 \leq \sqrt{3}^{2k+2}$ which implies $1 + 3 + \ldots + 3^{k-1} < m$ if $2m + 1 > \sqrt{3}^{2k}$ and $1 + 3 + \ldots + 3^{k-1} + 3^{k-1} < m$ if $2m + 1 > \sqrt{3}^{2k+1}$ Let q be the least number of distinct scores occuring in $\langle \mathcal{O}(u) \rangle$ or $\langle \mathcal{I}(u) \rangle$. Without loss of generality suppose q is the number of distinct scores in $\langle \mathcal{I}(u) \rangle$. If $q \geq k + 1$, then by the induction hypothesis $|G(T_n)| = (2m+1)|G_u| \leq (2m+1)|G(\langle \mathcal{I}(u) \rangle)||G(\langle \mathcal{O}(u) \rangle)| \leq \sqrt{3}^{2k+2} \sqrt{3}^{m-k-1} \sqrt{3}^{m-k-1} = \sqrt{3}^{2m} = \sqrt{3}^{n-1}$. If $q = k-r$ for $0 \leq r \leq k-1$, then using the equivalence classes relative to $\sim$ with the lemma and the induction hypothesis, we obtain $|H_e| \leq \sqrt{3}^{m-k-r-1}$ if $\sqrt{3}^{2k} < 2m+1 < \sqrt{3}^{2k+1}$ which implies $|G(T_n)| = (2m+1)|G_u| \leq \sqrt{3}^{2k+1} \sqrt{3}^{m-k-r-1} \sqrt{3}^{m-k+r} \leq \sqrt{3}^{2m} = \sqrt{3}^{n-1}$. Similarly, if $\sqrt{3}^{2k+1} < 2m + 1 \leq \sqrt{3}^{2k+2}$, the lemma implies $|G(T_n)| = (2m+1)|G_u| \leq \sqrt{3}^{2k+2} \sqrt{3}^{m-k-r-2} \sqrt{3}^{m-k-r} \leq \sqrt{3}^{2m} = \sqrt{3}^{n-1}$.

From the above it is immediate that if n is not a power of 3, then $g(n) < \sqrt{3}^{n-1}$. Formula (6) of [2] shows that $g(n) \geq \sqrt{3}^{n-1}$ if n is a power of 3 and thus $g(n) = \sqrt{3}^{n-1}$ if n is a power of 3. This completes the proof of the theorem.

4. Conclusion. As pointed out in [3], the following result is a consequence of the preceding theorem.

COROLLARY. For each $n = 1, 2, \ldots$ if G_n denotes a subgroup of S_n of maximum odd order, then $|G_n| \leq \sqrt{3}^{n-1}$ with equality if and only if $n = 3^k$ for some integer k.

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Simon Fraser University