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MULTIPLICATIVE FUNCTIONALS ON FRECHET ALGEBRAS WITH BASES

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1. Introduction. Let A denote a complex (or real) Fréchet algebra (i.e. a complete metrizable locally m-convex algebra, see [**2**] or [**3**]). It is known [**2**] that the topology of such an algebra can be defined by an increasing sequence $\{q_n\}$ (i.e. $q_n(x) \leqq q_{n+1}(x)$ for all $x \in A$ and $n \geqq 1$) of submultiplicative (i.e. $q_n(xy) \leqq q_n(x)q_n(y)$ for all $x, y \in A$ and for each $n \geqq 1$) seminorms.

A sequence $\{x_i\}$ in a topological vector space A is said to be a *basis* (see [**3**, p. 114]) if for each $x \in A$ there exists a unique sequence $\{\alpha_i\}$ of complex numbers such that $x = \sum_{i=1}^{\infty} \alpha_i x_i$. It is known that each coordinate functional $\alpha_i(x) = \alpha_i$ is continuous if A is a Fréchet space (i.e. a complete metrizable locally convex space ([**3**, p. 49]).

The purpose of this paper is to prove the continuity of each multiplicative linear functional on a certain Fréchet algebra A with a basis (Theorems 2, 3 and 4). For this we need a necessary and sufficient condition for the convergence of $\sum_{i=1}^{\infty} \alpha_i x_i$ in A for any complex sequence $\{\alpha_i\}$ in terms of the seminorms and the basis (Proposition 1). Also we prove some necessary and sufficient conditions for the existence of an identity in A (Proposision 3, Theorem 1). The problem of whether or not every multiplicative linear functional on a commutative Fréchet algebra is continuous is still unresolved (cf. [**1**; **2**]).

We shall use the definitions and notations from [**1**] and [**2**]. We shall throughout assume that the topology of a Fréchet algebra is given by an increasing sequence $\{q_n\}$ of submultiplicative seminorms.

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2. Some necessary and sufficient conditions. Let A be a Fréchet space with a sequence $\{x_i\}$. We first prove (Proposition 1) a necessary and a sufficient condition for $\sum_{i=1}^{\infty} \alpha_i x_i \in A$ for any complex sequence $\{\alpha_i\}$. This will lead us to a necessary and sufficient conditions for the existence of an identity in a Fréchet algebra (Proposition 3 and Theorem 1).

PROPOSITION 1. *Let $H = \{x_i\}$ be a sequence in a Fréchet space A. Then, for each complex sequence $\{\alpha_n\}$, $\sum_{i=1}^{\infty} \alpha_i x_i \in A$ if and only if for each q_n there is a positive integer N_n (depending on n) such that $q_n(x_i) = 0$ for all $i \geqq N_n$.*

Proof. (Sufficiency) Suppose for each complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty} \alpha_i x_i \in A$.

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Then for each n, $\lim_{i\to\infty} q_n(\alpha_i x_i) = \lim_{i\to\infty} |\alpha_i| q_n(x_i) = 0$. But then it implies that there exists N_n such that $q_n(x_i) = 0$ for all $i \geqq N_n$, because $\{\alpha_i\}$ is arbitrary.

(Necessity) Suppose that for each seminorm q_n there is a positive integer N_n such that $q_n(x_i) = 0$ for all $i \geqq N_n$. Then for any complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty} \alpha_i q_n(x_i)$ is convergent for each n and hence, $\sum_{i=1}^{\infty} \alpha_i x_i$ converges in A for each complex sequence $\{\alpha_i\}$.

PROPOSITION 2. *Let A be a Fréchet algebra with a basis $H = \{x_i\}$ such that, for each $x = \sum_{i=1}^{\infty} \alpha_i x_i \in A$, $\alpha_i \to 0$. Then there exists a q_n such that $q_n(x_i) > 0$ for all $x_i \in H$. Conversely, if A possesses a sequence $H = \{x_i\}$ such that*

i) *for each $x_i \in H$, $x_i^2 = c_i x_i$, where $\inf_{i>1} |c_i| > 0$, and*

ii) *there exists a q_n such that $q_n(x_i) > 0$ for all $x_i \in H$,*

then for each $x = \sum_{i=1}^{\infty} \alpha_i x_i \in A$, $\alpha_i \to 0$.

Proof. For each n, put $Z_n = \{i \in N^+ : q_n(x_i) = 0\}$. Since $\{q_n\}$ is an increasing sequence of seminorms, $Z_n \subset Z_m$ for $n \geqq m$. Suppose $Z_n \neq \emptyset$ for all $n \geqq 1$. Let $x_{k_n} \in Z_n$. But then $q_m(x_{k_n}) = 0$ for $n \geqq m$. Hence, by Proposition 1, $\sum_{n=1}^{\infty} \beta_n x_{k_n}$ converges in A for any complex sequence $\{\beta_n\}$ which is contrary to the assumption. Hence, $Z_{n_0} = \emptyset$ for some n_0 and so $q_{n_0}(x_i) \neq 0$ for $i \geqq 1$.

For the converse, suppose there exists n_0 such that $q_{n_0}(x_i) > 0$ for all $i \geqq 1$. Clearly $[q_{n_0}(x_i)]^2 \geqq q_{n_0}(x_i^2) = |c_i| q_{n_0}(x_i)$ by (i) and so $q_{n_0}(x_i) \geqq \inf |c_i| > 0$. If $x = \sum_{i=1}^{\infty} \alpha_i x_i \in A$, then $\lim_{i\to\infty} q_{n_0}(\alpha_i x_i) = \lim_{i\to\infty} |\alpha_i| q_{n_0}(x_i) = 0$ and hence $|\alpha_i| q_{n_0}(x_i) \geqq |\alpha_i| \inf |c_i|$ implies $\lim_{i\to\infty} |\alpha_i| = 0$.

COROLLARY 2.1. *Let $\{x_i\}$ be a basis in a Fréchet algebra such that $x_i^2 = c_i x_i$, where $\inf_{i\geqq 1} |c_i| > 0$. Then for any $x = \sum_{i=1}^{\infty} \alpha_i x_i \in A$, $\alpha_i \to 0$ if and only if there exists a q_n such that $q_n(x_i) \neq 0$ for all $i \geqq 1$.*

COROLLARY 2.2. *Let $\{x_i\}$ be a basis in a Fréchet space such that for each $x = \sum_{i=1}^{\infty} \alpha_i x_i \in A$, $\alpha_i \to 0$. Then there exists a q_n such that $q_n(x_i) \neq 0$ for all $i \geqq 1$.*

Conversely, if $\{x_i\}$ is a sequence in A such that $\inf_{i\geqq 1} q_n(x_i) > 0$ for some q_n, then for each $x = \sum_{i=1}^{\infty} \alpha_i x_i \in A$, $\alpha_i \to 0$.

PROPOSITION 3. *Let $\{x_i\}$ be a basis of a Fréchet algebra such that $x_i x_j = 0$ whenever $i \neq j$. Then A contains an identity e if and only if $x_i^2 \neq 0$ for all $x_i \in H$ and, for any sequence $\{\beta_i\}$ of complex numbers, $\sum_{i=1}^{\infty} \beta_i x_i \in A$.*

Proof. (Sufficiency) Suppose A has the identity e. Write $e = \sum_{i=1}^{\infty} \alpha_i x_i$. Then $x_j = x_j e = x_j \sum_{i=1}^{\infty} \alpha_i x_i = \alpha_j x_j$. Hence clearly $x_j^2 \neq 0$ and $|\alpha_j| [q_n(x_j)]^2 \geqq q_n(\alpha_j x_j^2) = q_n(x_j)$ and so $q_n(x_j)[|\alpha_j| q_n(x_j) - 1] \geqq 0$. Since $\lim_{j\to\infty} q_n(\alpha_j x_j) = \lim_{j\to\infty} |\alpha_j| q_n(x_j) = 0$ for each n, it follows that $q_n(x_j) = 0$ for each n and j (large enough) depending upon n. Hence by Proposition 1, the rest of the "only if" part follows.

(Necessity) Suppose $x_i^2 \neq 0$ for all $i \geqq 1$ and for each complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty} \alpha_i x_i \in A$. Since $\{x_i\}$ is a basis, for each i we have $x_i^2 = \sum_{j=1}^{\infty} \alpha_{ij} x_j$.

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Then $0 = x_k x_i = x_k(\sum_{j=1}^{\infty} \alpha_{ij} x_j) = \alpha_{ik} x_k^2$ for all $k \neq i$. But then $x_k^2 \neq 0$ implies $\alpha_{ik} = 0$ for all $k \neq i$ and $x_i^2 = \alpha_{ii} x_i$. Whence $\alpha_{ii} \neq 0$ and $\sum_{i=1}^{\infty} (1/\alpha_{ii}) x_i \in A$ by our assumption. If we put $e = \sum_{i=1}^{\infty} (1/\alpha_{ii}) x_i$, then for any $x = \sum_{i=1}^{\infty} \beta_i x_i \in A$, $ex = xe = \sum_{i=1}^{\infty} \beta_i \alpha_{ii}^{-1} x_i^2 = \sum_{i=1}^{\infty} \beta_i x_i = x$ proves that e is the identity.

THEOREM 1. *Let A be a Fréchet algebra with a basis $\{x_i\}$ such that $x_i x_j = 0$ for $i \neq j$, and $\neq 0$ for $i = j$. Then the following statements are equivalent:*

(i) *For each complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty} \alpha_i x_i \in A$.*

(ii) *For each seminorm q_n, there is a positive integer N_n such that $q_n(x_i) = 0$ for all $i > N_n$.*

(iii) *A contains an identity e.*

(iv) $A \simeq \mathbf{C}^{\mathbf{N}}$.

Proof. (i) $\Leftrightarrow$ (ii) follows from Proposition 1; (i) $\Leftrightarrow$ (iii) from Proposition 2. Hence (ii) $\Leftrightarrow$ (iii). Clearly (i) $\Leftrightarrow$ (iv).

Remark. Theorem 1 shows that the only Fréchet algebra with an orthogonal ($x_i x_j = 0$, $i \neq j$) basis and an identity is the algebra of all complex sequences, provided with the coordinate-wise multiplication.

3. On the continuity of multiplicative linear functionals. As mentioned before, it is not yet known whether or not every multiplicative linear functional on a commutative Fréchet algebra is continuous (Michael [**2**]). In this section we show that the answer to this question is in the affirmative for certain Fréchet algebras with a basis. See also [**1**].

THEOREM 2. *Let A be a Fréchet algebra with a basis $\{x_i\}$ such that $x_i x_j = 0$ for $i \neq j$. Then every multiplicative linear functional f on A is continuous, provided for some x_i, $f(x_i) \neq 0$.*

Proof. Let $x = \sum_{i=1}^{\infty} \alpha_i x_i$ be an arbitrary element of A. Clearly by hypothesis, $x\, x_i = \alpha_i x_i^2$. Since f is multiplicative and $f(x_i) \neq 0$, we have $f(xx_i) = f(x) f(x_i) = \alpha_i f^2(x_i)$, or $f(x) = \alpha_i f(x_i) = \alpha_i(x) f(x_i)$. Since each $\alpha_i(x)$ is continuous and $f(x_i)$ is a fixed complex number, this proves that f is continuous.

In order to prove a result similar to Theorem 2 under different conditions, we need the following:

LEMMA 3.1. *Let A be a Fréchet algebra with a sequence $\{x_i\} \subset A$ such that $x_i x_j = 0$ for $i \neq j$ and for each complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty} \alpha_i x_i \in A$. If f is a multiplicative linear functional on A such that $f(x_i) = 0$ for all $i \geqq 1$, then $f(\sum_{i=1}^{\infty} \alpha_i x_i) = 0$ for any complex sequence $\{\alpha_i\}$.*

Proof. First, we show that $f(\sum_{i=1}^{\infty} x_i) = 0$. If not, then by hypothesis, $f(\sum_{i=1}^{\infty} i x_i) f(\sum_{i=1}^{\infty} (1/i) x_i) = f(\sum_{i=1}^{\infty} x_i^2) = [f(\sum_{i=1}^{\infty} x_i)]^2 \neq 0$. Therefore, $f(\sum_{i=1}^{\infty} i x_i) \neq 0$. Put $f(\sum_{i=1}^{\infty} x_i) = \alpha$ and $f(\sum_{i=1}^{\infty} i x_i) = \alpha\beta$. Then $f(\sum_{i=1}^{\infty} (\beta - i) x_i) = f(\beta \sum_{i=1}^{\infty} x_i - \sum_{i=1}^{\infty} i x_i) = \beta f(\sum_{i=1}^{\infty} x_i) - f(\sum_{i=1}^{\infty} i x_i) = \beta\alpha - \beta\alpha = 0$. If β is not a positive integer, then $0 = f(\sum_{i=1}^{\infty} (\beta - i) x_i)$

$f(\sum_{i=1}^{\infty} (1/(\beta - i))x_i) = f(\sum_{i=1}^{\infty} x_i^2) = [f(\sum_{i=1}^{\infty} x_i)]^2 \neq 0$, which is impossible. If β is an integer, say $\beta = n$, then (since $f(x_i) = 0$ for all i)

$$0 = f\left(\sum_{i=1}^{\infty} (\beta - i)x_i\right)f\left(\sum_{\substack{i=1\\i\neq n}}^{\infty} \frac{1}{\beta - i}x_i\right) = f\left(\sum_{\substack{i=1\\i\neq n}}^{\infty} x_i^2\right)$$
$$= \left[f\left(\sum_{\substack{i=1\\i\neq n}}^{\infty} x_i\right)\right]^2 = \left[f\left(\sum_{i=1}^{\infty} x_i\right)\right]^2 \neq 0,$$

which is also impossible. Hence $f(\sum_{i=1}^{\infty} x_i) = 0$, and for any complex sequence $\{\alpha_i\}, f(\sum_{i=1}^{\infty} \alpha_i x_i)f(\sum_{i=1}^{\infty} \alpha_i x_i) = f(\sum_{i=1}^{\infty} \alpha_i^2 x_i^2) = f(\sum_{i=1}^{\infty} \alpha_i^2 x_i)f(\sum_{i=1}^{\infty} x_i) = 0$, which proves the lemma.

Remark. Lemma 3.1 shows that each multiplicative linear functional on the Fréchet algebra $\mathbf{C}^{\mathbf{N}}$ of all complex sequences is identically zero if it vanishes on the basis $\{e_i\}$, $e_i = \{\delta_{ij}\}_{j\geqq 1}$, where δ_{ij} is the Kronecker δ, which is a well-known fact, since $\mathbf{C}^{\mathbf{N}}$ is singly generated.

THEOREM 3. *Let A be a Fréchet algebra with a basis $H = \{x_i\}$ such that $x_i x_j = 0$ for $i \neq j$ and for any complex sequence $\{\alpha_i\}$, $\sum_{i=1}^{\infty} \alpha_i x_i \in A$. Then every multiplicative linear functional f on A is continuous.*

Proof. If there exists $x_i \in H$ such that $f(x_i) \neq 0$, then by Theorem 2, f is continuous. If $f(x_i) = 0$ for all $x_i \in H$, then by hypothesis coupled with Lemma 3.1 we see that f is identically zero and so continuous.

Before we prove the next theorem, we make the following observations: If $\{x_i\}$ is a basis of a Fréchet algebra A such that $x_i x_j = 0$ for $i \neq j$, then for each x_i^2 there exists a unique complex sequence $\{\alpha_{ij}\}$ such that

$$x_i^2 = \sum_{j=1}^{\infty} \alpha_{ij} x_j.$$

Multiplying the last equation both sides by x_i and using the orthogonality relation $x_i x_j = 0$ for $i \neq j$, we obtain:

(*) $\quad x_i^3 = \alpha_{ii} x_i^2.$

Again multiplying (*) by x_i we have:

(**) $\quad x_i^4 = \alpha_{ii} x_i^3.$

If $\alpha_{ii} \neq 0$, then by putting $y_i = \alpha_{ii}^{-1} x_i$ in (**), we obtain: $y_i^4 = y_i^3$. If $\alpha_{ii} = 0$, then from (*) and (**) we see that $x_i^3 = x_i^4 = 0$. Thus it may be assumed that a given orthogonal basis $\{x_i\}$ satisfies:

(***) $\quad x_i^4 = x_i^3.$

If an orthogonal basis $\{x_i\}$ satisfies (***), we say that it is *normalized*. Clearly from (***) we have:

(****) $\quad x_i^6 = x_i^4 \quad \text{and} \quad x_i^8 = x_i^6.$

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THEOREM 4. *Let A be a Fréchet algebra with a basis $H = \{x_i\}$ such that*
(i) $x_i x_j = 0$ *whenever* $j \neq i$, *and*
(ii) $\sum_{i=1}^{\infty} \alpha_i a_i x_i \in A$ *whenever* $\sum_{i=1}^{\infty} \alpha_i x_i \in A$ *and* $|a_i| \leqq 1$.
Then every multiplicative linear functional on A is continuous.

Proof. Let f be a multiplicative linear functional on A. Suppose there exists an $x_i \in H$ such that $f(x_i) \neq 0$. Then f is continuous by Theorem 2.

Suppose $f(x_i) = 0$ for all $x_i \in H$. We show then that f is identically zero.

It is easy to see that without any loss of generality we may assume $\{x_i\}$ is a normalized basis satisfying conditions (i) and (ii) of the hypothesis.

Let $B_n = \{x \in H: q_n(x^2) = 0\}$ for each $n \geqq 1$. Since $\{q_n\}_{n \geqq 1}$ is an increasing sequence of seminorms, it follows that $B_n \supset B_{n+1}$ for all $n \geqq 1$. Let $H_1 = \{x \in H: q_1(x^2) \neq 0\}$ and $H_n = B_{n-1} \backslash B_n$, for $n \geqq 2$, then, $H_n \cap B_n = \emptyset$ and so

(1) $\qquad q_n(x^2) \neq 0 \quad$ for all $x \in H_n$, $n \geqq 1$.

It is clear that for $m > n$, $H_m \subset B_{m-1} \subset B_n$, and so we have

(2) $\qquad q_n(x^2) = 0 \quad$ for all $x \in H_m$, $m > n$.

Put $H_0 = \bigcap_{i=1}^{\infty} B_i$. If $x \in H_0$ then $q_n(x^2) = 0$ for all $n \geqq 1$. Since A is a Hausdorff topological vector space, we have $x^2 = 0$. It is clear by definition that $\{H_i\}_{i \geqq 1}$ are pairwise disjoint and $\bigcup_{i=0}^{\infty} H_i = H$. Set $H_j = \{x_{jm}\}_{m \geqq 1}$. For each j, we can arrange $\{x_{jm}\}_{m \geqq 1}$ such that $jm \geqq jn$ if and only if $m > n$. We show that, for each $x = \sum_{i=1}^{\infty} \alpha_i x_i \in A$, we can write

$$x^2 = \sum_{i=1}^{\infty} \alpha_i^2 x_i^2 = \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{jm}^2 x_{jm}^2.$$

For each positive integer k, put

$$a_i = \begin{cases} 1 & \text{if } x_i \in H_k. \\ 0 & \text{if } x_i \notin H_k. \end{cases}$$

Clearly, $|a_i| \leqq 1$ and so by hypothesis (ii), we have:

$$\sum_{i=1}^{\infty} a_i \alpha_i x_i = \sum_{x_i \in H_k} \alpha_i x_i \in A.$$

Since $km > kn$ if and only if $m > n$, $\sum_{x_i \in H_k} \alpha_i x_i = \sum_{m=1}^{\infty} \alpha_{km} x_{kn} \in A$, and hence $(\sum_{m=1}^{\infty} \alpha_{km} x_{km})^2 = \sum_{m=1}^{\infty} \alpha_{kl}^2 x_{km}^2 \in A$ in view of hypothesis (i). Set $y_k = \sum_{m=1}^{\infty} \alpha_{km}^2 x_{km}^2$, for $k \geqq 1$. From (2), we observe that, for $k > j$, $q_j(x_{km}^2) = 0$ for all $x_{km} \in H_k$ and so $q_j(\sum_{m=1}^{p_k} x_{km}^2) = 0$ for $k > j$. Therefore, $\sum_{m=1}^{p_k} \alpha_{km}^2 x_{km}^2 \to \sum_{m=1}^{\infty} \alpha_{km}^2 x_{km}^2 = y_k$ as $p_k \to \infty$ implies that $q_j(y_k) = 0$ whenever $k > j$. Thus by Proposition 1, it follows that, for any complex sequence $\{\beta_k\}$, $\sum_{k=1}^{\infty} \beta_k y_k \in A$. In particular, $\sum_{k=1}^{\infty} y_n = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{km}^2 x_{km}^2 \in A$.

Since $y_k = \sum_{m=1}^{\infty} \alpha_{km}^2 x_{km}^2$, for each seminorm q_j, $q_j(y_k - \sum_{m=1}^{p_k} \alpha_{km}^2 x_{km}^2) \to 0$

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as $p_k \to \infty$ for each $k \geqq 1$. Observe that

$$q_j\left(\sum_{k=1}^{\infty} y_k - \sum_{i=1}^{p} \alpha_i^2 x_i^2\right) = q_j\left(\sum_{k=1}^{\infty}\left(\sum_{m=1}^{\infty} \alpha_{km}^2 x_{km}^2 - \sum_{m=1}^{p_k} \alpha_{km}^2 x_{km}^2\right)\right)$$
$$\leqq \sum_{k=1}^{j} q_j\left(\sum_{m=1}^{\infty} \alpha_{km}^2 x_{km}^2 - \sum_{m=1}^{p_k} \alpha_{km}^2 x_{km}^2\right),$$

because $q_j(x_{km}^2) = 0$ for $k > j$ by (2). Here p is split up into p_k's in the obvious way and $p \to \infty$ implies $p_k \to \infty$. Hence $q_j(\sum_{k=1}^{\infty} y_k - \sum_{i=1}^{p} \alpha_i^2 x_i^2) \to 0$ as $p \to \infty$ for $k > j$. Since A is Hausdorff, we have shown that

$$x^2 = \sum_{i=1}^{\infty} \alpha_i^2 x_i^2 = \sum_{k=1}^{\infty} y_k = \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \alpha_{km}^2 x_{km}^2.$$

But from $y_k = \sum_{m=1}^{\infty} \alpha_{km}^2 x_{km}^2$, it follows that $\lim_{m\to\infty} q_k(\alpha_{km}^2 x_{km}^2) = \lim_{m\to\infty} |\alpha_{km}|^2 q_{km}(x_{km}^2) = 0$. Whence we conclude that $\lim_{m\to\infty} |\alpha_{km}|^2 = 0$ for each k, because $q_k(x_{km}^2) \neq 0$ by (1).

Since $x_i x_j = 0$ for $i \neq j$, it is clear that $y_i y_j = 0$ for $i \neq j$.

We show that $f(y_i) = 0$ for all $i \geqq 1$. Suppose $f(y_i) \neq 0$ for some i. Without loss of generality, observe that we may assume that $f(y_i) = 1$ for some i. Since f is a multiplicative linear functional, we have $f(y_i^2 - y_i^3) = f(z_i) = 0$, where

$$z_i = y_i^2 - y_i^2 = \left(\sum_{m=1}^{\infty} \alpha_{im}^2 x_{im}^2\right)^2 - \left(\sum_{m=1}^{\infty} \alpha_{im}^2 x_{im}^2\right)^3$$
$$= \sum_{m=1}^{\infty} \alpha_{im}^4 x_{im}^4 - \sum_{m=1}^{\infty} \alpha_{im}^6 x_{im}^6.$$

Since by (****), $x_{im}^4 = x_{im}^6$, we have $z_i = \sum_{m=1}^{\infty} (\alpha_{im}^4 - \alpha_{im}^6) x_{im}^4$.

Since $\alpha_{im}^2 \to 0$ as $m \to \infty$, for large m, $|\alpha_{im}^2| < 1/2$ and so $|\alpha_{im}^6/(\alpha_{im}^4 - \alpha_{im}^6)| \leqq 1$, provided $\alpha_{im} \neq 0$. (Observe that $f(y_i) = 1$ implies that there exists $\alpha_{im} \neq 0$). Since $x = \sum_{i=1}^{\infty} \alpha_i x_i \in A$, by hypothesis (ii), $\sum_{m=1}^{\infty} \alpha_{im} x_{im} \in A$ and

$$\sum_{\substack{m=1\\ \alpha_{im}\neq 0}}^{\infty} \frac{\alpha_{im}^6}{\alpha_{im}^4 - \alpha_{im}^6} \alpha_{im} x_{im} \in A.$$

Hence using hypothesis (i), we have

$$z_i' = \left(\sum_{\substack{m=1\\ \alpha_{im}\neq 0}}^{\infty} \frac{\alpha_{im}^6}{\alpha_{im}^4 - \alpha_{im}^6} \alpha_{im} x_{im}\right) \times \left(\sum_{m=1}^{\infty} \alpha_{im} x_{im}\right)$$
$$= \sum_{\substack{m=1\\ \alpha_{im}\neq 0}}^{\infty} \frac{\alpha_{im}^8}{\alpha_{im}^4 - \alpha_{im}^6} x_{im}^2 \in A.$$

But then $z_i z_i' = \sum_{m=1}^{\infty} \alpha_{im}^8 x_{im}^6$. Again by (****), $x_{im}^6 = x_{im}^8$ and so $z_i z_i' = \sum_{m=1}^{\infty} \alpha_{im}^8 x_{im}^8 = (\sum_{m=1}^{\infty} \alpha_{im}^2 x_{im}^2)^4 = y_i^4$. Thus we have $0 = f(z_i) f(z_i') = f(z_i z_i') = [f(y_i)]^4 = 1$, which is absurd. Hence $f(y_i) = 0$ for all $i \geqq 1$, and by applying Lemma 3.1 to the sequence $\{y_i\}_{i\geqq 1}$ we obtain $0 = f(\sum_{i=1}^{\infty} y_i) = f(x^2) = (f(x))^2$. In other words, $f(x) = 0$ for an arbitrary $x \in A$. This completes the proof.

Remarks. Recall that a basis satisfying the hypothesis (ii) is called an *unconditional basis* ([**3**, p. 185]). Every basis in a Fréchet nuclear space is unconditional. Thus we have from Theorem 4 that every nuclear Fréchet algebra with an orthogonal basis is functionally continuous.

Theorem 4 says that every maximal ideal of codimension one in a Fréchet algebra with an orthogonal and unconditional basis (in particular, a Fréchet nuclear algebra with an orthogonal basis) is closed.

We note that some results of § 2 have appeared in a paper [**4**] by one of us with slightly different proofs. But they are given here for the sake of completeness.

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