BIQUASITRIANGULARITY AND SPECTRAL CONTINUITY

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In [**6**] Conway and Morrell characterized those operators on Hilbert space that are points of continuity of the spectrum. They also gave necessary and sufficient conditions that a biquasitriangular operator be a point of spectral continuity. Our point of view in this note is slightly different. Given a point T of spectral continuity, we ask what can then be inferred. Several of our results deal with invariant subspaces. We also give some conditions characterizing a biquasitriangular point of spectral continuity (Theorem 3). One of these is that the operator and its adjoint both have the single-valued extension property.

We first recall the following definitions and notations. Let H be a separable Hilbert space, and let $L(H)$ be the algebra of bounded linear operators on H. For $T \in L(H)$ let $\sigma(T)$ denote its spectrum, and let $\mathcal{K}$ denote the set of compact sets in the complex plane. The map $\sigma: L(H) \to \mathcal{K}$ defined by $T \to \sigma(T)$ is continuous at $T_0 \in L(H)$ if $T_n \to T_0$ uniformly in $L(H)$ implies that $\sigma(T_n) \to \sigma(T_0)$ in the Hausdorff metric of $\mathcal{K}$.

Now let $T \in L(H)$, and let $\sigma_p^o(T)$ be the set of isolated eigenvalues of T of finite multiplicity. Put $\sigma^o(T) = \sigma_p^o(T) \cup [\sigma_{re}(T) \cap \sigma_{le}(T)]$, where $\sigma_{re}(T)$ and $\sigma_{le}(T)$ are the right and left essential spectra of T. Let $P_0(T)$ denote the set of complex λ such that $\lambda - T$ is Fredholm of zero index, and let $P_1(T)$ denote the set of λ such that $\lambda - T$ is semi-Fredholm of nonzero index.

THEOREM 1 [**6**]. *Let $T \in L(H)$. Then σ is continuous at T if and only if for each $\lambda \in \sigma(T) \backslash P_1(T)^-$ and $\varepsilon > 0$, the ε-neighborhood of λ contains a component of $\sigma^o(T)$.*

A consequence of the Apostol-Foias-Voiculescu theorem [**3**] is that T is biquasitriangular if and only if $P_1(T) = \emptyset$. We shall therefore say that T is biquasitriangular if $P_1(T) = \emptyset$.

An obvious corollary of Theorem 1 is the following result [**6**, Corollary 3.3].

COROLLARY 1. *Let T be biquasitriangular. Then σ is continuous at T if and only if for each $\lambda \in \sigma(T)$ and $\varepsilon > 0$ the ε-neighborhood of λ contains a component of $\sigma^o(T)$.*

COROLLARY 2. *Let T be a point of continuity of σ. If T is not a scalar, then either T has a nontrivial hyperinvariant subspace or T is a translate of a quasinilpotent operator.*

Proof. Suppose $P_1(T) \neq \emptyset$. Since T is not scalar, either T or T^* has a nontrivial eigenspace; hence T has a nontrivial hyperinvariant subspace. If $P_1(T) = \emptyset$, then by Corollary 1 $\sigma(T)$ is the closure of its trivial components. Either $\sigma(T)$ is disconnected or $\sigma(T) = \{\lambda\}$. In the former case T has a nontrivial hyperinvariant subspace by the Riesz-Dunford functional calculus, and in the latter case $T = \lambda + Q$ for some quasinilpotent Q.

Corollary 3. *Let T be a point of continuity of σ. If T is nonscalar and essentially hyponormal, then T has a nontrivial hyperinvariant subspace.*

Proof. By Corollary 2 we may suppose that $\lambda - T$ is quasinilpotent for some λ. Since T is essentially hyponormal, its image in the Calkin algebra is (by definition) hyponormal (that is $\pi(T^*T - TT^*) \geq 0$). Hence $\pi(\lambda - T)$ is a quasinilpotent hyponormal operator, and it follows that $\pi(\lambda - T) = 0$. Thus T is the translate of a compact operator, and so T has a nontrivial hyperinvariant subspace, by Lomonosov's lemma.

A useful property in the theory of spectral decomposition is the single-valued extension property (svep). We say that $T \in L(H)$ has the svep if $f \equiv 0$ is the only analytic function $f: D \to H$ satisfying $(\lambda - T)f(\lambda) = 0$ for all $\lambda \in D$. The following is implicit in [**8**].

Theorem 2. *Let $T \in L(H)$ with adjoint T^*. If both T and T^* have the* svep, *then T is biquasitriangular.*

Proof. We prove that $P_1(T) = \varnothing$. Suppose that $\lambda_0 \in P_1(T)$ so that $\lambda_0 - T$ is semi-Fredholm with nonzero index. If $\text{ind}(\lambda_0 - T) > 0$, it follows from the results of [**8**] that there is a nonzero analytic function $f: D \to H$, where $\lambda_0 \in D$ and $(\lambda - T)f(\lambda) = 0$, for all $\lambda \in D$. This contradicts the hypothesis on T. If $\text{ind}(\lambda_0 - T) < 0$ a similar contradiction for T^* obtains. Hence $P_1(T) = \varnothing$ and the proof is complete.

It follows from the last theorem that every decomposable operator on H is biquasitriangular [**9**], since decomposable operators and their adjoints have the svep (see also [**2**]). A class of operators closely related to decomposable ones is that of (strongly) quasidecomposable operators [**11**]. Albrecht [**1**] showed that these two classes are distinct. The author has recently proved in [**12**, Corollary 2] that, on every Banach space, the adjoint of a strongly quasidecomposable operator has the svep. Since every quasidecomposable operator itself has the svep, the next corollary follows immediately from Theorem 2.

Corollary 4. *Every strongly quasidecomposable operator on a Hilbert space is biquasitriangular.*

Corollary 5. *Let $T \in L(H)$ be subnormal with minimal normal extension N. If T^* has the svep, then $\sigma(T) = \sigma(N)$.*

Proof. Since T has the svep (as the restriction of N), hence T is biquasitriangular by Theorem 2. If $\lambda \in \sigma(T) \backslash \sigma(N)$, then $\lambda - T$ is bounded below but is not surjective. This leads to the contradiction $\text{ind}(\lambda - T) < 0$, (that is $P_1(T) \neq \varnothing$). Thus $\sigma(T) \subset \sigma(N)$, and, because the opposite inclusion always holds, the proof is complete.

Corollary 6. *Let T be subnormal. If the cyclic vectors of T are dense in H, then T is a compact perturbation of a normal operator.*

Proof. By [**4**, Theorem 4] $T^*T - TT^*$ is a trace-class operator; hence $\pi(T)$ is normal in the Calkin algebra. But T has the svep; hence, since its cyclic vectors are dense by hypothesis, it follows by [**10**, Theorem 1] that T^* also has the svep. By Theorem 2, T is biquasitriangular, and this in turn implies that $T = N + K$ with N normal and K compact by [**5**, Corollary 11.2].

COROLLARY 7. *Let $T = S + K$, where S is subnormal and K is compact. If the sets of cyclic vectors for T and T^* are both dense, then T is biquasitriangular. If, in addition, S has a cyclic vector, then $T = N + K'$, where N is normal and K' is compact.*

Proof. For the first conclusion, the hypothesis implies, by [**10**, Theorem 1], that T and T^* both have the svep. Hence T is biquasitriangular by Theorem 2. If S is also cyclic, then S is essentially normal [**4**, Theorem 4]. But then T is also essentially normal, and so $T = N + K'$ by [**5**, Corollary 11.2].

THEOREM 3. *Let $T \in L(H)$ be a point of continuity of σ. Then the following are equivalent*:

(1) $P_1(T) = \varnothing$

(2) *$\sigma(T)$ is nowhere dense*;

(3) *T and T^* both have the* svep;

(4) *$\sigma(T) = \sigma_e(T) \cup \sigma_p^o(T)$, where the essential spectrum $\sigma_e(T)$ is nowhere dense*;

(5) *$\sigma(T)$ is the closure of its isolated points.*

Proof. Implication (3)⇒(1) follows from Theorem 2. If (1) holds, then by Corollary 1 $\sigma(T)$ is the closure of its singleton components of $\sigma^o(T)$. Hence $\sigma(T)$ is nowhere dense, and so (1)⇒(2). Now (2)⇒(3) always holds by [**7**, Lemma XVI, 5, 1, p. 2149]; hence (1)–(3) are equivalent.

Next suppose that $P_1(T) \neq \varnothing$ with $\lambda_0 \in P_1(T)$. If $\text{ind}(\lambda_0 - T) > 0$, then $\text{ind}(\lambda - T) > 0$ for all λ sufficiently near λ_0 (see [**8**, p. 66]). Then clearly (5) fails. Hence (5)⇒(1), and the converse (1)⇒(5) follows from Corollary 1. Obviously, (4)⇒(2). Suppose that (2) holds. Then $\sigma_e(T) \subset \sigma(T)$ is nowhere dense. If $\lambda \in \sigma(T) \backslash \sigma_e(T)$, then $\lambda \in P_0(T)$ by the equivalence (1)⇔(2). By [**6**, Corollary 3.2] $\lambda \in \sigma_p^o(T)$. Hence (2)⇒(4), and the proof is complete.

REMARK. If σ is not continuous at T, the equivalence (1)–(5) in Theorem 3 may fail. The bilateral shift on H is not a point of continuity of σ, and (1)–(4) are true but (5) fails. On the other hand, σ is continuous at the *unilateral* shift and (1)–(5) are *all* false in this case.

To state our final result recall that operators $T, S \in L(H)$ are quasisimilar if there exist bounded injective operators A, B with dense ranges such that $TA = AS$ and $BT = SB$.

COROLLARY 8. *Let T be biquasitriangular, and let S and T be quasisimilar. If σ is continuous at T, then S is biquasitriangular*; *further, if the quasimilarity preserves essential spectrum, then σ is also continuous at S.*

Proof. By Theorem 3, T and T^* both have the svep. It follows by an easy argument that S and S^* also both have the svep. By Theorem 2, S is biquasitriangular. Next suppose that $\sigma_e(S) = \sigma_e(T)$. Since it is easy to check that $\sigma_p^o(S) = \sigma_p^o(T)$ for any quasisimilar operators, it follows that $\sigma(S)$ is the closure of its isolated points. By Corollary 1, S is a point of continuity of the spectrum.

REFERENCES

1. E. Albrecht, An example of a weakly decomposable operator that is not decomposable, *Rev. Roum. Math Pures Appl.* **20** (1975), 555–561.

2. C. Apostol, Quasitriangularity in Hilbert space, *Indiana Univ. Math. J.* **22** (1972/73), 817–825.

3. C. Apostol, C. Foias and D. Voiculescu, Some results on non-quasitriangular operators. IV, *Rev. Roum. Math. Pures Appl.* **18** (1973), 487–514.

4. C. A. Berger and B. I. Shaw, Intertwining, analytic structure, and the trace norm estimate, Lecture Notes in Mathematics No. 345 (Springer-Verlag, 1973).

5. L. Brown, R. G. Douglas and P. Fillmore, Unitary equivalence modulo the compact operators and extensions of C*-algebras, Lecture Notes in Mathematics No. 345 (Springer-Verlag, 1973).

6. J. B. Conway and B. Morrell, Operators that are points of spectral continuity, *Integral Equations and Operator Theory*, **2** (1979), 174–198.

7. N. Dunford and J. T. Schwartz, *Linear operators Part III* (Wiley, 1971).

8. J. Finch, The single valued extension property in a Banach space, *Pacific J. Math.* **58** (1975), 61–69.

9. S. Frunza, A duality theorem for decomposable operators. *Rev. Roum. Math. Pures Appl.* **16** (1971), 1055–1058.

10. D. Herrero, Possible structure for the set of cyclic vectors, *Indiana Univ. Math. J.* **28** (1978), 913–926.

11. A. Jafarian, Weak and quasidecomposable operators, *Rev. Roum. Math. Pures Appl.* **22** (1977), 195–212.

12. R. Lange, Duality and asymptotic spectral decomposition, preprint.

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