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Braid group actions via categorified Heisenberg complexes

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ABSTRACT

We construct categorical braid group actions from 2-representations of a Heisenberg algebra. These actions are induced by certain complexes which generalize spherical (Seidel–Thomas) twists and are reminiscent of the Rickard complexes defined by Chuang–Rouquier. Conjecturally, one can relate our complexes to Rickard complexes using categorical vertex operators.

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1. Introduction

Let $\mathcal{D}(0)$ and $\mathcal{D}(1)$ be graded, triangulated categories and let $P : \mathcal{D}(0) \rightarrow \mathcal{D}(1)$ and $Q : \mathcal{D}(1) \rightarrow \mathcal{D}(0)$ be bi-adjoint functors up to a grading shift (which we take to be equal to 2 for convenience). If $Q \circ P \cong \mathbf{1}_0 \langle -1 \rangle \oplus \mathbf{1}_0 \langle 1 \rangle$, where $\mathbf{1}_0$ denotes the identity functor of $\mathcal{D}(0)$, then P is called a spherical functor. This notion is due (in various levels of generality) to Seidel–Thomas [ST01], Horja [Hor05], Anno [Ann07] and Rouquier [Rou04].

The general theory of spherical twists states that $\Sigma := \text{Cone}(P \circ Q \langle -1 \rangle \rightarrow \mathbf{1}_1)$ is an autoequivalence of $\mathcal{D}(1)$. One important reason to consider equivalences coming from spherical functors is that if $\{P_i\}$ is a configuration of spherical functors (meaning that there is one P_i for each node i of a simply laced Kac–Moody graph D and $\text{Hom}^*(P_i, P_j)$ is either one-dimensional

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or zero-dimensional depending on whether or not i and j are joined by edge), then the associated autoequivalences Σ_i will define an action of the corresponding braid group Br_D on $\mathcal{D}(1)$.

The notion of a spherical twist was generalized in [CK12, CR08] to that of categorical \mathfrak{g} actions, with \mathfrak{g} a symmetric Kac–Moody algebra. Such categorical \mathfrak{g} actions can be used to construct further examples of braid group actions. Another generalization of spherical twists, which replaces the role of the Kac–Moody algebra by a Heisenberg algebra, is the subject of the current paper. Namely, fix a simply laced Kac–Moody Dynkin diagram D with vertex set I . For each $n \geq 0$, suppose that we have a collection of additive categories $\mathcal{D}(n)$ together with bi-adjoint functors

$$P_i : \mathcal{D}(n) \rightarrow \mathcal{D}(n + 1) \quad \text{and} \quad Q_i : \mathcal{D}(n + 1) \rightarrow \mathcal{D}(n)$$

for any $i \in I$ which give a 2-representation of a particular Heisenberg algebra (see § 4). Roughly speaking, this means that we have isomorphisms

$$Q_i \circ P_i \cong P_i \circ Q_i \oplus \mathbf{1}_n\langle -1 \rangle \oplus \mathbf{1}_n\langle 1 \rangle, \tag{1}$$

along with a precise collection of natural transformations of functors. Such a 2-representation generalizes the notion of a spherical functor since P_i and Q_i are spherical functors between $\mathcal{D}(0)$ and $\mathcal{D}(1)$.

However, the data of a Heisenberg 2-representation contains more than just a spherical functor. For instance, the action by natural transformations includes an action of the symmetric group S_k on the composition P_i^k . This splits P_i^k into a direct sum of indecomposable functors $P_i^{(\lambda)}$ corresponding to irreducible representations of S_k (Q_i^k also splits analogously). We may then form a complex

$$\Sigma_i \mathbf{1}_n := \left[\cdots \rightarrow \bigoplus_{\lambda \vdash d} P_i^{(\lambda)} Q_i^{(\lambda^t)} \langle -d \rangle \mathbf{1}_n \rightarrow \bigoplus_{\lambda \vdash d-1} P_i^{(\lambda)} Q_i^{(\lambda^t)} \langle -d + 1 \rangle \mathbf{1}_n \rightarrow \cdots \rightarrow P_i Q_i \langle -1 \rangle \mathbf{1}_n \rightarrow \mathbf{1}_n \right] \tag{2}$$

(we have omitted the symbol \circ for composition of functors in the above, as we will do for the remainder of the paper). Theorem 1.1 of the current paper states that these complexes define a categorical action of the associated braid group on the homotopy category of each $\mathcal{D}(n)$. In particular, each Σ_i defines an equivalence of categories.

An important example where the setup above holds is the following. Let A be the skew zig-zag algebra (defined in § 2.3), which is the quadratic dual of the deformed preprojective algebra of a quiver. For $n \geq 0$, we let $A^{[n]}$ denote the wreath product of A with the group algebra of S_n (by convention, we take $A^{[0]} = \mathbb{C}$). By a formal construction, the braid group action of [HK01, KS02] on the homotopy category $\text{Kom}(A\text{-mod})$ by spherical twists lifts to a braid group action on $\text{Kom}(A^{[n]}\text{-mod})$ for each n .

On the other hand, from the point of view of representation theory of infinite-dimensional algebras, it is natural to consider the categories $\text{Kom}(A^{[n]}\text{-mod})$ together. In particular, in [CL12] we define 2-representations of a Heisenberg algebra on $\bigoplus_n A^{[n]}\text{-mod}$. Thus, there are two algebraic objects of interest: the braid group action (which is somewhat complicated) and the Heisenberg action (which is simpler). The constructions of the current paper explain precisely the relationship between these two actions. In particular, we prove that integrable 2-representations of the Heisenberg algebra always induce braid group actions.

There is also a geometric version of this setup, where the algebra A is replaced by a surface X , the algebra $A^{[n]}$ is replaced by the Hilbert scheme $X^{[n]}$ of n points on X , and the triangulated category $\text{Kom}(A^{[n]}\text{-mod})$ is replaced by the derived category of coherent sheaves $D(X^{[n]})$. Then, as studied by Ploog [Plo07], if a group G acts on $D(X)$ then there is an induced action of G

on $D(X^{[n]})$. In particular, if G is an affine braid group of simply laced type, one can take the surface X to be the minimal resolution $\widehat{\mathbb{C}^2/\Gamma}$, where $\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ is the finite subgroup associated to the affine quiver by the McKay correspondence. A 2-representation of the associated Heisenberg algebra on $\bigoplus_n D(X^{[n]})$ was defined in [CL12], and Theorem 6.1 then describes the relationship between this 2-representation and the associated affine braid group action on $D(X^{[n]})$.

In the remainder of the introduction we will give a more detailed exposition of the content in this paper.

1.1 Heisenberg actions and braid groups

To any simply laced Dynkin diagram D one can associate a quantum Heisenberg algebra $\widehat{\mathfrak{h}}$. This algebra has generators $P_i^{(n)}, Q_i^{(n)}$ satisfying relations described in §3. A representation V of this algebra breaks up into weight spaces $V = \bigoplus_{\ell \in \mathbb{N}} V(\ell)$ with

$$P_i^{(n)} : V(\ell) \rightarrow V(\ell + n) \quad \text{and} \quad Q_i^{(n)} : V(\ell + n) \rightarrow V(\ell).$$

In [CL12] we define a 2-category \mathcal{H} whose Grothendieck group is isomorphic to $\widehat{\mathfrak{h}}$. A 2-representation of \mathcal{H} consists of graded, additive categories $\mathcal{D}(\ell)$ where $\ell \in \mathbb{N}$ and, for any partition λ , functors

$$P_i^{(\lambda)} : \mathcal{D}(\ell) \rightarrow \mathcal{D}(\ell + |\lambda|) \quad \text{and} \quad Q_i^{(\lambda)} : \mathcal{D}(\ell + |\lambda|) \rightarrow \mathcal{D}(\ell)$$

satisfying various relations described in §§4.1 and 4.2.

Now, in the homotopy category $\mathrm{Kom}(\mathcal{H})$ of \mathcal{H} one can define complexes as in (2) where the differentials are given by certain explicit 2-morphisms described in §4.5. Given a 2-representation, each $\Sigma_i \mathbf{1}_n$ defines an endofunctor of $\mathrm{Kom}(\mathcal{D}(n))$. The following is the main result of this paper.

THEOREM 1.1. *For each $n \in \mathbb{N}$, the map $\sigma_i \mapsto \Sigma_i \mathbf{1}_n$ defines a morphism $\mathrm{Br}(D) \rightarrow \mathrm{Aut}(\mathrm{Kom}(\mathcal{D}(n)))$ where $\mathrm{Br}(D)$ denotes the braid group associated with the Dynkin diagram D and the σ_i are its standard generators.*

Now, if we take A to be the zig-zag algebra from [HK01], then, following [CL12], we can define a 2-representation of $\widehat{\mathfrak{h}}$ where $\mathcal{D}(n) = A^{[n]\text{-mod}}$. Theorem 1.1 above then gives us a morphism $\mathrm{Br}(D) \rightarrow \mathrm{Aut}(\mathrm{Kom}(A^{[n]\text{-mod}}))$. Applying Theorem 6.1 (see also Remark 6.4) this also gives us a morphism $\mathrm{Br}(D) \rightarrow \mathrm{Aut}(D(A^{[n]\text{-mod}}))$.

When $n = 1$ this gives the braid group action of Khovanov–Seidel [KS02] via spherical twists. For $n > 1$ we recover the action on $D(A^{[n]\text{-mod}}$) induced from that on $D(A\text{-mod})$ (see Theorem 5.3).

1.2 Another braid group action

The complexes Σ_i also act on the 2-category $\mathrm{Kom}(\mathcal{H})$ by conjugation. It would be interesting to describe this braid group action explicitly.

While we do not address the conjugation action on the entire category $\mathrm{Kom}(\mathcal{H})$ here, in §7 we define another braid group action on $\mathrm{Kom}(\mathcal{H})$ and conjecture (Conjecture 8.4) that it agrees with the conjugation action. This additional action is defined explicitly by describing how each generator $\sigma_i^{\pm 1}$ of the braid group acts on the generating 1- and 2-morphisms.

Although conjecturally related to conjugation by the complexes of Theorem 1.1, §7 is independent of the rest of the paper. The proofs of §7 are postponed until Appendix A.

1.3 Lie algebra actions and braid groups

The story above closely parallels (and is directly related to) that of quantum groups. Recall that for any Dynkin diagram D one has the associated quantum group $U_q(\mathfrak{g})$. One can consider 2-representations of \mathfrak{g} , which consist of various additive categories $\mathcal{D}(\lambda)$ indexed by weights λ and functors

$$E_i^{(r)} \mathbf{1}_\lambda : \mathcal{D}(\lambda) \rightarrow \mathcal{D}(\lambda + r\alpha_i) \quad \text{and} \quad \mathbf{1}_\lambda F_i^{(r)} : \mathcal{D}(\lambda + r\alpha_i) \rightarrow \mathcal{D}(\lambda).$$

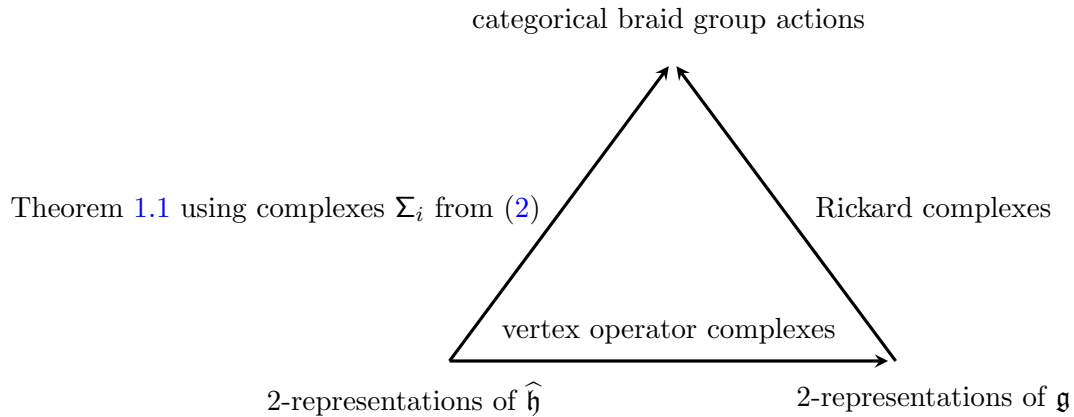
These functors satisfy certain relations lifting those in $U_q(\mathfrak{g})$. For more details see [CL11a, KL09, KL10, KL11, Lau10, Rou08].

In analogy with (2), one can then define the Rickard complexes

$$T_i \mathbf{1}_\lambda := [\dots \rightarrow F_i^{(\langle \lambda, \alpha_i \rangle + s)} E_i^{(s)} \langle -s \rangle \mathbf{1}_\lambda \rightarrow \dots \rightarrow F_i^{(\langle \lambda, \alpha_i \rangle + 1)} E_i \langle -1 \rangle \mathbf{1}_\lambda \rightarrow F_i^{(\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda].$$

These complexes define a morphism $\text{Br}(D) \rightarrow \text{Aut}(\bigoplus_\lambda \text{Kom}(\mathcal{D}(\lambda)))$ just like that in Theorem 1.1. See [CK12, CKL13, CR08] for more details.

The relationship between 2-representations of $\widehat{\mathfrak{h}}$ and 2-representations of \mathfrak{g} is given by the vertex operator constructions from [CL11b]. Thus, we expect to have the following diagram.



As the above diagram indicates, we should be able to deduce Theorem 1.1 as a consequence of the braid group actions arising from 2-representations of \mathfrak{g} [CK12] (this essentially amounts to checking that the diagram above commutes). However, there are several technical details required to give a proof in this way (see §8 for more details), so in this paper we choose to give a direct construction of the left arrow.

2. Preliminaries

We will always work over a base field \mathbb{k} of characteristic zero.

2.1 Dynkin data

Let D be a finite graph without edge loops or multiple edges between vertices. This means that D is associated with a symmetric simply laced Kac–Moody algebra. We let I denote the vertex set of D , and E the edge set. We define a pairing $\langle \cdot, \cdot \rangle : I \times I \rightarrow \mathbb{Z}$ by $\langle i, j \rangle := C_{i,j}$ where $C_{i,j}$ is

the Cartan matrix associated to our Dynkin diagram. More precisely:

$$\langle i, j \rangle = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i \neq j \text{ are joined by an edge} \\ 0 & \text{if } i \neq j \text{ are not joined by an edge.} \end{cases}$$

Associated to this data there is the braid group $\text{Br}(D)$ which is generated by $\{\sigma_i\}_{i \in I}$ subject to the relations $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $i, j \in I$ are not joined by an edge, and $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ if they are joined.

Fix an orientation ϵ of D . For $i, j \in I$ with $\langle i, j \rangle = -1$, we set $\epsilon_{ij} = 1$ if the edge is oriented $i \rightarrow j$ by ϵ and $\epsilon_{ij} = -1$ if oriented $j \rightarrow i$. If $\langle i, j \rangle = 0$, then we set $\epsilon_{ij} = 0$. Note that in both cases we have $\epsilon_{ij} = -\epsilon_{ji}$.

2.2 Partitions

Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \geq 0)$ be a partition. We denote the size of λ by $|\lambda| := \sum_i \lambda_i$; we write $\lambda \vdash n$ if λ is a partition of n , and denote the transposed partition by λ^t . If the number of $\lambda_i = k$ is a_k , we also write $\lambda = (1^{a_1}, 2^{a_2}, \dots, s^{a_s} \dots)$. For example, in this notation, $(n)^t = (1^n)$. We write $\lambda' \subset \lambda$ if $\lambda'_i \geq \lambda_i$ for all i .

We denote by $\mathbb{k}[S_n]$ the group algebra of the symmetric group and $s_k = (k, k + 1) \in S_n$ the simple transposition. Since the characteristic of \mathbb{k} is zero, $\mathbb{k}[S_n]$ is isomorphic to a direct sum of matrix algebras, $\mathbb{k}[S_n] = \bigoplus_{\lambda \vdash n} M_{h_\lambda}(\mathbb{k})$. Here $\{h_\lambda\}_{\lambda \vdash n}$ are positive integers, and $M_s(\mathbb{k})$ is the algebra of $s \times s$ matrices over \mathbb{k} . For any partition λ of n , we denote by $e_\lambda \in \mathbb{k}[S_n]$ a minimal idempotent (a matrix unit) in the matrix algebra M_{h_λ} corresponding to λ . We denote by $\tau : \mathbb{k}[S_n] \rightarrow \mathbb{k}[S_n]$ the involution which sends $s_i \mapsto -s_i$ for all $i \in I$. The minimal idempotents e_λ may be chosen so as to have $\tau(e_\lambda) = e_{\lambda^t}$.

2.3 Zig-zag algebras

Let cD denote the doubled quiver, with the same vertex set as D and with two oriented edges (one in each orientation) for each edge of D . Let $\mathbb{k}[dD]$ denote the path algebra of dD . A path in dD is described as a sequence of vertices $(i_1|i_2| \dots |i_m)$ where i_k and i_{k+1} are connected by an edge in D . If D has more than two nodes then we define B_ϵ^D to be the quotient of $\mathbb{C}[dD]$ by the two sided ideal generated by:

- $(a|b|c)$ if $a \neq c$; and
- $\epsilon_{ab}(a|b|a) - \epsilon_{ac}(a|c|a)$ whenever a is connected to both b and c .

In the above, e_i denotes the constant path which starts and ends at the vertex $i \in I$. If D consists of the single vertex only, we let B_ϵ^D be the algebra generated by 1 and x with $x^2 = 0$. If D consists of two points joined by a single edge, we dene B_ϵ^D to be the quotient of $\mathbb{k}[dD]$ by the two-sided ideal spanned by all paths of length greater than two. Note that B_ϵ^D is \mathbb{Z} -graded by the length of the path (we denote the degree of a path by $|\cdot|$). The \mathbb{Z}_2 -grading induced from the \mathbb{Z} -grading makes B_ϵ^D into a \mathbb{Z} -graded superalgebra.

Remark 2.1. The algebras B_ϵ^D first appeared in [HK01] in the context of categorifying the adjoint representation of the Lie algebra associated to D .

For $n \geq 0$, we define \mathbb{Z} -graded superalgebras $B_\epsilon^D(n) := (B_\epsilon^D)^{\otimes n} \rtimes \mathbb{k}[S_n]$. As a vector space, we have $B_\epsilon^D(n) = (B_\epsilon^D)^{\otimes n} \otimes_{\mathbb{k}} \mathbb{k}[S_n]$, but for the algebra structure the tensor product is in the

category of superalgebras (see [CL12, §9.1]). Thus,

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{|b||a'|} (aa' \otimes bb'),$$

while S_n acts by superpermutations $s_k \cdot (b_1 \otimes \dots \otimes b_k \otimes b_{k+1} \otimes \dots \otimes b_n) = (-1)^{|b_k||b_{k+1}|} b_1 \otimes \dots \otimes b_{k+1} \otimes b_k \otimes \dots \otimes b_n$. By convention, $B_\epsilon^D(0) = \mathbb{k}$. To shorten notation we will write

$$e_{i,m} := (1 \otimes \dots \otimes 1 \otimes e_i \otimes 1 \otimes \dots \otimes 1, 1) \in B_\epsilon^D(n)$$

for the idempotent where e_i occurs in the m th tensor factor on the right-hand side. The \mathbb{Z} -grading on $B_\epsilon^D(n)$ is induced from that on B_ϵ^D , with the factor $\mathbb{k}[S_n]$ placed in degree 0.

Remark 2.2. All of the constructions in the remainder of the paper will involve \mathbb{Z} -graded superalgebras over \mathbb{k} and graded supermodules or superbimodules over such superalgebras. For simplicity, we will write ‘algebra’, ‘module’ and ‘bimodule’, omitting the understood prefixes ‘ \mathbb{Z} -graded’ and ‘super’.

2.4 The wreath functor $(\cdot)^{[n]}$

If A is a algebra, then we can define a new algebra $A^{[n]} := A^{\otimes n} \rtimes \mathbb{k}[S_n]$. The grading and superstructure on $A^{[n]}$ are inherited from that on A , with the understanding that S_n acts on A by superpermutations and that the subalgebra $\mathbb{k}[S_n] \subset A^{[n]}$ is in degree 0. Similarly, if A_1 and A_2 are algebras and M is an (A_2, A_1) -bimodule, then we can define the $(A_2^{[n]}, A_1^{[n]})$ -bimodule $M^{[n]} := M^{\otimes n} \rtimes \mathbb{k}[S_n]$.

To describe $(\cdot)^{[n]}$ as a functor, it is convenient to use the language of 2-categories. Let \mathcal{C}_a be the 2-category whose objects are algebras, 1-morphisms are bimodules and 2-morphisms are bimodule maps. Composition of 1-morphisms is the tensor product of bimodules and composition of 2-morphisms is composition of bimodule maps.

LEMMA 2.3. *The map $(\cdot) \mapsto (\cdot)^{[n]}$ defines a 2-functor*

$$(\cdot)^{[n]} : \mathcal{C}_a \longrightarrow \mathcal{C}_a.$$

Proof. For M_1 is an (A_2, A_1) -bimodule and M_2 is an (A_3, A_2) bimodule, we define $M_2^{[n]} \otimes_{A_2^{[n]}} M_1^{[n]} \rightarrow (M_2 \otimes_{A_2} M_1)^{[n]}$ by

$$(m_1 \otimes \dots \otimes m_n, \sigma) \otimes (m'_1 \otimes \dots \otimes m'_n, \sigma') \mapsto ((m_1 \otimes m'_{\sigma(1)}) \otimes \dots \otimes (m_n \otimes m'_{\sigma(n)}), \sigma\sigma').$$

It is not difficult to check that this map is an isomorphism. Thus, $(\cdot) \mapsto (\cdot)^{[n]}$ respects composition of 1-morphisms. It is also clear that $(\cdot) \mapsto (\cdot)^{[n]}$ intertwines compositions of 2-morphisms, because if $f : M_1 \rightarrow M_2$ is a map of bimodules, then

$$f^{[n]} := (f \otimes \dots \otimes f, 1) : M_1^{\otimes n} \rtimes \mathbb{k}[S_n] \rightarrow M_2^{\otimes n} \rtimes \mathbb{k}[S_n]$$

is a morphism $M_1^{[n]} \rightarrow M_2^{[n]}$ with $(f_2 \circ f_1)^{[n]} = f_2^{[n]} \circ f_1^{[n]}$. □

The functor $(\cdot)^{[n]}$ is somewhat subtle. In particular:

- $(\cdot)^{[n]}$ is not linear: if $f, g \in \text{Hom}(M_1, M_2)$, then both $(f + g)^{[n]}$ and $f^{[n]} + g^{[n]}$ are well-defined elements of $\text{Hom}(M_1^{[n]}, M_2^{[n]})$ but in general they are not equal to each other;
- $(\cdot)^{[n]}$ is not additive: in general $(M_1 \oplus M_2)^{[n]}$ and $M_1^{[n]} \oplus M_2^{[n]}$ are not isomorphic (this is already clear at the level of vector spaces via a dimension count); subsequently, $(\cdot)^{[n]}$ is neither left exact nor right exact.

However, $(\cdot)^{[n]}$ behaves well with respect to homotopies of complexes. Suppose $M_\bullet = M_0 \rightarrow \dots \rightarrow M_\ell$ is a complex of (A_1, A_2) -bimodules. Then the complex $M_\bullet^{[n]}$ is a complex of $(A_1^{[n]}, A_2^{[n]})$ -bimodules. The slightly subtle part of this definition is the definition of the boundary map in the complex $M_\bullet^{[n]}$; the easiest way to define it is to consider M_\bullet as a supermodule over the superalgebra $A \otimes_{\mathbb{k}} \mathbb{k}[d]/d^2$, where d has superdegree one. Then $M_\bullet^{[n]}$ is naturally an $(A \otimes_{\mathbb{k}} \mathbb{k}[d]/d^2)^{[n]} \cong A^{[n]} \otimes_{\mathbb{k}[S_n]} (\mathbb{k}[d]/d^2)^{[n]}$ supermodule. Now the coproduct

$$\Delta : \mathbb{k}[d]/d^2 \rightarrow (\mathbb{k}[d]/d^2)^{\otimes n} \subset (\mathbb{k}[d]/d^2)^{[n]}$$

given by

$$\Delta(d) = (1 \otimes 1 \otimes \dots \otimes d) + (1 \otimes d \otimes 1 \otimes \dots \otimes 1) + \dots + (d \otimes 1 \otimes \dots \otimes 1)$$

embeds $A^{[n]} \otimes_{\mathbb{k}} \mathbb{k}[d]/d^2$ as a subalgebra of $(A \otimes_{\mathbb{k}} \mathbb{k}[d]/d^2)^{[n]}$. Thus, the $(A \otimes_{\mathbb{k}} \mathbb{k}[d]/d^2)^{[n]}$ supermodule $M_\bullet^{[n]}$ can be restricted to $A^{[n]} \otimes_{\mathbb{k}} \mathbb{k}[d]/d^2$, and thus $M_\bullet^{[n]}$ may be considered as a complex of $A^{[n]}$ -modules.

An important point to keep in mind is that, because all constructions take place in the category of supermodules, the action of $\mathbb{k}[S_n]$ on an n -fold tensor product of graded vector spaces is via superpermutations. Thus, spelling this out, the action of the simple transposition s_i on the complex $M_\bullet^{[n]}$ is

$$s_i \cdot (m_1 \otimes \dots \otimes m_i \otimes m_{i+1} \otimes \dots \otimes m_n) = (-1)^{\deg(m_i) \deg(m_{i+1})} m_1 \otimes \dots \otimes m_{i+1} \otimes m_i \otimes \dots \otimes m_n,$$

where $\deg(m_i) = |m_i| + |m_i|_h$ where $|m_i|$ denotes the inner graded degree of m_i and $|m_i|_h$ denotes the homological degree of m_i .

Now, the following lemma shows that the functor $(\cdot)^{[n]}$ behaves well with respect to homotopies. (This is not immediately obvious since $(\cdot)^{[n]}$ is not linear and chain homotopies involve linear combinations of maps.)

LEMMA 2.4. *Let C_\bullet, D_\bullet be complexes of (A_1, A_2) -bimodules, and suppose that $f, g : C_\bullet \rightarrow D_\bullet$ are homotopic maps. Then $f^{[n]}, g^{[n]} : C_\bullet^{[n]} \rightarrow D_\bullet^{[n]}$ are homotopic.*

Proof. By assumption, there exists a chain homotopy h with $f - g = d_D h + h d_C$. We set

$$h' = \sum_{i+j=n-1} (f^{\otimes i} \otimes h \otimes g^{\otimes j}, 1).$$

Then one can check that $f^{[n]} - g^{[n]} = d_{D^{[n]}} h' + h' d_{C^{[n]}}$. □

2.5 Graded 2-categories

By a graded category we will mean a category equipped with an auto-equivalence $\langle 1 \rangle$. A graded additive \mathbb{k} -linear 2-category \mathcal{K} is a category enriched over graded additive \mathbb{k} -linear categories. This means that for any two objects $A, B \in \mathcal{K}$ the Hom category $\text{Hom}_{\mathcal{K}}(A, B)$ is a graded additive \mathbb{k} -linear category. Moreover, the composition map $\text{Hom}_{\mathcal{K}}(A, B) \times \text{Hom}_{\mathcal{K}}(B, C) \rightarrow \text{Hom}_{\mathcal{K}}(A, C)$ is a graded additive \mathbb{k} -linear functor.

Example. Suppose that B_n is a sequence of graded \mathbb{k} -algebras indexed by $n \in \mathbb{N}$. Then one can define a 2-category \mathcal{K} whose objects (0-morphisms) are indexed by \mathbb{N} , the 1-morphisms are graded (B_m, B_n) -bimodules and the 2-morphisms are maps of graded (B_m, B_n) -bimodules.

A graded additive \mathbb{k} -linear 2-functor $F : \mathcal{K} \rightarrow \mathcal{K}'$ is a (weak) 2-functor that maps the Hom categories $\text{Hom}_{\mathcal{K}}(A, B)$ to $\text{Hom}_{\mathcal{K}'}(FA, FB)$ by additive functors that commute with the auto-equivalence $\langle 1 \rangle$.

An additive category \mathcal{C} is said to be idempotent complete when every idempotent 1-morphism splits in \mathcal{C} . Similarly, we say that the additive 2-category \mathcal{K} is idempotent complete when the Hom categories $\text{Hom}_{\mathcal{K}}(A, B)$ are idempotent complete for any pair of objects $A, B \in \mathcal{K}$ (so that all idempotent 2-morphisms split).

2.5.1 The homotopy 2-category. If \mathcal{K} is an additive \mathbb{k} -linear 2-category then one can define its homotopy 2-category $\text{Kom}(\mathcal{K})$ as follows. The objects are the same. The 1-morphisms are unbounded complexes of 1-morphisms in \mathcal{K} while the 2-morphisms are maps of complexes. Two complexes of 1-morphisms are then deemed isomorphic if they are homotopy equivalent.

Example. Denote by \mathcal{C}_a the 2-category of algebras. Then in $\text{Kom}(\mathcal{C}_a)$:

- objects are algebras over \mathbb{k} ;
- 1-morphisms from A to B are complexes of (A, B) -bimodules;
- 2-morphisms are chain maps up to homotopy.

Combining Lemmas 2.3 and 2.4 implies the following.

PROPOSITION 2.5. *For each $n \in \mathbb{N}$, the 2-functor $(\cdot) \mapsto (\cdot)^{[n]}$ defines an endofunctor of the 2-category $\text{Kom}(\mathcal{C}_a)$.*

Remark 2.6. The above endofunctors appeared earlier in [Kho06], which emphasized their relevance for constructing group actions on categories.

2.5.2 Triangulated 2-categories. A graded triangulated category is a graded category equipped with a triangulated structure where the autoequivalence $\langle 1 \rangle$ takes exact triangles to exact triangles. We denote the homological shift by $[-]$ where $[1]$ denotes a downward shift by one.

A graded triangulated \mathbb{k} -linear 2-category \mathcal{K}' is a category enriched over graded triangulated \mathbb{k} -linear categories. This means that for any two objects $A, B \in \mathcal{K}'$ the Hom category $\text{Hom}_{\mathcal{K}'}(A, B)$ is a graded additive \mathbb{k} -linear triangulated category.

Example. If \mathcal{K} is a \mathbb{k} -linear 2-category, then $\text{Kom}(\mathcal{K})$ is a triangulated 2-category. In the remainder of the paper this extra triangulated structure of $\text{Kom}(\mathcal{K})$ will not play a role and will usually be ignored.

3. Quantum Heisenberg algebras

Here we recall the quantum Heisenberg algebra $\widehat{\mathfrak{h}}$ and its Fock space representation. We will denote the quantum integers by $[n] := (t^n - t^{-n}) / (t - t^{-1})$. This means that if $n \geq 0$ then $[n] = t^{-n+1} + t^{-n+3} + \dots + t^{n-3} + t^{n-1}$ and $[-n] = -[n]$.

The traditional presentation for the quantum Heisenberg algebra is as a unital algebra over $\mathbb{k}[t, t^{-1}]$ generated by $a_i(n)$, where $i \in I$ and $n \in \mathbb{Z} \setminus \{0\}$. The relations are

$$a_i(m)a_j(n) - a_j(n)a_i(m) = \delta_{m,-n}[n\langle i, j \rangle] \frac{[n]}{n}. \tag{3}$$

When $t = 1$, this presentation specializes to the standard presentation of the non-quantum Heisenberg algebra.

For our purposes, a more convenient presentation of $\widehat{\mathfrak{h}}$ takes as generators $\{P_i^{(n)}, Q_i^{(n)}\}_{i \in I, n \geq 0}$ subject to the following relations:

$$P_i^{(n)} P_j^{(m)} = P_j^{(m)} P_i^{(n)} \quad \text{and} \quad Q_i^{(n)} Q_j^{(m)} = Q_j^{(m)} Q_i^{(n)} \quad \text{for all } i, j \in I,$$

$$Q_i^{(n)} P_j^{(m)} = \begin{cases} \sum_{k \geq 0} [k+1] P_i^{(m-k)} Q_i^{(n-k)} & \text{if } i = j \\ P_j^{(m)} Q_i^{(n)} + P_j^{(m-1)} Q_i^{(n-1)} & \text{if } \langle i, j \rangle = -1 \\ P_j^{(m)} Q_i^{(n)} & \text{if } \langle i, j \rangle = 0. \end{cases}$$

By convention $P_i^{(0)} = Q_j^{(0)} = 1$ and $P_i^{(k)} = Q_i^{(k)} = 0$ when $k < 0$ so the summations in the relations above are all finite. Note that $\widehat{\mathfrak{h}}$ has a natural \mathbb{Z} -grading where $\deg P_i^{(n)} = n$ and $\deg Q_i^{(n)} = -n$.

An explicit isomorphism between these two presentations is given in [CL12] and [CL11b, § 3.1].

3.1 The Fock space

Let $\widehat{\mathfrak{h}}^- \subset \widehat{\mathfrak{h}}$ denote the subalgebra generated $\{Q_i^{(n)}\}_{i \in I, n \geq 0}$. Let triv_0 denote the trivial (one-dimensional) representation of $\widehat{\mathfrak{h}}^-$, where all $Q_i^{(n)}$ ($n > 0$) act by zero. Then $V_{\text{Fock}} := \text{Ind}_{\widehat{\mathfrak{h}}^-}^{\widehat{\mathfrak{h}}}(\text{triv}_0)$ is called the Fock space representation of $\widehat{\mathfrak{h}}$.

The Fock space has a basis given by elements of the form $P_{i_k}^{(n_k)} \dots P_{i_1}^{(n_1)}(v)$ where v is a vector spanning triv_0 . This gives a decomposition $V_{\text{Fock}} = \bigoplus_{n \geq 0} V_{\text{Fock}}(n)$. To simplify notation, we will denote $P_{i_k}^{(n_k)} \dots P_{i_1}^{(n_1)}(v)$ by $P_{i_k}^{(n_k)} \dots P_{i_1}^{(n_1)}$.

For any partition $\lambda \vdash n$ on can define $P_i^{(\lambda)}$ using Giambelli's formula as the determinant

$$[P_i^\lambda] = \det_{kl} [P_i^{(\lambda_k + l - k)}].$$

For example, $P_i^{(1^2)} = P_i P_i - P_i^{(2)}$. See [CL12, § 7] for more details.

3.2 The braid group action on V_{Fock}

We now describe a braid group action on V_{Fock} . Since V_{Fock} is multiplicatively generated by elements $P_i^{(n)}$ it suffices to describe this action on these generators and extend multiplicatively. On generators, the action is given by

$$\sigma_i(P_j^{(n)}) = \begin{cases} (-t^{-2})^n P_i^{(1^n)} & \text{if } i = j \\ \sum_{k=0}^n (-t^{-1})^{n-k} P_j^{(k)} P_i^{(n-k)} & \text{if } \langle i, j \rangle = -1 \\ P_j^{(n)} & \text{if } \langle i, j \rangle = 0 \end{cases}$$

and

$$\sigma_i^{-1}(P_j^{(n)}) = \begin{cases} (-t^2)^n P_i^{(1^n)} & \text{if } i = j \\ \sum_{k=0}^n (-t)^{n-k} P_j^{(k)} P_i^{(n-k)} & \text{if } \langle i, j \rangle = -1 \\ P_j^{(n)} & \text{if } \langle i, j \rangle = 0. \end{cases}$$

PROPOSITION 3.1. *The endomorphisms σ_i and σ_i^{-1} for $i \in I$ define a representation of the braid group $\text{Br}(D)$ on each weight space $V_{\text{Fock}}(n)$ of the Fock space.*

Proof. This is a consequence of Proposition 8.1 and Remark 8.2 following it. □

It is already interesting to see this braid action on various basis vectors, as in the example below.

LEMMA 3.2. *Suppose $i, j, k \in I$ are different with $\langle i, j \rangle = -1 = \langle j, k \rangle$ and $\langle i, k \rangle = 0$. Then*

$$\begin{aligned} \sigma_i \sigma_j \sigma_i (P_i^{(n)}) &= t^{-3n} P_j^{(1^n)} \\ \sigma_j \sigma_i \sigma_j (P_k^{(n)}) &= \sum_{a=0}^n \sum_{b=0}^{n-a} (-1)^b t^{-2(n-a)+b} P_k^{(a)} P_j^{(b)} P_i^{(n-a-b)}. \end{aligned}$$

Proof. By definition we have

$$\sigma_i \sigma_j \sigma_i (P_i^{(n)}) = (-t^{-2})^n \sum_{a=0}^n \sum_{b=0}^{n-a} (-1)^{a+n} t^{-2n+b} P_j^{(1^b)} P_i^{(a)} P_i^{(1^{n-a-b})}.$$

This simplifies to give $t^{-3n} P_j^{(1^n)}$ if we use the identity $P_i^{(m)} P_i^{(1^n)} = P_i^{(m,1^n)} + P_i^{(m+1,1^{n-1})}$ (see for instance [CL11b, Proposition 1]). The second identity is similar but more involved so we omit the proof. \square

The action of $\sigma_i^{\pm 1}$ on the usual generators $a_j(-n)$ of the Fock space has a somewhat easier description. The proof is a straightforward calculation using the generating functions in [CL12, § 2.2.1]:

$$\sigma_i(a_j(-n)) = \begin{cases} -t^{-2n} a_i(-n) & \text{if } i = j \\ a_j(-n) + (-1)^n t^{-n} a_i(-n) & \text{if } \langle i, j \rangle = -1 \\ a_j(-n) & \text{if } \langle i, j \rangle = 0 \end{cases}$$

and

$$\sigma_i^{-1}(a_j(-n)) = \begin{cases} -t^{2n} a_i(-n) & \text{if } i = j \\ a_j(-n) + (-1)^n t^n a_i(-n) & \text{if } \langle i, j \rangle = -1 \\ a_j(-n) & \text{if } \langle i, j \rangle = 0. \end{cases}$$

4. 2-representations of $\widehat{\mathfrak{h}}$ and the braid complex

In this section we review some facts about 2-representations of $\widehat{\mathfrak{h}}$ and define the braid complex $\Sigma_i \mathbf{1}_n$.

4.1 The 2-category \mathcal{H}

In [CL12] we introduced a 2-category \mathcal{H}^Γ associated to any finite subgroup $\Gamma \subset \text{SL}_2(\mathbb{C})$. In our current language this 2-category is associated to the pair (D, ϵ) where D is the affine Dynkin diagram corresponding to Γ by the McKay correspondence and ϵ is an appropriately chosen orientation of D .

That definition generalizes with no effort to give a 2-category \mathcal{H}_ϵ^D associated to any simply laced Dynkin diagram D and orientation ϵ . More precisely, \mathcal{H}_ϵ^D is the (idempotent closure of) the additive, graded, \mathbb{k} -linear 2-category where:

- 0-morphisms (objects) are indexed by the integers \mathbb{Z} ;
- 1-morphisms consist of the identity 1-morphisms $\mathbf{1}_n$ of $n \in \mathbb{Z}$ and compositions, direct sums and grading shifts of $P_i \mathbf{1}_n : n \rightarrow n + 1$ and $\mathbf{1}_n Q_i : n + 1 \rightarrow n$ for $i \in I$;

- the 2-morphisms are generated by adjunction maps, making P_i and Q_i bi-adjoint up to shift, together with certain maps $X_i^j \in \text{Hom}(P_i, P_j)$ and $T_{ij} \in \text{Hom}(P_i P_j, P_i P_j)$ satisfying a series of relations.

The generating 2-morphisms and their relations were described diagrammatically in [CL12] and reviewed in [CL11b, § 3.2]. In the interest of space, we have elected not to spell them out again. However, the essential algebraic structure of these 2-morphisms is straightforward to summarize, and is enough for the purposes of the current paper.

- The 1-morphisms P_i and Q_i are left and right adjoint to one another, up to a grading shift.
- For each $n \geq 0$, there is a natural injective map

$$B_\epsilon^D(n) \longrightarrow \text{End}\left(\left(\bigoplus_{i \in I} P_i\right)^n\right).$$

In particular, for each $i \in I$ there is an embedding $\mathbb{k}[S_n] \longrightarrow \text{End}(P_i^n)$. Since \mathcal{H}_ϵ^D is idempotent complete, for any partition $\lambda \vdash n$ we can define the 1-morphism $P_i^{(\lambda)}$ as the image of the idempotent $e_\lambda \in \mathbb{k}[S_{|\lambda|}]$ acting on $P_i^{|\lambda|}$.

From hereon we will fix D and ϵ and denote \mathcal{H}_ϵ^D simply by \mathcal{H} in order to simplify notation.

4.2 2-representation of $\widehat{\mathfrak{h}}$

A 2-representation of $\widehat{\mathfrak{h}}$ consists of a graded, idempotent complete \mathbb{k} -linear category \mathcal{K} where:

- 0-morphisms are graded, \mathbb{k} -linear, additive categories $\mathcal{D}(n)$;
- 1-morphisms are (certain types of) functors between these categories;
- 2-morphisms are natural transformations of these functors;

together with a 2-functor $\mathcal{H} \rightarrow \mathcal{K}$. We also require that the space of 2-morphisms between any two 1-morphisms in \mathcal{K} be finite-dimensional and that $\text{Hom}_{\mathcal{K}}(\mathbf{1}_n, \mathbf{1}_n \langle \ell \rangle)$ is zero if $\ell < 0$ and one-dimensional if $\ell = 0$.

The fact that the space of maps between any two 1-morphisms is finite-dimensional means that the Krull–Schmidt property holds. Thus, any 1-morphism has a unique direct sum decomposition (see [Rin84, § 2.2]). Note that if \mathcal{K} satisfies the Krull–Schmidt property then so does $\text{Kom}(\mathcal{K})$.

A 2-representation of $\widehat{\mathfrak{h}}$ is said to be *integrable* if $\mathbf{1}_n = 0$ are zero for $n \ll 0$. For example, in [CL12] we constructed an integrable 2-representation where n corresponded to the category of coherent sheaves on the Hilbert scheme of points $\text{Hilb}^n(\widehat{\mathbb{C}^2/\Gamma})$ where $\Gamma \subset \text{SL}_2(\mathbb{C})$ is the finite subgroup associated to our Dynkin diagram.

4.3 The 2-representation $\mathcal{K}_{\text{Fock}}$

The 2-representation $\mathcal{K}_{\text{Fock}}$ categorifies the Fock space representation V_{Fock} . It consists of:

- 0-morphisms: n indexes the category of projective $B_\epsilon^D(n)$ -modules;
- 1-morphisms: $(B_\epsilon^D(n), B_\epsilon^D(n'))$ -bimodules which are direct summands of tensor products of the bimodules $P_i \mathbf{1}_n \langle s \rangle$ and $\mathbf{1}_n Q_i \langle s \rangle$ for arbitrary shifts $\langle s \rangle$;
- 2-morphisms: all degree-preserving maps of bimodules.

The 2-functor $\mathcal{H} \rightarrow \mathcal{K}_{\text{Fock}}$ was defined in [CL12, § 9].

4.4 Some technical facts

We gather some useful facts about 2-representations of $\widehat{\mathfrak{h}}$.

LEMMA 4.1. For an arbitrary partition λ we have

$$Q_i^{(\lambda)}P_i \cong P_i Q_i^{(\lambda)} \bigoplus_{\lambda' \subset \lambda} Q_i^{(\lambda')} \langle -1, 1 \rangle \quad \text{and} \quad Q_i P_i^{(\lambda)} \cong P_i^{(\lambda)} Q_i \bigoplus_{\lambda' \subset \lambda} P_i^{(\lambda')} \langle -1, 1 \rangle$$

where the sums are over all $\lambda' \subset \lambda$ with $|\lambda'| = |\lambda| - 1$.

Proof. See Lemma 3.3 of [CL11b]. □

LEMMA 4.2. Suppose that λ, λ', μ and μ' are partitions such that $|\lambda| > |\lambda'|$ and $|\mu| > |\mu'|$. Then $\dim \text{Hom}(P_i^{(\lambda)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i^{(\mu')} \langle 1 \rangle) \leq 1$ with equality if and only if $\lambda' \subset \lambda$ and $\mu' \subset \mu$ with $|\lambda| = |\lambda'| + 1$ and $|\mu| = |\mu'| + 1$. In this case, the map is spanned by the composition

$$P_i^{(\lambda)} Q_i^{(\mu)} \longrightarrow P_i^{(\lambda')} P_i Q_i Q_i^{(\mu')} \longrightarrow P_i^{(\lambda')} Q_i^{(\mu')} \langle 1 \rangle$$

where the second map is given by adjunction.

Proof. See Lemma 3.4 of [CL11b]. □

LEMMA 4.3. Consider partitions $\lambda, \lambda', \mu, \mu'$ such that $|\lambda| - |\lambda'| = 2 = |\mu| - |\mu'|$. Then

$$\text{Hom}(P_i^{(\lambda)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i^{(\mu')} \langle 2 \rangle) \tag{4}$$

is zero unless $\lambda' \subset \lambda$ and $\mu' \subset \mu$ in which case its dimension is equal to

$$\begin{cases} 2 & \text{if } \lambda \setminus \lambda' \text{ and } \mu \setminus \mu' \text{ both consist of two boxes in different rows and columns,} \\ 0 & \text{if } \lambda \setminus \lambda' \text{ consists of two boxes in same row (respectively column)} \\ & \text{while } \mu \setminus \mu' \text{ consists of two boxes in same column (respectively row),} \\ 1 & \text{otherwise.} \end{cases}$$

Remark 4.4. Here $\lambda \setminus \lambda'$ and $\mu \setminus \mu'$ are skew partitions.

Proof. First one notes that a map in (4) consists of a composition of two degree 1 maps. Since the only degree 1 maps involving only vertex i are induced from the adjunction map $P_i Q_i \rightarrow \text{id} \langle 1 \rangle$, it follows that any map in (4) factors

$$P_i^{(\lambda)} Q_i^{(\mu)} \rightarrow P_i^{|\lambda|-1} Q_i^{|\mu|-1} \langle 1 \rangle \rightarrow P_i^{(\lambda')} Q_i^{(\mu')} \langle 2 \rangle.$$

Since $P_i^{|\lambda|-1} Q_i^{|\mu|-1} \langle 1 \rangle$ is isomorphic to a direct sum of indecomposable terms $P_i^{(\lambda'')} Q_i^{(\mu'')} \langle 1 \rangle$ for partitions λ'', μ'' , we see that the space of maps in (4) is spanned by maps which factor through some $P_i^{(\lambda'')} Q_i^{(\mu'')} \langle 1 \rangle$. Subsequently, by Lemma 4.2, it follows that (4) is zero unless $\lambda' \subset \lambda$ and $\mu' \subset \mu$.

Now, we proceed by induction on $|\lambda| + |\mu|$. We prove the first case above, the others follow similarly.

First, suppose that there exists some $\lambda' \not\subset \nu \subset \lambda$. Then $P_i P_i^{(\nu)} \cong P_i^{(\lambda)} \bigoplus_{\gamma} P_i^{(\gamma)}$. Subsequently

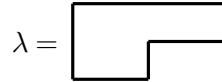
$$\text{Hom}(P_i^{(\lambda)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i^{(\mu')} \langle 2 \rangle) \cong \text{Hom}(P_i P_i^{(\nu)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i^{(\mu')} \langle 2 \rangle)$$

since, for all γ in the sum above, $\text{Hom}(P_i^{(\gamma)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i^{(\mu')} \langle 2 \rangle) = 0$ because $\lambda' \not\subset \gamma$. Hence,

$$\begin{aligned} (4) &\cong \text{Hom}(P_i^{(\nu)} Q_i^{(\mu)}, Q_i P_i^{(\lambda')} Q_i^{(\mu')} \langle 1 \rangle) \\ &\cong \text{Hom}(P_i^{(\nu)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i Q_i^{(\mu')} \langle 1 \rangle) \bigoplus_{\rho \subset \lambda'} \text{Hom}(P_i^{(\nu)} Q_i^{(\mu)}, P_i^{(\rho)} Q_i^{(\mu')} \langle 0, 2 \rangle). \end{aligned}$$

The left term above is zero since $\lambda' \not\subset \nu$. Using induction, every term in the summation on the right is also zero with the exception of the ρ which satisfies $\rho \subset \nu$. The result now follows by induction.

Now, suppose that no such ν as above exists. Thus means that λ is a partition whose Young diagram is a union of at most two rectangles, as illustrated by the following diagram.



Now, we choose $\nu \subset \lambda$ but this time $\lambda' \subset \nu$ so that once again we have $P_i P_i^{(\nu)} \cong P_i^{(\lambda)} \oplus_{\gamma} P_i^{(\gamma)}$. On the one hand, we have

$$\begin{aligned} \text{Hom}(P_i P_i^{(\nu)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i^{(\mu')} \langle 2 \rangle) &\cong (4) \bigoplus_{\gamma} \text{Hom}(P_i^{(\gamma)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i^{(\mu')} \langle 2 \rangle) \\ &\cong (4) \oplus \mathbb{k}^{2\#\{\gamma\}} \end{aligned}$$

where the second line follows by applying the first step above and then induction. On the other hand, by adjunction

$$\begin{aligned} &\text{Hom}(P_i P_i^{(\nu)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i^{(\mu')} \langle 2 \rangle) \\ &\cong \text{Hom}(P_i^{(\nu)} Q_i^{(\mu)}, Q_i P_i^{(\lambda')} Q_i^{(\mu')} \langle 1 \rangle) \\ &\cong \text{Hom}(P_i^{(\nu)} Q_i^{(\mu)}, P_i^{(\lambda')} Q_i Q_i^{(\mu')} \langle 1 \rangle) \bigoplus_{\rho \subset \lambda'} \text{Hom}(P_i^{(\nu)} Q_i^{(\mu)}, P_i^{(\rho)} Q_i^{(\mu')} \langle 0, 2 \rangle) \\ &\cong (\mathbb{k} \oplus \mathbb{k}) \oplus (\mathbb{k}^{2(\#\{\gamma\}-1)} \oplus \mathbb{k} \oplus \mathbb{k}) \end{aligned}$$

where the last line requires a quick case by case analysis of the possible ρ . Comparing these two expressions gives us that (4) $\cong \mathbb{k}^2$ and the induction is complete. \square

COROLLARY 4.5. *Suppose $\nu \subset \mu \subset \lambda$ are partitions with $|\lambda| = |\mu| + 1 = |\nu| + 2$ such that $\lambda \setminus \nu$ consists of two boxes which are not in the same row or column. Then the composition*

$$P_i^{(\lambda)} Q_i^{(\lambda^t)} \rightarrow P_i^{(\mu)} Q_i^{(\mu^t)} \langle 1 \rangle \rightarrow P_i^{(\nu)} Q_i^{(\nu^t)} \langle 2 \rangle \tag{5}$$

consisting of maps from Lemma 4.2 is nonzero.

Proof. Fix λ, ν as above. Any map $P_i^{(\lambda)} Q_i^{(\lambda^t)} \rightarrow P_i^{(\nu)} Q_i^{(\nu^t)} \langle 2 \rangle$ must factor as

$$P_i^{(\lambda)} Q_i^{(\lambda^t)} \rightarrow P_i^{(\nu)} P_i P_i Q_i Q_i Q_i^{(\nu^t)} \rightarrow P_i^{(\nu)} Q_i^{(\nu^t)} \langle 2 \rangle$$

where the rightmost map is given by two adjunctions. Now, since

$$\text{Hom}(P_i^{(2)} Q_i^{(1^2)}, \mathbf{1} \langle 2 \rangle) = 0 = \text{Hom}(P_i^{(1^2)} Q_i^{(2)}, \mathbf{1} \langle 2 \rangle)$$

this map factors as

$$P_i^{(\lambda)} Q_i^{(\lambda^t)} \rightarrow P_i^{(\nu)} P_i^{(2)} Q_i^{(2)} Q_i^{(\nu^t)} \oplus P_i^{(\nu)} P_i^{(1^2)} Q_i^{(1^2)} Q_i^{(\nu^t)} \rightarrow P_i^{(\nu)} Q_i^{(\nu^t)} \langle 2 \rangle.$$

Now,

$$\text{Hom}(P_i^{(\lambda)} Q_i^{(\lambda^t)}, P_i^{(\nu)} P_i^{(2)} Q_i^{(2)} Q_i^{(\nu^t)}) \cong \mathbb{k} \cong \text{Hom}(P_i^{(\lambda)} Q_i^{(\lambda^t)}, P_i^{(\nu)} P_i^{(1^2)} Q_i^{(1^2)} Q_i^{(\nu^t)})$$

and, by Lemma 4.3, $\dim_{\mathbb{k}} \text{Hom}(P_i^{(\lambda)} Q_i^{(\lambda^t)}, P_i^{(\nu)} Q_i^{(\nu^t)} \langle 2 \rangle) = 2$. Hence, the compositions

$$\begin{aligned} P_i^{(\lambda)} Q_i^{(\lambda^t)} &\rightarrow P_i^{(\nu)} P_i^{(2)} Q_i^{(2)} Q_i^{(\nu^t)} \rightarrow P_i^{(\nu)} Q_i^{(\nu^t)} \langle 2 \rangle \\ P_i^{(\lambda)} Q_i^{(\lambda^t)} &\rightarrow P_i^{(\nu)} P_i^{(1^2)} Q_i^{(1^2)} Q_i^{(\nu^t)} \rightarrow P_i^{(\nu)} Q_i^{(\nu^t)} \langle 2 \rangle \end{aligned}$$

are both nonzero. Fortunately, the unique map $P_i^{(\lambda)} \rightarrow P_i^{(\nu)}P_i^{(2)}$ factors through $P_i^{(\mu)}P_i$ and likewise $Q_i^{(\lambda^t)} \rightarrow Q_i^{(2)}Q_i^{(\nu^t)}$ factors through $Q_iQ_i^{(\mu)}$ for any $\nu \subset \mu \subset \lambda$. Hence,

$$P_i^{(\lambda)}Q_i^{(\lambda^t)} \rightarrow P_i^{(\mu)}P_iQ_iQ_i^{(\mu^t)} \rightarrow P_i^{(\nu)}P_iP_iQ_iQ_i^{(\nu^t)} \rightarrow P_i^{(\nu)}Q_i^{(\nu^t)}\langle 2 \rangle$$

is nonzero and the result follows since this composition is the same as that in (5). □

4.5 The braid complex $\Sigma_i \mathbf{1}_n$

Suppose that \mathcal{K} is some integrable 2-representation of $\widehat{\mathfrak{h}}$. The main object of study in this paper is the following complex of 1-morphisms

$$\Sigma_i \mathbf{1}_n := \left[\cdots \rightarrow \bigoplus_{\lambda \vdash d} P_i^{(\lambda)}Q_i^{(\lambda^t)}\langle -d \rangle \mathbf{1}_n \rightarrow \bigoplus_{\lambda \vdash d-1} P_i^{(\lambda)}Q_i^{(\lambda^t)}\langle -d+1 \rangle \mathbf{1}_n \rightarrow \cdots \rightarrow P_iQ_i\langle -1 \rangle \mathbf{1}_n \rightarrow \mathbf{1}_n \right] \tag{6}$$

which lives naturally in $\text{Kom}(\mathcal{K})$. The right-hand term $\mathbf{1}_n$ of this complex is in cohomological degree zero. Note that since \mathcal{K} is integrable this complex is finite.

The differential in (6) is defined as the composition

$$P_i^{(\lambda)}Q_i^{(\lambda^t)} \rightarrow P_i^{(\mu)}P_iQ_iQ_i^{(\mu^t)} \rightarrow P_i^{(\mu)}Q_i^{(\mu^t)}\langle 1 \rangle \tag{7}$$

where the first map is inclusion and the second is given by adjunction (note that we must have $\mu \subset \lambda$ in order for this map to be nonzero). The inclusion map is unique but only up to multiple. Likewise, by Lemma 4.2, the composition is unique but only up to multiple. Fortunately, Proposition 4.7 and Remark 4.8 below shows that there is a unique way (up to homotopy) to choose these multiples in order to get an indecomposable complex.

Remark 4.6. Note that we did not check directly that the compositions in (7) define a differential (i.e. square to zero). It is possible to check this directly by generalizing the statement in Lemma 4.3 but we avoid doing this extra work because later we will conclude for free that there exists an indecomposable complex with terms as in (6). Then Proposition 4.7 will tell us that the differentials are indeed given by (7).

PROPOSITION 4.7. *Any complex whose terms are the same as those of Σ_i and which is indecomposable in $\text{Kom}(\mathcal{K})$ is homotopic to Σ_i .*

Proof. Let us fix $n \in \mathbb{Z}$ and look at $\Sigma_i \mathbf{1}_n$. For any λ, μ such that $\text{Hom}(P_i^{(\lambda)}Q_i^{(\lambda^t)}\mathbf{1}_n, P_i^{(\mu)}Q_i^{(\mu^t)}\mathbf{1}_n\langle 1 \rangle) = 1$ fix a spanning map $f_{\lambda, \mu}$. A complex as in (6) will have differentials of the form $a_{\lambda, \mu}f_{\lambda, \mu}$ for some scalars $a_{\lambda, \mu} \in \mathbb{k}$. We need to show that:

- (i) if the complex in (6) is indecomposable, then $a_{\lambda, \mu} \neq 0$;
- (ii) if $\{a_{\lambda, \mu}\}$ and $\{a'_{\lambda, \mu}\}$ are two choices of scalars then they are equivalent via some homotopy.

Proof of (1). We proceed by (decreasing) induction on the cohomological degree. The base case is the map $a_{1,1}f_{1,1} : P_iQ_i\mathbf{1}_n\langle -1 \rangle \rightarrow \mathbf{1}_n$. Clearly $a_{1,1} \neq 0$ because otherwise $\Sigma_i \mathbf{1}_n$ would decompose. Now suppose $a_{\beta, \alpha} \neq 0$ for all β with $|\beta| > \ell$ (for some ℓ).

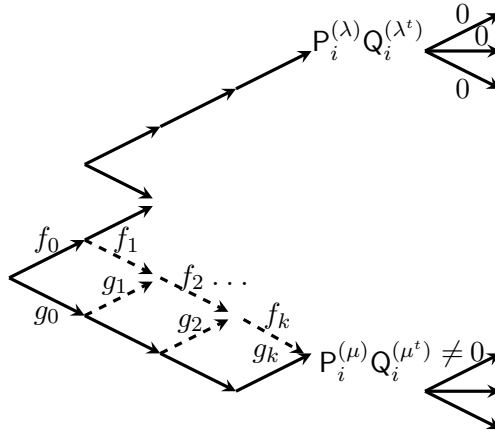
CLAIM. If $a_{\lambda, \mu} = 0$ for some λ with $|\lambda| = \ell$ then all differentials out of $P_i^{(\lambda)}Q_i^{(\lambda^t)}\mathbf{1}_n$ are zero.

To see this consider another differential $a_{\lambda,\mu'}f_{\lambda,\mu'}$. These two maps are part of a unique skew-commutative square as follows.

$$\begin{array}{ccc}
 & P_i^{(\mu)}Q_i^{(\mu^t)}\mathbf{1}_n\langle 1 \rangle & \\
 a_{\lambda,\mu}f_{\lambda,\mu}=0 \nearrow & & \searrow a_{\mu,\nu}f_{\mu,\nu} \\
 P_i^{(\lambda)}Q_i^{(\lambda^t)}\mathbf{1}_n & & P_i^{(\nu)}Q_i^{(\nu^t)}\mathbf{1}_n\langle 2 \rangle \\
 a_{\lambda,\mu'}f_{\lambda,\mu'} \searrow & & \nearrow a_{\mu',\nu}f_{\mu',\nu} \\
 & P_i^{(\mu')}Q_i^{(\mu'^t)}\mathbf{1}_n\langle 1 \rangle &
 \end{array} \tag{8}$$

By induction the right two maps are nonzero, which means that $a_{\mu',\nu} = 0$ since $f_{\lambda',\nu}f_{\lambda,\mu'} \neq 0$ by Lemma 4.3. This proves our claim.

Now, since Σ_i is indecomposable and all maps out of $P_i^{(\lambda)}Q_i^{(\lambda^t)}\mathbf{1}_n$ are zero there must be a sequence of nonzero differentials which connect $P_i^{(\lambda)}Q_i^{(\lambda^t)}\mathbf{1}_n$ to some other $P_i^{(\mu)}Q_i^{(\mu^t)}\mathbf{1}_n$. Such a path is depicted by the solid arrows in the following diagram.



By assumption, in the path connecting $P_i^{(\lambda)}Q_i^{(\lambda^t)}\mathbf{1}_n$ and $P_i^{(\mu)}Q_i^{(\mu^t)}\mathbf{1}_n$, the outward differentials (and in particular the map g_k at the end) are assumed to be nonzero. Without loss of generality we may assume that the path is minimal with respect to the area to its right. Now consider the square formed by f_0, g_0, f_1, g_1 (note that f_1, g_1 are uniquely determined by f_0, g_0). The skew-commutativity of this square means that f_1 and g_1 are either both zero or both nonzero. If they were both nonzero then one would obtain a smaller path which uses f_1, g_1 in place of f_0, g_0 . So we must have $f_1 = 0 = g_1$.

Now, the skew-commutativity of the subsequent squares means that $g_1 = 0 \Rightarrow g_2 = 0 \Rightarrow \dots \Rightarrow g_k = 0$ which is a contradiction. Thus, $a_{\lambda,\mu} \neq 0$ and we are done.

Proof of (2). By (1) we know that $a_{1,1} \neq 0 \neq a'_{1,1}$. So we can rescale $P_iQ_i\mathbf{1}_n$ (the second term from the right in $\Sigma_i\mathbf{1}_n$) so that $a_{1,1} = a'_{1,1}$. Now suppose by induction that $a_{\alpha,\beta} = a'_{\alpha,\beta}$ for all $|\beta| > \ell$. For any λ with $|\lambda| = \ell$ and differential $P_i^{(\lambda)}Q_i^{(\lambda^t)}\mathbf{1}_n \rightarrow P_i^{(\mu)}Q_i^{(\mu^t)}\mathbf{1}_n$ we can rescale $P_i^{(\lambda)}Q_i^{(\lambda^t)}$ so that $a_{\lambda,\mu} = a'_{\lambda,\mu}$.

CLAIM. For any other differential out of $P_i^{(\lambda)}Q_i^{(\lambda^t)}$ we have $a_{\lambda,\mu'} = a'_{\lambda,\mu'}$. To see this consider again the square in (8). By induction we know that $a_{\mu,\nu} = a'_{\mu,\nu}$ and $a_{\mu',\nu} = a'_{\mu',\nu}$. Then by the rescaling above we also have $a_{\lambda,\mu} = a'_{\lambda,\mu}$ so the skew-commutativity of the square also gives $a_{\lambda,\mu'} = a'_{\lambda,\mu'}$.

Thus, we can continue this way and scale things so that $a_{\lambda,\mu} = a'_{\lambda,\mu}$ for all $\mu \subset \lambda$ (which proves (2)). □

Remark 4.8. To define the differentials in (6) one can choose arbitrary nonzero maps $a_{\lambda,\mu} f_{\lambda,\mu} : P_i^{(\lambda)} Q_i^{(\lambda^t)} \mathbf{1}_n \rightarrow P_i^{(\mu)} Q_i^{(\mu^t)} \langle 1 \rangle \mathbf{1}_n$ and then rescale them as in the proof above to obtain a complex (i.e. so the square in (8) is skew-commutative).

5. The braid relations

5.1 The braid relations in $\mathcal{K}_{\text{Fock}}$

We first work with the 2-category $\text{Kom}(K_{\text{Fock}})$. To simplify notation we will denote $B := B_\epsilon^D(1)$ and $B_i := B e_{i,1}$ and ${}_i B := e_{i,1} B$ the natural (B, \mathbb{k}) and (\mathbb{k}, B) bimodules. Define the complex $\Sigma_i \mathbf{1}_1$ of (B, B) -bimodules as

$$\Sigma_i \mathbf{1}_1 := B_i \otimes_{\mathbb{k}} {}_i B \langle -2 \rangle \rightarrow B$$

where the map is multiplication (or equivalently given by a cap) and the right-hand term B is in cohomological degree zero. Similarly, we define

$$\Sigma_i^{-1} \mathbf{1}_1 := B \rightarrow B_i \otimes_{\mathbb{k}} {}_i B \langle 2 \rangle$$

where the map is given by a cap and the left-hand term B is in cohomological degree zero.

PROPOSITION 5.1. *The complexes $\Sigma_i \mathbf{1}_1$ satisfy the braid relations of $\text{Br}(D)$ where the inverse of $\Sigma_i \mathbf{1}_1$ is the complex $\Sigma_i^{-1} \mathbf{1}_1$. In other words, in the homotopy category of (B, B) -bimodules we have the following homotopy equivalences:*

- $\Sigma_i \Sigma_i^{-1} \mathbf{1}_1 \xrightarrow{\sim} \mathbf{1}_1$ and $\Sigma_i^{-1} \Sigma_i \mathbf{1}_1 \xrightarrow{\sim} \mathbf{1}_1$;
- $\Sigma_i \Sigma_j \Sigma_i \mathbf{1}_1 \xrightarrow{\sim} \Sigma_j \Sigma_i \Sigma_j \mathbf{1}_1$ if $\langle i, j \rangle = -1$;
- $\Sigma_i \Sigma_j \mathbf{1}_1 \xrightarrow{\sim} \Sigma_j \Sigma_i \mathbf{1}_1$ if $\langle i, j \rangle = 0$.

Proof. These relations are proven in [HK01], following earlier work [KS02] in type A . □

COROLLARY 5.2. *In the homotopy category $\text{Kom}(\mathcal{K}_{\text{Fock}})$ we have the following homotopy equivalences:*

- $\Sigma_i^{[n]} (\Sigma_i^{-1})^{[n]} \mathbf{1}_n \xrightarrow{\sim} \mathbf{1}_n$ and $(\Sigma_i^{-1})^{[n]} \Sigma_i^{[n]} \mathbf{1}_n \xrightarrow{\sim} \mathbf{1}_n$;
- $\Sigma_i^{[n]} \Sigma_j^{[n]} \Sigma_i^{[n]} \mathbf{1}_n \xrightarrow{\sim} \Sigma_j^{[n]} \Sigma_i^{[n]} \Sigma_j^{[n]} \mathbf{1}_n$ if $\langle i, j \rangle = -1$;
- $\Sigma_i^{[n]} \Sigma_j^{[n]} \mathbf{1}_n \xrightarrow{\sim} \Sigma_j^{[n]} \Sigma_i^{[n]} \mathbf{1}_n$ if $\langle i, j \rangle = 0$.

Proof. This is an immediate consequence of the naturality of the wreath functor $(\cdot) \mapsto (\cdot)^{[n]}$ as discussed in § 2.4. For example, the homotopy equivalence $\Sigma_1^{-1} \Sigma_1 \mathbf{1}_1 \xrightarrow{\sim} \mathbf{1}_1$ induces

$$\Sigma_i^{[n]} (\Sigma_i^{-1})^{[n]} \mathbf{1}_n \cong (\Sigma_i \Sigma_i^{-1} \mathbf{1}_1)^{[n]} \xrightarrow{\sim} (\mathbf{1}_1)^{[n]} = \mathbf{1}_n. \quad \square$$

The next theorem relates $\Sigma_i^{[n]} \mathbf{1}_n$ and the complex $\Sigma_i \mathbf{1}_n$ defined in (6).

THEOREM 5.3. *The complex $\Sigma_i^{[n]} \mathbf{1}_n$ is isomorphic to the complex of $(B_\epsilon^D(n), B_\epsilon^D(n))$ -bimodules*

$$\bigoplus_{\lambda \vdash n} P_i^{(\lambda)} Q_i^{(\lambda^t)} \langle -n \rangle \mathbf{1}_n \rightarrow \cdots \rightarrow \bigoplus_{\lambda \vdash d} P_i^{(\lambda)} Q_i^{(\lambda^t)} \langle -d \rangle \mathbf{1}_n \rightarrow \cdots \rightarrow P_i Q_i \langle -1 \rangle \mathbf{1}_n \rightarrow \mathbf{1}_n \quad (9)$$

defined in § 4.5.

5.2 Proof of Theorem 5.3

The statement of the Theorem 5.3 only involves one vertex of the Dynkin diagram D . To simplify notation we will assume that D contains only one vertex so that $B_\epsilon^D \cong \mathbb{k}[x]/x^2$. Note that $\deg x = 2$ so B_ϵ^D is commutative (not supercommutative). We will use the notation $B := B_\epsilon^D$ and $B_n := B_\epsilon^D(n) = B^{\otimes n} \rtimes \mathbb{k}[S_n]$. Thus, $\Sigma 1_1 = B' \rightarrow B$ as a (B, B) -bimodule where we denote $B' := B \otimes_{\mathbb{k}} B$.

By definition the term of the complex $\Sigma_i^{[n]}$ in cohomological degree $-d$ is $\bigoplus_{\underline{\ell}} B_{\underline{\ell}} \rtimes \mathbb{k}[S_n]$ where $\underline{\ell} = (\ell_1, \dots, \ell_d)$ with $0 \leq \ell_1 < \dots < \ell_d \leq n$ and $B_{\underline{\ell}}$ is a tensor product over \mathbb{k} of B and B' where the B' occur exactly in positions ℓ_1, \dots, ℓ_d . For example,

$$B_{(2,3)} = B \otimes_{\mathbb{k}} B' \otimes_{\mathbb{k}} B' \otimes_{\mathbb{k}} B \quad \text{where } n = 4, d = 2.$$

Remark 5.4. The tensor product above is that of supervector spaces where $\deg B = 0$ and $\deg B' = 1$. So, for instance, the involution $(23) \in S_4$ acts on $B_{(2,3)}$ by

$$(23) \cdot (b_1 \otimes b_2 \otimes b_3 \otimes b_4) \mapsto -(b_1 \otimes b_3 \otimes b_2 \otimes b_4)$$

while $(12) \in S_4$ acts as a map $B_{(2,3)} \rightarrow B_{(1,3)}$ by

$$(12) \cdot (b_1 \otimes b_2 \otimes b_3 \otimes b_4) \mapsto (b_2 \otimes b_1 \otimes b_3 \otimes b_4).$$

PROPOSITION 5.5. *There is an isomorphism of (B_n, B_n) -bimodules*

$$\bigoplus_{\lambda \vdash d} B_n e_\lambda \otimes_{B_{n-d}} e_{\lambda'} B_n \xrightarrow{\sim} \bigoplus_{\underline{\ell}} B_{\underline{\ell}} \rtimes \mathbb{k}[S_n]$$

where B_{n-d} acts on B_n via the embedding

$$B^{\otimes n-d} \ni b_1 \otimes \dots \otimes b_{n-d} \mapsto 1 \otimes \dots \otimes 1 \otimes b_1 \otimes \dots \otimes b_{n-d} \in B^{\otimes n}$$

and $e_\lambda \in \mathbb{k}[S_d] \subset B_d \subset B_n$ acts on the first d factors (so it commutes with B_{n-d}).

Proof. Consider the map $\phi: B_n \otimes_{B_{n-d}} B_n \rightarrow \bigoplus_{\underline{\ell}} B_{\underline{\ell}} \rtimes \mathbb{k}[S_n]$ given by

$$(1^{\otimes n}, 1) \otimes (1^{\otimes n}, 1) \mapsto (1^{\otimes d} \otimes 1^{\otimes n-d}, 1) \in B_{(1,2,\dots,d)} = (B')^{\otimes d} \otimes B^{\otimes n-d} \subset B_{\underline{\ell}}.$$

This map is well defined since if $(b_1 \otimes \dots \otimes b_{n-d}, \sigma) \in B_{n-d}$, then

$$\begin{aligned} \phi((1^{\otimes d} \otimes b_1 \otimes \dots \otimes b_{n-d}, \sigma) \otimes (1^{\otimes n}, 1)) &= (1^{\otimes d} \otimes b_1 \otimes \dots \otimes b_{n-d}, \sigma) \\ &= \phi((1^{\otimes n}, 1) \otimes (1^{\otimes d} \otimes b_1 \otimes \dots \otimes b_{n-d}, \sigma)). \end{aligned}$$

Moreover, since $(1^{\otimes d} \otimes 1^{\otimes n-d}, 1) \in B_{(1,2,\dots,d)}$ generates $B_{\underline{\ell}}$ as a (B_n, B_n) -bimodule ϕ is also surjective.

Now,

$$B_n \otimes_{B_{n-d}} B_n \cong \bigoplus_{\lambda, \lambda' \vdash d} (B_n e_\lambda \otimes_{B_{n-d}} e_{\lambda'} B_n)^{\oplus d_\lambda d_{\lambda'}}$$

where d_λ (respectively $d_{\lambda'}$) is the dimension of the irreducible $\mathbb{k}[S_d]$ -module V_λ (respectively $V_{\lambda'}$) indexed by the partition λ (respectively λ'). Now, for $s_i \in S_d$ we have

$$\phi((1^{\otimes n}, s_i) \otimes (1^{\otimes n}, 1)) = -(1^{\otimes n}, s_i) = (1^{\otimes n}, \tau(s_i))$$

where the minus sign is because s_i acts on $B' \otimes_{\mathbb{k}} B'$ by $s_i \cdot (1 \otimes 1) = -(1 \otimes 1)$. Subsequently,

$$\phi((1^{\otimes n}, e_\lambda) \otimes (1^{\otimes n}, e_{\lambda'})) = (1^{\otimes n}, \tau(e_\lambda)e_{\lambda'}) = \delta_{\lambda', \lambda}(1^{\otimes n}, e_{\lambda'}).$$

This means that $B_n e_\lambda \otimes_{B_{n-d}} e_{\lambda'} B_n$ is in the kernel of ϕ unless $\lambda' = \lambda^t$. We conclude that

$$\phi : \bigoplus_{\lambda \vdash d} B_n e_\lambda \otimes_{B_{n-d}} e_{\lambda^t} B_n \longrightarrow \bigoplus_{\underline{\ell}} B_{\underline{\ell}} \rtimes \mathbb{k}[S_n] \tag{10}$$

is surjective.

To show that the map in (10) is an isomorphism we compute the dimensions over \mathbb{k} of both sides. On the one hand, $\dim_{\mathbb{k}} B_{\underline{\ell}} = (\dim_{\mathbb{k}} B')^d (\dim_{\mathbb{k}} B)^{n-d} = 4^d 2^{n-d}$. Thus,

$$\dim_{\mathbb{k}} \left(\bigoplus_{\underline{\ell}} B_{\underline{\ell}} \rtimes \mathbb{k}[S_n] \right) = \binom{n}{d} 2^{n+d} n!. \tag{11}$$

On the other hand,

$$\dim_{\mathbb{k}}(B_n e_\lambda) = \dim_{\mathbb{k}} B_n \cdot \frac{\dim_{\mathbb{k}} V_\lambda}{d!} = \frac{2^n n! \dim_{\mathbb{k}} V_\lambda}{d!}$$

and likewise $\dim_{\mathbb{k}}(e_{\lambda^t} B_n) = (2^n n! \dim_{\mathbb{k}} V_\lambda)/d!$. Since $B_n e_\lambda$ and $e_{\lambda^t} B_n$ are free B_{n-d} modules it follows that

$$\dim_{\mathbb{k}}(B_n e_\lambda \otimes_{B_{n-d}} e_{\lambda^t} B_n) = \frac{1}{\dim_{\mathbb{k}} B_{n-d}} \frac{2^{2n} n! n! (\dim_{\mathbb{k}} V_\lambda)^2}{d! d!} = \binom{n}{d} 2^{n+d} n! \frac{(\dim_{\mathbb{k}} V_\lambda)^2}{d!}.$$

Summing over all $\lambda \vdash d$ and using that $\sum_{\lambda \vdash d} (\dim_{\mathbb{k}} V_\lambda)^2 = d!$ we get the same dimension as in (11). Thus, (10) must be an isomorphism. \square

Theorem 5.3 now follows by combining Proposition 5.5 with Proposition 4.7 which says that any indecomposable complex such as that in (9) is unique up to homotopy.

COROLLARY 5.6. *In $\text{Kom}(\mathcal{K}_{\text{Fock}})$ the complexes from (9) satisfy the braid relations of $\text{Br}(D)$.*

5.3 The braid relations in integrable 2-representations

We now consider an arbitrary integrable 2-representation \mathcal{K} of $\widehat{\mathfrak{h}}$ and prove Theorem 1.1 using Theorem 5.3. We will show that $\sum_i \Sigma_i^{-1} \mathbf{1}_n \xrightarrow{\sim} \mathbf{1}_n$ in $\text{Kom}(\mathcal{H})$ (the proof of the other braid relations is similar).

The composition $\sum_i \Sigma_i^{-1} \mathbf{1}_n$ is a (finite) complex of 1-morphisms in \mathcal{H} . Decompose each term in this complex into indecomposables of the form $P_i^{(\lambda)} Q_i^{(\mu)} \langle \ell \rangle \mathbf{1}_n$ where $\ell \in \mathbb{Z}$ and λ, μ are partitions. Since $\text{End}^k(P_i^{(\lambda)} Q_i^{(\mu)} \langle \ell \rangle \mathbf{1}_n)$ is zero if $k < 0$ and one-dimensional if $k = 0$ we can restrict $\sum_i \Sigma_i^{-1} \mathbf{1}_n$ to these terms to obtain a complex of the form $P_i^{(\lambda)} Q_i^{(\mu)} \langle \ell \rangle \mathbf{1}_n \otimes_{\mathbb{k}} V_\bullet$ where V_\bullet is a complex of vector spaces.

Using the cancellation Lemma 5.7 it suffices to show that V_\bullet is exact (unless $\lambda = \mu = \emptyset$ and $\ell = 0$ in which case V_\bullet should have one-dimensional cohomology in degree zero). To see this consider the image of $\sum_i \Sigma_i^{-1} \mathbf{1}_n$ in $\text{Kom}(\mathcal{K}_{\text{Fock}})$. By Theorem 5.3 this complex is homotopic to $\mathbf{1}_n$. Thus, the image of V_\bullet is exact and so V_\bullet must also be exact.

LEMMA 5.7. *Let X, Y, Z, W, U, V be six objects in an additive category and consider a complex*

$$\dots \rightarrow U \xrightarrow{u} X \oplus Y \xrightarrow{f} Z \oplus W \xrightarrow{v} V \rightarrow \dots \tag{12}$$

where $f = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and u, v are arbitrary morphisms. If $D : Y \rightarrow W$ is an isomorphism, then (12) is homotopic to a complex

$$\dots \rightarrow U \xrightarrow{u} X \xrightarrow{A-BD^{-1}C} Z \xrightarrow{v|_Z} V \rightarrow \dots \quad (13)$$

Proof. The following result is a slight generalization of a lemma which Bar-Natan [Bar07] calls ‘Gaussian elimination’. For a proof see [CL11b, Lemma 6.2]. \square

6. Convolution in triangulated 2-representations and Hilbert schemes

Consider an affine Dynkin diagram where $\Gamma \subset \widehat{\text{SL}}_2(\mathbb{C})$ is the finite subgroup associated to it via the McKay correspondence. Let $X_\Gamma = \widehat{\mathbb{C}^2}/\Gamma$ be the minimal resolution and $X_\Gamma^{[n]}$ the Hilbert scheme of n points.

In [CL12] we constructed a Heisenberg 2-representation on the derived categories of coherent sheaves $\bigoplus_{n \geq 0} D_{\mathbb{C}^\times}(X_\Gamma^{[n]})$. Subsequently, Theorem 1.1 induces an action of the affine braid group on $\text{Kom}(D_{\mathbb{C}^\times}(X_\Gamma^{[n]}))$. We would like to explain now how this action descends to one on $D_{\mathbb{C}^\times}(X_\Gamma^{[n]})$. The main result we prove is the following.

THEOREM 6.1. *For each $n \geq 0$, the complexes $\Sigma_i \mathbf{1}_n$ have a unique convolution*

$$\text{Conv}(\Sigma_i \mathbf{1}_n) \in D_{\mathbb{C}^\times \times \mathbb{C}^\times}(X_\Gamma^{[n]} \times X_\Gamma^{[n]}).$$

These convolutions define an action of the affine braid group on $D_{\mathbb{C}^\times}(X_\Gamma^{[n]})$.

The existence of such an affine braid group action is known by combining [Plo07, ST01]. The convolutions above give another interpretation of this action by showing that it arises from a categorical Heisenberg action.

6.1 Convolutions

Let $A_\bullet = A_0 \xrightarrow{f_1} A_1 \rightarrow \dots \xrightarrow{f_n} A_n$ be a sequence of objects and morphisms in a triangulated category \mathcal{D} such that $f_{i+1} \circ f_i = 0$. Such a sequence is called a complex.

A (right) convolution of a complex A_\bullet is any object B such that there exist:

- (i) objects $A_0 = B_0, B_1, \dots, B_{n-1}, B_n = B$; and
- (ii) morphisms $g_j : B_{j-1} \rightarrow A_j, h_j : A_j \rightarrow B_j$ (with $h_0 = \text{id}$);

such that $B_{j-1} \xrightarrow{g_j} A_j \xrightarrow{h_j} B_j$ is a distinguished triangle for each i and $g_j \circ h_{j-1} = f_j$. Such a collection of data is called a Postnikov system. We will denote B_n by $\text{Conv}(A_\bullet)$. When $n = 1$ then B_n is isomorphic to the usual cone.

PROPOSITION 6.2 (Cautis and Kamnitzer[CK08, Proposition 8.3]). *Consider a complex A_\bullet .*

- (i) *If $\text{Hom}(A_j[k], A_{j+k+1}) = 0$ for all $j \geq 0, k \geq 1$, then any two convolutions of (A_\bullet, f_\bullet) are isomorphic.*
- (ii) *If $\text{Hom}(A_j[k], A_{j+k+2}) = 0$ for all $j \geq 0, k \geq 1$, then (A_\bullet, f_\bullet) has a convolution.*

PROPOSITION 6.3. *Suppose that \mathcal{D} is a triangulated category which satisfies the Krull–Schmidt property. If A_\bullet is a complex of objects in \mathcal{D} which is homotopic to zero, then $\text{Conv}(A_\bullet) \cong 0$.*

Proof. The proof is by induction on the length of the complex. The base case is trivial.

Now, if $A_\bullet = A_0 \xrightarrow{f_1} A_1 \rightarrow B_\bullet$ is homotopic to zero, then there exists a map $g_1 : A_1 \rightarrow A_0$ such that $g_1 \circ f_1 = \text{id}_{A_0}$. Since \mathcal{D} is Krull–Schmidt this means that $A_1 \cong A_0 \oplus A'_1$ and we can

rewrite A_\bullet as

$$A_0 \xrightarrow{(f'_1, f''_1)} A_0 \oplus A'_1 \xrightarrow{h} B_\bullet$$

where f'_1 is an isomorphism. Now, if we take the first cone we obtain a commutative diagram

$$\begin{CD} A_0 @>(f'_1, f''_1)>> A_0 \oplus A'_1 @>h>> B_\bullet \\ @VVV @VV(\alpha, \beta)V @VV\text{id}V \\ 0 @>>> \text{Cone}(f'_1, f''_1) @>u>> B_\bullet \end{CD}$$

where u is some map. Note that $\text{Conv}(A_\bullet) \cong \text{Conv}(\text{Cone}(f'_1, f''_1) \xrightarrow{u} B_\bullet)$ so, by induction, it suffices to show that $\text{Cone}(f'_1, f''_1) \xrightarrow{u} B_\bullet$ is homotopic to zero.

Since f'_1 is an isomorphism this means β is an isomorphism. Thus, precomposing with β we get

$$[\text{Cone}(f'_1, f''_1) \xrightarrow{u} B_\bullet] \cong [A'_1 \xrightarrow{u \circ \beta} B_\bullet]. \tag{14}$$

Note that by commutativity of the square $u \circ \beta = h$.

On the other hand, using the cancellation lemma (Lemma 5.7)

$$[A_0 \xrightarrow{(f'_1, f''_1)} A_0 \oplus A'_1 \xrightarrow{h} B_\bullet] \cong [0 \rightarrow A'_1 \xrightarrow{h} B_\bullet].$$

Thus, combining this with (14) we get that $\text{Cone}(f'_1, f''_1) \xrightarrow{u} B_\bullet$ is homotopic to zero. □

6.2 Proof of Theorem 6.1

First we show that $\Sigma_i \mathbf{1}_n$ has a unique convolution. By Proposition 6.2 it suffices to check that for any partitions λ, μ, ν with $|\lambda| - |\mu| \geq 2$ and $|\lambda| - |\nu| \geq 3$ we have

$$\text{Hom}(\mathbb{P}_i^{(\lambda)} \mathbb{Q}_i^{(\lambda^t)}, \mathbb{P}_i^{(\mu)} \mathbb{Q}_i^{(\mu^t)} \langle 1 \rangle) = 0 = \text{Hom}(\mathbb{P}_i^{(\lambda)} \mathbb{Q}_i^{(\lambda^t)}, \mathbb{P}_i^{(\nu)} \mathbb{Q}_i^{(\nu^t)} \langle 2 \rangle) \tag{15}$$

where $\langle 1 \rangle = [1]$. In the 2-category \mathcal{H} this is clear because any map in the first (respectively second) Hom space above, must contain two (respectively three) adjunction maps and must therefore have degree at least two (respectively three). On the other hand, this can also be seen by using adjunction and the commutation relations in \mathcal{H} to reduce the statements above to the statement that $\text{End}(\mathbf{1}_0, \mathbf{1}_0(s)) = 0$ for $s < 0$. Since this is also true in our 2-representation consisting of coherent sheaves on $X_\Gamma^{[n]}$ the vanishing of (15) holds there too.

It remains to show that these unique convolutions $\text{Conv}(\Sigma_i \mathbf{1}_n)$ satisfy the affine braid relations. This follows from the affine braid relations satisfied in $\text{Kom}(\mathcal{H})$. We illustrate this by proving that

$$\text{Conv}(\Sigma_i \mathbf{1}_n) \circ \text{Conv}(\Sigma_i^{-1} \mathbf{1}_n) \cong \mathbf{1}_n.$$

First, we have that $\Sigma_i \mathbf{1}_n \circ \Sigma_i^{-1} \mathbf{1}_n \cong \mathbf{1}_n$ in the homotopy category. This means that we have a map $\Sigma_i \mathbf{1}_n \circ \Sigma_i^{-1} \mathbf{1}_n \rightarrow \mathbf{1}_n$ whose cone is homotopic to zero. By Proposition 6.3 we know that any convolution of this cone is zero and hence there is an isomorphism

$$\text{Conv}(\Sigma_i \mathbf{1}_n \circ \Sigma_i^{-1} \mathbf{1}_n) \xrightarrow{\sim} \mathbf{1}_n$$

for any choice of convolution (we do not know that it has a unique convolution). On the other hand, $\text{Conv}(\Sigma_i \mathbf{1}_n) \circ \text{Conv}(\Sigma_i^{-1} \mathbf{1}_n)$ is *some* convolution of $\Sigma_i \mathbf{1}_n \circ \Sigma_i^{-1} \mathbf{1}_n$. Thus,

$$\text{Conv}(\Sigma_i \mathbf{1}_n) \circ \text{Conv}(\Sigma_i^{-1} \mathbf{1}_n) \cong \text{Conv}(\Sigma_i \mathbf{1}_n \circ \Sigma_i^{-1} \mathbf{1}_n) \cong \mathbf{1}_n$$

and we are done.

Remark 6.4. Although Theorem 6.1 involves the triangulated category of coherent sheaves on a surface, the same proof works to show that the conclusions in that theorem hold for any (graded) triangulated category where

$$\text{Hom}(\mathbf{1}_n, \mathbf{1}_n\langle \ell \rangle) = \begin{cases} 0 & \text{if } \ell < 0 \\ \mathbb{C} & \text{if } \ell = 0 \\ \text{finite-dimensional} & \text{if } \ell > 0. \end{cases}$$

6.3 A larger group

As mentioned in the introduction, $D_{\mathbb{C}^\times}(X_\Gamma)$ carries an action of the affine braid group defined using Seidel–Thomas twists [ST01]. The affine braid group action constructed above coincides with its lift from $D_{\mathbb{C}^\times}(X_\Gamma)$ to $D_{\mathbb{C}^\times}(X_\Gamma^{[n]})$ using the results in [Pl07].

On the other hand, there is actually a larger group acting on $D_{\mathbb{C}^\times}(X_\Gamma^{[n]})$. It is generated by certain complexes very similar to our $\Sigma_i \mathbf{1}_n$. These complexes are briefly discussed in § 8.3 (equation (21) is an example of such a complex). However, the autoequivalence such a complex generates is not the lift of any automorphism of $D_{\mathbb{C}^\times}(X_\Gamma)$. It would be interesting to give a more geometric description of these equivalences.

7. A braid group action on $\text{Kom}(\mathcal{H})$

In this section we define an abstract action of $\text{Br}(D)$ on $\text{Kom}(\mathcal{H}')$ where \mathcal{H}' is the full subcategory of \mathcal{H} generated by the P_i . The quotient of \mathcal{H}' by the ideal generated by $\mathbf{1}_n$ for $n < 0$ may be thought of as another categorification of Fock space.

Recall that the 2-category \mathcal{H} contains generating 2-morphisms $X_j^k : P_j \rightarrow P_k\langle 1 \rangle$ for $\langle j, k \rangle = -1$ and $T_{jk} : P_j P_k \rightarrow P_k P_j$ for any $j, k \in I$. In [CL12] we encode these maps diagrammatically as a dot and a crossing respectively. To define a braid group action on \mathcal{H}' we need to explain how the generators $\sigma_i^{\pm 1}$ act on 1-morphisms P_j and Q_j as well as on 2-morphisms X_j^k and T_{jk} . We will then extend this action monoidally; for example, $\sigma_i(P_j P_j) = \sigma_i(P_j) \sigma_i(P_j)$.

In Appendix B we will prove the following result.

THEOREM 7.1. *The actions of $\sigma_i^{\pm 1}$ defined in §§ 7.1–7.5 induce 2-endofunctors of the 2-category $\text{Kom}(\mathcal{H}')$ which satisfy the braid relations of $\text{Br}(D)$.*

One can extend the results of this section to describe an action of $\text{Br}(D)$ on the entire category $\text{Kom}(\mathcal{H})$, rather than on just the upper half $\text{Kom}(\mathcal{H}')$. However, this would require checking even more relations and our interest in this paper is to understand braid group actions arising from integrable 2-representations of \mathcal{H} . For this purpose, Theorem 7.1 is sufficient.

7.1 The action of $\sigma_i^{\pm 1}$ on 1-morphisms

As usual, we will use $[k]$ to denote a cohomological shift to the left by k and $\langle k \rangle$ to denote the internal grading shift of \mathcal{H} .

We define

$$\sigma_i(P_j) := \begin{cases} P_i\langle -2 \rangle \oplus P_i \xrightarrow{(X_i^i \ 1_i)} P_i & \text{if } i = j \in I \\ P_i\langle -1 \rangle \xrightarrow{X_i^j} P_j & \text{if } \langle i, j \rangle = -1 \\ P_j & \text{if } \langle i, j \rangle = 0 \end{cases}$$

where the right-hand terms are all in homological degree zero. Likewise, we define

$$\sigma_i^{-1}(P_j) := \begin{cases} P_i \xrightarrow{(1_i \ X_i^i)} P_i \oplus P_i\langle 2 \rangle & \text{if } i = j \in I \\ P_j \xrightarrow{X_j^i} P_i\langle 1 \rangle & \text{if } \langle i, j \rangle = -1 \\ P_j & \text{if } \langle i, j \rangle = 0 \end{cases}$$

where this time the left-hand terms are all in degree zero.

7.2 The action of σ_i on X

Suppose $\langle i, j \rangle = -1$. We define

$$\begin{array}{ccc} \sigma_i(P_i\langle -1 \rangle) & = & P_i\langle -3 \rangle \oplus P_i\langle -1 \rangle \xrightarrow{(X_i^i \ 1_i)} P_i\langle -1 \rangle \\ \sigma_i(X_j^i) \downarrow & & (0 \ 1_i) \downarrow \qquad \qquad \qquad \downarrow X_i^j \\ \sigma_i(P_j) & = & P_i\langle -1 \rangle \xrightarrow{X_i^j} P_j \end{array}$$

$$\begin{array}{ccc} \sigma_i(P_j\langle -1 \rangle) & = & P_i\langle -2 \rangle \xrightarrow{X_i^j} P_j\langle -1 \rangle \\ \sigma_i(X_j^i) \downarrow & & (\epsilon_{ij} 1_i \ 0) \downarrow \qquad \qquad \qquad \downarrow X_j^i \\ \sigma_i(P_i) & = & P_i\langle -2 \rangle \oplus P_i \xrightarrow{(X_i^i \ 1_i)} P_i \end{array}$$

where the rightmost columns are both in cohomological degree zero.

Next, suppose further that $j \neq k$ and $\langle j, k \rangle = -1$, $\langle i, k \rangle = 0$. Then define

$$\begin{array}{ccc} \sigma_i(P_j\langle -1 \rangle) & = & P_i\langle -2 \rangle \xrightarrow{X_i^j} P_j\langle -1 \rangle \\ \sigma_i(X_j^k) \downarrow & & \downarrow \qquad \qquad \qquad \downarrow X_j^k \\ \sigma_i(P_k) & = & 0 \longrightarrow P_k \end{array}$$

$$\begin{array}{ccc} \sigma_i(P_k) & = & 0 \longrightarrow P_k\langle -1 \rangle \\ \sigma_i(X_k^j) \downarrow & & \downarrow \qquad \qquad \qquad \downarrow X_k^j \\ \sigma_i(P_j) & = & P_i\langle -1 \rangle \xrightarrow{X_i^j} P_j \end{array}$$

where again the rightmost columns are in cohomological degree zero.

An unusual situation is when i, j, k form a triangle, meaning $\langle i, j \rangle = \langle j, k \rangle = \langle i, k \rangle = -1$. In this case we define

$$\begin{array}{ccc} \sigma_i(P_j) & = & P_i\langle -1 \rangle \xrightarrow{X_i^j} P_j \\ \sigma_i(X_j^k) \downarrow & & \downarrow 0 \qquad \qquad \qquad \downarrow X_j^k \\ \sigma_i(P_k) & = & P_i \xrightarrow{X_i^k} P_k\langle 1 \rangle \end{array}$$

where the rightmost column is in cohomological degree zero.

Finally, if $\langle i, j \rangle = \langle i, k \rangle = 0$, then $\sigma_i(X_j^k) = X_j^k$.

7.3 The action of σ_i^{-1} on X

Suppose $\langle i, j \rangle = -1$. We define

$$\begin{array}{ccc} \sigma_i^{-1}(P_i\langle -1 \rangle) & = & P_i\langle -1 \rangle \xrightarrow{(1_i \ X_i^i)} P_i\langle -1 \rangle \oplus P_i\langle 1 \rangle \\ \downarrow \sigma_i^{-1}(X_i^j) & & \downarrow X_i^j \qquad \qquad \downarrow (0 \ \epsilon_{ij} 1_i) \\ \sigma_i^{-1}(P_j) & = & P_j \xrightarrow{X_j^i} P_i\langle 1 \rangle \\ \\ \sigma_i^{-1}(P_j\langle -1 \rangle) & = & P_j\langle -1 \rangle \xrightarrow{X_j^i} P_i \\ \downarrow \sigma_i^{-1}(X_j^j) & & \downarrow X_j^i \qquad \qquad \downarrow (1_i \ 0) \\ \sigma_i^{-1}(P_i) & = & P_i \xrightarrow{(1_i \ X_i^i)} P_i \oplus P_i\langle 2 \rangle \end{array}$$

where the leftmost columns are in cohomological degree zero.

In the case when i, j, k form a triangle so that $\langle i, j \rangle = \langle j, k \rangle = \langle i, k \rangle = -1$ we define

$$\begin{array}{ccc} \sigma_i^{-1}(P_j\langle -1 \rangle) & = & P_j\langle -1 \rangle \xrightarrow{X_j^i} P_i \\ \downarrow \sigma_i^{-1}(X_j^k) & & \downarrow X_j^k \qquad \downarrow 0 \\ \sigma_i^{-1}(P_k) & = & P_k \xrightarrow{X_k^i} P_i\langle 1 \rangle \end{array}$$

where the leftmost column is in cohomological degree zero.

Next, suppose $\langle j, k \rangle = -1, \langle i, j \rangle = -1, \langle i, k \rangle = 0$. Then define

$$\begin{array}{ccc} \sigma_i^{-1}(P_j\langle -1 \rangle) & = & P_j\langle -1 \rangle \xrightarrow{X_j^i} P_i \\ \downarrow \sigma_i^{-1}(X_j^k) & & \downarrow X_j^k \qquad \downarrow 0 \\ \sigma_i^{-1}(P_k) & = & P_k \longrightarrow 0 \\ \\ \sigma_i^{-1}(P_k\langle -1 \rangle) & = & P_k\langle -1 \rangle \longrightarrow 0 \\ \downarrow \sigma_i^{-1}(X_k^j) & & \downarrow X_k^j \qquad \downarrow 0 \\ \sigma_i^{-1}(P_j) & = & P_j \xrightarrow{X_j^i} P_i\langle 1 \rangle \end{array}$$

where the leftmost columns are in cohomological degree zero.

Finally if $\langle i, j \rangle = \langle i, k \rangle = 0$, then define $\sigma_i^{-1}(X_j^k) = X_j^k$.

7.4 The action of σ_i on T

We define $\sigma_i(T_{ii})$ as the map of complexes

$$\begin{array}{ccccc} P_i P_i\langle -4, -2, -2, 0 \rangle & \xrightarrow{A} & P_i P_i\langle -2, 0, -2, 0 \rangle & \xrightarrow{B} & P_i P_i \\ \left(\begin{array}{cccc} T_{ii} & 0 & 0 & 0 \\ 0 & 0 & T_{ii} & 0 \\ 0 & T_{ii} & 0 & 0 \\ 0 & 0 & 0 & T_{ii} \end{array} \right) \downarrow & & \left(\begin{array}{cccc} 0 & 0 & T_{ii} & 0 \\ 0 & 0 & 0 & T_{ii} \\ T_{ii} & 0 & 0 & 0 \\ 0 & T_{ii} & 0 & 0 \end{array} \right) \downarrow & & \downarrow T_{ii} \\ P_i P_i\langle -4, -2, -2, 0 \rangle & \xrightarrow{A} & P_i P_i\langle -2, 0, -2, 0 \rangle & \xrightarrow{B} & P_i P_i \end{array}$$

where, for instance, $P_i P_i \langle -4, -2, -2, 0 \rangle$ is shorthand for $P_i P_i \langle -4 \rangle \oplus P_i P_i \langle -2 \rangle \oplus P_i P_i \langle -2 \rangle \oplus P_i P_i$ and the rightmost column is in homological degree zero. The maps A and B are given by

$$A = \begin{pmatrix} -1_i X_i^i & -1_i 1_i & 0 & 0 \\ 0 & 0 & -1_i X_i^i & -1_i 1_i \\ X_i^i 1_i & 0 & 1_i 1_i & 0 \\ 0 & -X_i^i 1_i & 0 & 1_i 1_i \end{pmatrix}$$

$$B = (X_i^i 1_i \quad 1_i 1_i \quad 1_i X_i^i \quad 1_i 1_i).$$

Now suppose $\langle i, j \rangle = -1$. Then we define $\sigma_i(T_{ij})$ by

$$\begin{array}{ccccc} \sigma_i(P_i P_j) & = & P_i P_i \langle -3 \rangle \oplus P_i P_i \langle -1 \rangle & \xrightarrow{A} & P_i P_j \langle -2 \rangle \oplus P_i P_j \oplus P_i P_i \langle -1 \rangle & \xrightarrow{B} & P_i P_j \\ \downarrow \sigma_i(T_{ij}) & & \downarrow \begin{pmatrix} -T_{ii} & 0 \\ 0 & -T_{ii} \end{pmatrix} & & \downarrow \begin{pmatrix} T_{ij} & 0 & 0 \\ 0 & T_{ij} & 0 \\ 0 & 0 & T_{ij} \end{pmatrix} & & \downarrow T_{ij} \\ \sigma_i(P_j P_i) & = & P_i P_i \langle -3 \rangle \oplus P_i P_i \langle -1 \rangle & \xrightarrow{C} & P_j P_i \langle -2 \rangle \oplus P_j P_i \oplus P_i P_i \langle -1 \rangle & \xrightarrow{D} & P_j P_i \end{array}$$

where the rightmost column is in cohomological degree zero and

$$A = \begin{pmatrix} -1_i X_i^k & 0 \\ 0 & -1_i X_i^k \\ X_i^i 1_i & 1_i 1_i \end{pmatrix}, \quad C = \begin{pmatrix} X_i^k 1_i & 0 \\ 0 & X_i^k 1_i \\ -1_i X_i^i & -1_i 1_i \end{pmatrix},$$

$$B = (X_i^i 1_k \quad 1_i 1_k \quad 1_i X_i^k), \quad D = (1_k X_i^i \quad 1_k 1_i \quad X_i^k 1_i).$$

Likewise, if $\langle i, j \rangle = -1$, then we define $\sigma_i(T_{ji})$ by

$$\begin{array}{ccccc} \sigma_i(P_j P_i) & = & P_i P_i \langle -3 \rangle \oplus P_i P_i \langle -1 \rangle & \xrightarrow{A} & P_j P_i \langle -2 \rangle \oplus P_j P_i \oplus P_i P_i \langle -1 \rangle & \xrightarrow{B} & P_j P_i \\ \downarrow \sigma_i(T_{ji}) & & \downarrow \begin{pmatrix} -T_{ii} & 0 \\ 0 & -T_{ii} \end{pmatrix} & & \downarrow \begin{pmatrix} T_{ji} & 0 & 0 \\ 0 & T_{ji} & 0 \\ 0 & 0 & T_{ii} \end{pmatrix} & & \downarrow T_{ji} \\ \sigma_i(P_i P_j) & = & P_i P_i \langle -3 \rangle \oplus P_i P_i \langle -1 \rangle & \xrightarrow{C} & P_i P_j \langle -2 \rangle \oplus P_i P_j \oplus P_i P_i \langle -1 \rangle & \xrightarrow{D} & P_i P_j \end{array}$$

where the rightmost column is in cohomological degree zero and

$$A = \begin{pmatrix} X_i^j 1_i & 0 \\ 0 & X_i^j 1_i \\ -1_i X_i^i & -1_i 1_i \end{pmatrix}, \quad C = \begin{pmatrix} -1_i X_i^j & 0 \\ 0 & -1_i X_i^j \\ X_i^i 1_i & 1_i 1_i \end{pmatrix}$$

$$B = (1_j X_i^i \quad 1_j 1_i \quad X_i^j 1_i), \quad D = (X_i^i 1_j \quad 1_i 1_j \quad 1_i X_i^j).$$

On the other hand, if $\langle i, j \rangle = 0$ then $\sigma_i(T_{ij})$ is given by

$$\begin{array}{ccc} \sigma_i(\mathbb{P}_i\mathbb{P}_j) & = & \mathbb{P}_i\mathbb{P}_j\langle -2 \rangle \oplus \mathbb{P}_i\mathbb{P}_j \xrightarrow{\begin{pmatrix} X_i^i 1_j & 1_i 1_j \end{pmatrix}} \mathbb{P}_i\mathbb{P}_j \\ \sigma_i(T_{ij}) \downarrow & & \downarrow \begin{pmatrix} T_{ij} & 0 \\ 0 & T_{ij} \end{pmatrix} \downarrow T_{ij} \\ \sigma_i(\mathbb{P}_j\mathbb{P}_i) & = & \mathbb{P}_j\mathbb{P}_i\langle -2 \rangle \oplus \mathbb{P}_j\mathbb{P}_i \xrightarrow{\begin{pmatrix} 1_j X_i^i & 1_j 1_i \end{pmatrix}} \mathbb{P}_j\mathbb{P}_i \end{array}$$

where the rightmost column is in cohomological degree zero.

Likewise, if $\langle i, j \rangle = 0$, then $\sigma_i(T_{ji})$ is given by

$$\begin{array}{ccc} \sigma_i(\mathbb{P}_j\mathbb{P}_i) & = & \mathbb{P}_j\mathbb{P}_i\langle -2 \rangle \oplus \mathbb{P}_j\mathbb{P}_i \xrightarrow{\begin{pmatrix} 1_j X_i^i & 1_j 1_i \end{pmatrix}} \mathbb{P}_j\mathbb{P}_i \\ \sigma_i(T_{ji}) \downarrow & & \downarrow \begin{pmatrix} T_{ji} & 0 \\ 0 & T_{ji} \end{pmatrix} \downarrow T_{ji} \\ \sigma_i(\mathbb{P}_i\mathbb{P}_j) & = & \mathbb{P}_i\mathbb{P}_j\langle -2 \rangle \oplus \mathbb{P}_i\mathbb{P}_j \xrightarrow{\begin{pmatrix} X_i^i 1_j & 1_i 1_j \end{pmatrix}} \mathbb{P}_i\mathbb{P}_j \end{array}$$

where the rightmost column is in cohomological degree zero.

Next, suppose $\langle i, j \rangle = -1 = \langle i, k \rangle$. Define $\sigma_i(T_{jk})$ by

$$\begin{array}{ccccccc} \sigma_i(\mathbb{P}_j\mathbb{P}_k) & = & \mathbb{P}_i\mathbb{P}_i\langle -2 \rangle & \xrightarrow{\begin{pmatrix} 1_i X_i^k \\ X_i^j 1_i \end{pmatrix}} & \mathbb{P}_i\mathbb{P}_k\langle -1 \rangle \oplus \mathbb{P}_j\mathbb{P}_i\langle -1 \rangle & \xrightarrow{\begin{pmatrix} X_i^j 1_k & 1_j X_i^k \end{pmatrix}} & \mathbb{P}_j\mathbb{P}_k \\ \sigma_i(T_{jk}) \downarrow & & \downarrow T_{ii} & & \downarrow \begin{pmatrix} 0 & T_{ji} \\ T_{ik} & 0 \end{pmatrix} & & \downarrow T_{jk} \\ \sigma_i(\mathbb{P}_j\mathbb{P}_k) & = & \mathbb{P}_i\mathbb{P}_i\langle -2 \rangle & \xrightarrow{\begin{pmatrix} 1_i X_i^j \\ X_i^k 1_i \end{pmatrix}} & \mathbb{P}_i\mathbb{P}_j\langle -1 \rangle \oplus \mathbb{P}_k\mathbb{P}_i\langle -1 \rangle & \xrightarrow{\begin{pmatrix} X_i^k 1_j & 1_k X_i^j \end{pmatrix}} & \mathbb{P}_k\mathbb{P}_j \end{array}$$

where the rightmost column is in cohomological degree zero.

On the other hand, if $\langle i, j \rangle = -1$ and $\langle i, k \rangle = 0$ then $\sigma_i(T_{jk})$ is defined by

$$\begin{array}{ccc} \sigma_i(\mathbb{P}_j\mathbb{P}_k) & = & \mathbb{P}_i\mathbb{P}_k\langle -1 \rangle \xrightarrow{X_i^j 1_k} \mathbb{P}_j\mathbb{P}_k \\ \sigma_i(T_{jk}) \downarrow & & \downarrow T_{ik} \quad \downarrow T_{jk} \\ \sigma_i(\mathbb{P}_k\mathbb{P}_j) & = & \mathbb{P}_k\mathbb{P}_i\langle -1 \rangle \xrightarrow{1_k X_i^j} \mathbb{P}_k\mathbb{P}_j \end{array}$$

while $\sigma_i(T_{kj})$ is defined by

$$\begin{array}{ccc} \sigma_i(\mathbb{P}_k\mathbb{P}_j) & = & \mathbb{P}_k\mathbb{P}_i\langle -1 \rangle \xrightarrow{1_k X_i^j} \mathbb{P}_k\mathbb{P}_j \\ \sigma_i(T_{kj}) \downarrow & & \downarrow T_{ki} \quad \downarrow T_{kj} \\ \sigma_i(\mathbb{P}_j\mathbb{P}_k) & = & \mathbb{P}_i\mathbb{P}_k\langle -1 \rangle \xrightarrow{X_i^j 1_k} \mathbb{P}_j\mathbb{P}_k \end{array}$$

where both the rightmost columns are in cohomological degree zero.

Finally, if $\langle i, j \rangle = 0 = \langle i, k \rangle$, then $\sigma_i(T_{jk}) = T_{jk}$.

7.5 The action of σ_i^{-1} on T

We define $\sigma_i^{-1}(T_{ii})$ as a map of complexes:

$$\begin{array}{ccccc}
 P_i P_i & \xrightarrow{A} & P_i P_i \oplus P_i P_i \langle 2 \rangle \oplus P_i P_i \oplus P_i P_i \langle 2 \rangle & \xrightarrow{B} & P_i P_i \langle 0, 2, 2, 4 \rangle \\
 \downarrow T_{ii} & & \downarrow \begin{pmatrix} 0 & 0 & T_{ii} & 0 \\ 0 & 0 & 0 & T_{ii} \\ T_{ii} & 0 & 0 & 0 \\ 0 & T_{ii} & 0 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} T_{ii} & 0 & 0 & 0 \\ 0 & 0 & T_{ii} & 0 \\ 0 & T_{ii} & 0 & 0 \\ 0 & 0 & 0 & T_{ii} \end{pmatrix} \\
 P_i P_i & \xrightarrow{A} & P_i P_i \oplus P_i P_i \langle 2 \rangle \oplus P_i P_i \oplus P_i P_i \langle 2 \rangle & \xrightarrow{B} & P_i P_i \langle 0, 2, 2, 4 \rangle
 \end{array}$$

where the leftmost column is in cohomological degree zero. The maps A and B are given by

$$A = \begin{pmatrix} 1_i 1_i \\ 1_i X_i^i \\ 1_i 1_i \\ X_i^i 1_i \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1_i 1_i & 0 & -1_i 1_i & 0 \\ 0 & 1_i 1_i & -1_i X_i^i & 0 \\ X_i^i 1_i & 0 & 0 & -1_i 1_i \\ 0 & X_i^i 1_i & 0 & -1_i X_i^i \end{pmatrix}.$$

Now suppose $\langle i, j \rangle = -1$. Then we define $\sigma_i^{-1}(T_{ij})$ as a map of complexes by

$$\begin{array}{ccccc}
 \sigma_i^{-1}(P_i P_j) & = & P_i P_j \xrightarrow{A} P_i P_j \oplus P_i P_j \langle 2 \rangle \oplus P_i P_i \langle 1 \rangle & \xrightarrow{B} & P_i P_i \langle 1 \rangle \oplus P_i P_i \langle 3 \rangle \\
 \downarrow \sigma_i^{-1}(T_{ij}) & & \downarrow T_{ij} & & \downarrow \begin{pmatrix} -T_{ii} & 0 \\ 0 & -T_{ii} \end{pmatrix} \\
 \sigma_i^{-1}(P_j P_i) & = & P_j P_i \xrightarrow{C} P_j P_i \oplus P_j P_i \langle 2 \rangle \oplus P_i P_i \langle 1 \rangle & \xrightarrow{D} & P_i P_i \langle 1 \rangle \oplus P_i P_i \langle 3 \rangle
 \end{array}$$

where the leftmost column is in cohomological degree zero and

$$A = \begin{pmatrix} 1_i 1_k \\ X_i^i 1_k \\ 1_i X_k^i \end{pmatrix}, \quad B = \begin{pmatrix} -1_i X_k^i & 0 & 1_i 1_i \\ 0 & -1_i X_k^i & X_i^i 1_i \end{pmatrix}, \\
 C = \begin{pmatrix} 1_k 1_i \\ 1_k X_i^i \\ X_k^i 1_i \end{pmatrix}, \quad D = \begin{pmatrix} X_k^i 1_i & 0 & -1_i 1_i \\ 0 & X_k^i 1_i & -1_i X_i^i \end{pmatrix}.$$

Likewise, if $\langle i, j \rangle = -1$ then we define $\sigma_i^{-1}(T_{ji})$ by

$$\begin{array}{ccccc}
 \sigma_i^{-1}(P_j P_i) & = & P_j P_i \xrightarrow{A} P_j P_i \oplus P_j P_i \langle 2 \rangle \oplus P_i P_i \langle 1 \rangle & \xrightarrow{B} & P_i P_i \langle 1 \rangle \oplus P_i P_i \langle 3 \rangle \\
 \downarrow \sigma_i^{-1}(T_{ji}) & & \downarrow T_{ji} & & \downarrow \begin{pmatrix} -T_{ii} & 0 \\ 0 & -T_{ii} \end{pmatrix} \\
 \sigma_i^{-1}(P_i P_j) & = & P_i P_j \xrightarrow{C} P_i P_j \oplus P_i P_j \langle 2 \rangle \oplus P_i P_i \langle 1 \rangle & \xrightarrow{D} & P_i P_i \langle 1 \rangle \oplus P_i P_i \langle 3 \rangle
 \end{array}$$

where the leftmost column is in degree zero and

$$A = \begin{pmatrix} 1_j 1_i \\ 1_j X_i^i \\ X_j^i 1_i \end{pmatrix}, \quad B = \begin{pmatrix} X_j^i 1_i & 0 & -1_i 1_i \\ 0 & X_j^i 1_i & -1_i X_i^i \end{pmatrix},$$

$$C = \begin{pmatrix} 1_i 1_j \\ X_i^i 1_j \\ 1_i X_j^i \end{pmatrix}, \quad D = \begin{pmatrix} -1_i X_j^i & 0 & 1_i 1_i \\ 0 & -1_i X_j^i & X_i^i 1_i \end{pmatrix}.$$

On the other hand, if $\langle i, j \rangle = 0$, then $\sigma_i^{-1}(T_{ij})$ and $\sigma_i^{-1}(T_{ji})$ are given by

$$\begin{array}{ccc} \sigma_i^{-1}(\mathbb{P}_i \mathbb{P}_j) & = & \mathbb{P}_i \mathbb{P}_j \xrightarrow{(1_i 1_j \quad X_i^i 1_j)} \mathbb{P}_i \mathbb{P}_j \oplus \mathbb{P}_i \mathbb{P}_j \langle 2 \rangle \\ \downarrow \sigma_i^{-1}(T_{ij}) & & \downarrow T_{ij} \qquad \qquad \downarrow \begin{pmatrix} T_{ij} & 0 \\ 0 & T_{ij} \end{pmatrix} \\ \sigma_i^{-1}(\mathbb{P}_j \mathbb{P}_i) & = & \mathbb{P}_j \mathbb{P}_i \xrightarrow{(1_j 1_i \quad 1_j X_i^i)} \mathbb{P}_j \mathbb{P}_i \oplus \mathbb{P}_k \mathbb{P}_i \langle 2 \rangle \\ \sigma_i^{-1}(\mathbb{P}_j \mathbb{P}_i) & = & \mathbb{P}_j \mathbb{P}_i \xrightarrow{(1_j 1_i \quad 1_j X_i^i)} \mathbb{P}_j \mathbb{P}_i \oplus \mathbb{P}_j \mathbb{P}_i \langle 2 \rangle \\ \downarrow \sigma_i^{-1}(T_{ji}) & & \downarrow T_{ji} \qquad \qquad \downarrow \begin{pmatrix} T_{ji} & 0 \\ 0 & T_{ji} \end{pmatrix} \\ \sigma_i^{-1}(\mathbb{P}_i \mathbb{P}_j) & = & \mathbb{P}_i \mathbb{P}_j \xrightarrow{(1_i 1_j \quad X_i^i 1_j)} \mathbb{P}_i \mathbb{P}_j \oplus \mathbb{P}_i \mathbb{P}_j \langle 2 \rangle \end{array}$$

where the leftmost columns are in cohomological degree zero.

Next, suppose $\langle i, j \rangle = -1 = \langle i, k \rangle$. Define $\sigma_i^{-1}(T_{jk})$ by

$$\begin{array}{ccccc} \sigma_i^{-1}(\mathbb{P}_j \mathbb{P}_k) & = & \mathbb{P}_j \mathbb{P}_k \xrightarrow{(X_j^i 1_k \quad 1_j X_k^i)} \mathbb{P}_i \mathbb{P}_k \langle 1 \rangle \oplus \mathbb{P}_j \mathbb{P}_i \langle 1 \rangle \xrightarrow{(1_i X_k^i \quad X_j^i 1_i)} \mathbb{P}_i \mathbb{P}_i \langle 2 \rangle \\ \downarrow \sigma_i^{-1}(T_{jk}) & & \downarrow T_{jk} \qquad \qquad \downarrow \begin{pmatrix} 0 & T_{ji} \\ T_{ik} & 0 \end{pmatrix} & & \downarrow T_{ii} \\ \sigma_i^{-1}(\mathbb{P}_k \mathbb{P}_j) & = & \mathbb{P}_k \mathbb{P}_j \xrightarrow{(X_k^i 1_j \quad 1_k X_j^i)} \mathbb{P}_i \mathbb{P}_j \langle 1 \rangle \oplus \mathbb{P}_k \mathbb{P}_i \langle 1 \rangle \xrightarrow{(1_i X_j^i \quad X_k^i 1_i)} \mathbb{P}_i \mathbb{P}_i \langle 2 \rangle \end{array}$$

where the leftmost column is in cohomological degree zero.

On the other hand, if $\langle i, j \rangle = -1$ and $\langle i, k \rangle = 0$ then $\sigma_i^{-1}(T_{jk})$ and $\sigma_i^{-1}(T_{kj})$ are defined by

$$\begin{array}{ccc} \sigma_i^{-1}(\mathbb{P}_j \mathbb{P}_k) & = & \mathbb{P}_j \mathbb{P}_k \xrightarrow{X_j^i 1_k} \mathbb{P}_i \mathbb{P}_k \langle 1 \rangle \\ \downarrow \sigma_i^{-1}(T_{jk}) & & \downarrow T_{jk} \qquad \qquad \downarrow T_{ik} \\ \sigma_i^{-1}(\mathbb{P}_k \mathbb{P}_j) & = & \mathbb{P}_k \mathbb{P}_j \xrightarrow{1_k X_j^i} \mathbb{P}_k \mathbb{P}_i \langle 1 \rangle \\ \sigma_i^{-1}(\mathbb{P}_k \mathbb{P}_j) & = & \mathbb{P}_k \mathbb{P}_j \xrightarrow{1_k X_j^i} \mathbb{P}_k \mathbb{P}_i \langle 1 \rangle \\ \downarrow \sigma_i^{-1}(T_{kj}) & & \downarrow T_{kj} \qquad \qquad \downarrow T_{ki} \\ \sigma_i^{-1}(\mathbb{P}_j \mathbb{P}_k) & = & \mathbb{P}_j \mathbb{P}_k \xrightarrow{X_j^i 1_k} \mathbb{P}_i \mathbb{P}_k \langle 1 \rangle \end{array}$$

where both leftmost columns are in cohomological degree zero.

Finally if $\langle i, j \rangle = 0 = \langle i, k \rangle$, then $\sigma_i^{-1}(T_{jk}) = T_{jk}$.

PROPOSITION 7.2. *The definitions above give well-defined endomorphisms of the 2-category $\text{Kom}(\mathcal{H})$.*

7.6 Some homotopy equivalences

Some of the definitions above can be simplified, as we now explain. The reason we do not use these simpler definitions is that in practice they are more difficult to work with when checking the braid relations in the next section.

The complex $\sigma_i(\mathbb{P}_i)$ is homotopy equivalent to $\mathbb{P}_i\langle -2 \rangle[1]$ via the maps $\nu_{\mathbb{P}_i}$ and $\bar{\nu}_{\mathbb{P}_i}$ defined as follows.

$$\begin{array}{ccc}
 \sigma_i(\mathbb{P}_i) & = & \mathbb{P}_i\langle -2 \rangle \oplus \mathbb{P}_i \xrightarrow{(X_i^i \ 1_i)} \mathbb{P}_i \\
 \nu_{\mathbb{P}_i} \downarrow & & \begin{array}{ccc} (1_i \ 0) \downarrow & & \downarrow \\ \mathbb{P}_i\langle -2 \rangle & \xrightarrow{\quad\quad\quad} & 0 \\ (1_i \ -X_i^i) \downarrow & & \downarrow \\ \sigma_i(\mathbb{P}_i) & = & \mathbb{P}_i\langle -2 \rangle \oplus \mathbb{P}_i \xrightarrow{(X_i^i \ 1_i)} \mathbb{P}_i \end{array}
 \end{array}$$

Clearly $\nu_{\mathbb{P}_i} \bar{\nu}_{\mathbb{P}_i}$ is the identity map. On the other hand, $\bar{\nu}_{\mathbb{P}_i} \nu_{\mathbb{P}_i}$ is homotopic to the identity using $(0 \ -1_i) : \mathbb{P}_i \rightarrow \mathbb{P}_i\langle -2 \rangle \oplus \mathbb{P}_i$.

Likewise, $\sigma_i^{-1}(\mathbb{P}_i)$ is homotopy equivalent to $\mathbb{P}_i\langle 2 \rangle[-1]$ via the maps $\zeta_{\mathbb{P}_i}$ and $\bar{\zeta}_{\mathbb{P}_i}$ defined by

$$\begin{array}{ccc}
 \sigma_i^{-1}(\mathbb{P}_i) & = & \mathbb{P}_i \xrightarrow{(1_i \ X_i^i)} \mathbb{P}_i \oplus \mathbb{P}_i\langle 2 \rangle \\
 \zeta_{\mathbb{P}_i} \downarrow & & \begin{array}{ccc} \downarrow & & \downarrow (-X_i^i \ 1_i) \\ \mathbb{P}_i\langle 2 \rangle[-1] & = & 0 \xrightarrow{\quad\quad\quad} \mathbb{P}_i\langle 2 \rangle \\ \bar{\zeta}_{\mathbb{P}_i} \downarrow & & \downarrow (0 \ 1_i) \\ \sigma_i^{-1}(\mathbb{P}_i) & = & \mathbb{P}_i \xrightarrow{(1_i \ X_i^i)} \mathbb{P}_i \oplus \mathbb{P}_i\langle 2 \rangle \end{array}
 \end{array}$$

Using these homotopy equivalences one can simplify some of the definitions above. In particular, if $\langle i, j \rangle = -1$, then we get the following diagrams.

$$\begin{array}{ccc}
 \nu_{\mathbb{P}_i} \sigma_i(\mathbb{P}_i\langle -1 \rangle) & = & \mathbb{P}_i\langle -3 \rangle \longrightarrow 0 \\
 \sigma_i(X_i^j) \circ \bar{\nu}_{\mathbb{P}_i} \downarrow & & \begin{array}{ccc} \downarrow -X_i^i & & \downarrow \\ \sigma_i(\mathbb{P}_j) & = & \mathbb{P}_i\langle -1 \rangle \xrightarrow{X_i^j} \mathbb{P}_j \end{array} \\
 \\
 \sigma_i(\mathbb{P}_j\langle -1 \rangle) & = & \mathbb{P}_i\langle -2 \rangle \xrightarrow{X_i^j} \mathbb{P}_j\langle -1 \rangle \\
 \nu_{\mathbb{P}_i} \circ \sigma_i(X_i^j) \downarrow & & \begin{array}{ccc} \downarrow \epsilon_{ij} 1_i & & \downarrow \\ \nu_{\mathbb{P}_i} \sigma_i(\mathbb{P}_i) & = & \mathbb{P}_i\langle -2 \rangle \longrightarrow 0 \end{array}
 \end{array}$$

Moreover, it is easy to check that

$$(\nu_{\mathbb{P}_i} \nu_{\mathbb{P}_i}) \circ \sigma_i(T_{ii}) \circ (\bar{\nu}_{\mathbb{P}_i} \bar{\nu}_{\mathbb{P}_i}) = -T_{ii} : \mathbb{P}_i\mathbb{P}_i\langle -4 \rangle[2] \rightarrow \mathbb{P}_i\mathbb{P}_i\langle -4 \rangle[2]$$

while if $\langle i, j \rangle = -1$, then

$$\begin{aligned}
 (\nu_{P_i} \sigma_i(P_i)) \sigma_i(P_j) &= P_i P_i \langle -3 \rangle \xrightarrow{-1_i X_i^j} P_i P_j \langle -2 \rangle \\
 \downarrow (1_{\sigma_i(P_j)} \nu_{P_i}) \circ (\sigma_i(T_{ij})) \circ (\bar{\nu}_{P_i} 1_{\sigma_i(P_j)}) &\quad \downarrow -T_{ii} \quad \downarrow T_{ij} \\
 \sigma_i(P_j) (\nu_{P_i} \sigma_i(P_i)) &= P_i P_i \langle -3 \rangle \xrightarrow{X_i^j 1_i} P_j P_i \langle -2 \rangle \\
 \\
 \sigma_i(P_j) (\nu_{P_i} \sigma_i(P_i)) &= P_i P_i \langle -3 \rangle \xrightarrow{X_i^j 1_i} P_j P_i \langle -2 \rangle \\
 \downarrow (\nu_{P_i} 1_{\sigma_i(P_j)}) \circ (\sigma_i(T_{ji})) \circ (1_{\sigma_i(P_j)} \bar{\nu}_{P_i}) &\quad \downarrow -T_{ii} \quad \downarrow T_{ji} \\
 (\nu_{P_i} \sigma_i(P_i)) \sigma_i(P_j) &= P_i P_i \langle -3 \rangle \xrightarrow{-1_i X_i^j} P_i P_j \langle -2 \rangle
 \end{aligned}$$

where the rightmost columns are in cohomological degree -1 .

Finally, if $\langle i, j \rangle = 0$, then

$$\begin{aligned}
 1_j \nu_{P_i} \circ \sigma_i(T_{ij}) \circ \bar{\nu}_{P_i} 1_j &= T_{ij} : P_i P_j \langle -2 \rangle [1] \rightarrow P_j P_i \langle -2 \rangle [1] \\
 \nu_{P_i} 1_j \circ \sigma_i(T_{ji}) \circ 1_j \bar{\nu}_{P_i} &= T_{ji} : P_j P_i \langle -2 \rangle [1] \rightarrow P_i P_j \langle -2 \rangle [1].
 \end{aligned}$$

There are similar simplifications involving σ_i^{-1} which we omit.

8. Some remarks and conjectures

8.1 The action on 1-morphisms in $\text{Kom}(\mathcal{H}')$

The action of $\text{Br}(D)$ on $\text{Kom}(\mathcal{H}')$ from §7 was defined explicitly only on 1-morphisms P_i . On some more general 1-morphisms it acts as follows.

PROPOSITION 8.1. *The braid group action from §7 on the 2-category $\text{Kom}(\mathcal{H}')$ acts on $P_j^{(n)}$ by*

$$\sigma_i(P_j^{(n)}) \cong \begin{cases} P_i^{(1^n)} \langle -2n \rangle [n] & \text{if } i = j \\ [P_i^{(n)} \langle -n \rangle \rightarrow P_i^{(n-1)} P_j \langle -n + 1 \rangle \rightarrow \dots \rightarrow P_i P_j^{(n-1)} \langle -1 \rangle \rightarrow P_j^{(n)}] & \text{if } \langle i, j \rangle = -1 \\ P_j^{(n)} & \text{if } \langle i, j \rangle = 0 \end{cases}$$

where the rightmost term is in cohomological degree zero and by

$$\sigma_i^{-1}(P_j^{(n)}) \cong \begin{cases} P_i^{(1^n)} \langle 2n \rangle [-n] & \text{if } i = j \\ [P_j^{(n)} \rightarrow P_j^{(n-1)} P_i \langle 1 \rangle \rightarrow \dots \rightarrow P_j P_i^{(n-1)} \langle n - 1 \rangle \rightarrow P_i^{(n)} \langle n \rangle] & \text{if } \langle i, j \rangle = -1 \\ P_j^{(n)} & \text{if } \langle i, j \rangle = 0 \end{cases}$$

where the leftmost term is in degree zero. The differentials in the complexes are the unique morphisms (up to a scalar) given by the compositions

$$\begin{aligned}
 P_i^{(r)} P_j^{(n-r)} &\rightarrow P_i^{(r-1)} P_i P_j^{(n-r)} \xrightarrow{IX_i^j I} P_i^{(r-1)} P_j P_j^{(n-r)} \langle 1 \rangle \rightarrow P_i^{(r-1)} P_j^{(n-r+1)} \langle 1 \rangle \\
 P_i^{(r)} P_j^{(n-r)} &\rightarrow P_i^{(r)} P_j P_j^{(n-r-1)} \xrightarrow{IX_j^i I} P_i^{(r)} P_i P_j^{(n-r-1)} \langle 1 \rangle \rightarrow P_i^{(r+1)} P_j^{(n-r-1)} \langle 1 \rangle.
 \end{aligned}$$

Proof. In order to compute $\sigma_i(P_j^{(n)})$ one must compute $\sigma_i(P_j^n)$ and determine the image of $\sigma_i(e_{(n)})$ where $e_{(n)}$ is the trivial idempotent in S_n acting on n strands colored by j .

The case that $\langle i, j \rangle = 0$ is trivial since $\sigma_i(P_j^n) = P_j^n$ and $\sigma_i(T_{jj}) = T_{jj}$. Thus, the image of $\sigma_i(e_{(n)})$ on P_j^n is $P_j^{(n)}$ by definition.

If $i = j$, then, up to homotopy, $\sigma_i(\mathbb{P}_i^n) \cong \mathbb{P}_i^n \langle -2n \rangle [n]$ and $\sigma_i(e_{(n)}) = e_{(1^n)}$ since $\sigma_i(T_{ii}) = -T_{ii}$. Thus, the image of $\sigma_i(e_{(n)})$ on $\sigma_i(\mathbb{P}_i^n)$ is $\mathbb{P}_i^{(1^n)} \langle -2n \rangle [n]$.

If $\langle i, j \rangle = -1$, then $\sigma_i(\mathbb{P}_j) = [\mathbb{P}_i \langle -1 \rangle \rightarrow \mathbb{P}_j]$ which means that $\sigma_i(\mathbb{P}_j^n)$ is now a complex. To simplify notation, for a sequence $\mathbf{d} = (d_1, \dots, d_n)$ where each entry is i or j we denote $\mathbb{P}_{\mathbf{d}} = \mathbb{P}_{d_1} \cdots \mathbb{P}_{d_n}$. Then the term in $\sigma_i(\mathbb{P}_j^n)$ lying in cohomological degree $-r$ (where $r \geq 0$) is $\bigoplus_{\mathbf{d}} \mathbb{P}_{\mathbf{d}} \langle -r \rangle$ where r of the entries of \mathbf{d} are i and $n - r$ are j .

Since S_n permutes the entries of \mathbf{d} we may consider the subgroup $S_{\mathbf{d}} \subset S_n$ which stabilizes \mathbf{d} . Let \mathbf{d}' be another sequence where r of the entries are i and $n - r$ are j . Then, using the definition of $\sigma_i(T_{jj})$ from § 7.4, the 2-morphism $\sigma_i(e_{(n)})$ induces a map $\mathbb{P}_{\mathbf{d}} \langle -r \rangle \rightarrow \mathbb{P}_{\mathbf{d}'} \langle -r \rangle$ which (up to a nonzero scalar) is the sum over all elements in the coset of $S_n/S_{\mathbf{d}}$ which transforms \mathbf{d} to \mathbf{d}' .

Since for any \mathbf{d}, \mathbf{d}' as above there is an isomorphism $\phi: \mathbb{P}_{\mathbf{d}} \rightarrow \mathbb{P}_{\mathbf{d}'}$ such that $\sigma_i(e_{(n)})|_{\mathbb{P}_{\mathbf{d}}} = \sigma_i(e_{(n)})|_{\mathbb{P}_{\mathbf{d}'}} \circ \phi$ it suffices to consider the compute the image of $\sigma_i(e_{(n)})$ on $\mathbb{P}_{(\mathbf{i}, \mathbf{j})}$ where $(\mathbf{i}, \mathbf{j}) := (\underbrace{i, \dots, i}_r, \underbrace{j, \dots, j}_{n-r})$.

So consider now $\sigma_i(e_{(n)})|_{\mathbb{P}_{(\mathbf{i}, \mathbf{j})}}: \mathbb{P}_{(\mathbf{i}, \mathbf{j})} \rightarrow \bigoplus_{\mathbf{d}} \mathbb{P}_{\mathbf{d}}$. We must show that the image is isomorphic to $\mathbb{P}_i^{(r)} \mathbb{P}_j^{(n-r)}$. Let $w_{(\mathbf{i}, \mathbf{j}), \mathbf{d}}$ be a minimal length representative in this coset. The component of this map $\mathbb{P}_{(\mathbf{i}, \mathbf{j})} \rightarrow \mathbb{P}_{\mathbf{d}}$ is a sum over all elements in the coset of $S_n/S_r \times S_{n-r}$ which transforms (\mathbf{i}, \mathbf{j}) to \mathbf{d} .

Consider the composition $\mathbb{P}_{(\mathbf{i}, \mathbf{j})} \rightarrow \bigoplus_{\mathbf{d}} \mathbb{P}_{\mathbf{d}} \rightarrow \bigoplus_{\mathbf{d}} \mathbb{P}_{(\mathbf{i}, \mathbf{j})}$ where the first map is the restriction of $\sigma_i(e_{(n)})$ and the second map is the isomorphism $\bigoplus_{\mathbf{d}} w_{(\mathbf{i}, \mathbf{j}), \mathbf{d}}^{-1}$. The composition is the diagonal map where each entry is the sum over all elements in $S_r \times S_{n-r}$. Thus, the image of $\sigma_i(e_{(n)})$ in $\mathbb{P}_{(\mathbf{i}, \mathbf{j})}$ is $\mathbb{P}_i^{(r)} \mathbb{P}_j^{(n-r)}$.

For example, take $r = 2, n = 3$ so that $(\mathbf{i}, \mathbf{j}) = (i, i, j)$. Then the restriction of $\sigma_i(e_{(3)})$ to $\mathbb{P}_{(i, i, j)}$ is the map

$$\begin{pmatrix} 1 + s_1 & & \\ s_2 + s_2 s_1 & & \\ s_1 s_2 s_1 + s_1 s_2 & & \end{pmatrix} : \mathbb{P}_{(i, i, j)} \rightarrow \mathbb{P}_{(i, i, j)} \oplus \mathbb{P}_{(i, j, i)} \oplus \mathbb{P}_{(j, i, i)}$$

where the s_1, s_2 are the standard transpositions. Composing this map with

$$\bigoplus_{\mathbf{d}} w_{(\mathbf{i}, \mathbf{j}), \mathbf{d}}^{-1} = \begin{pmatrix} 1 & & \\ & s_2 & \\ & & s_2 s_1 \end{pmatrix} : \mathbb{P}_{(i, i, j)} \oplus \mathbb{P}_{(i, j, i)} \oplus \mathbb{P}_{(j, i, i)} \rightarrow \mathbb{P}_{(i, i, j)} \oplus \mathbb{P}_{(i, i, j)} \oplus \mathbb{P}_{(i, i, j)}$$

we obtain

$$\begin{pmatrix} 1 + s_1 & & \\ 1 + s_1 & & \\ 1 + s_1 & & \end{pmatrix} : \mathbb{P}_{(i, i, j)} \rightarrow \mathbb{P}_{(i, i, j)} \oplus \mathbb{P}_{(i, i, j)} \oplus \mathbb{P}_{(i, i, j)}.$$

The computation of $\sigma_i^{-1}(\mathbb{P}_j^{(n)})$ is similar so we omit it. □

Remark 8.2. It was shown in [CL12] that $\mathcal{K}_{\text{Fock}}$ categorifies V_{Fock} (the Fock space). This means that there exists an isomorphism

$$\Phi: K_0(\text{Kom}(\mathcal{K}_{\text{Fock}})) = K_0(\mathcal{K}_{\text{Fock}}) \rightarrow V_{\text{Fock}}$$

which takes $\mathbb{P}_i^{(n)} \mathbf{1}_0 \mapsto P_i^{(n)}$ in the notation of § 3.2. Subsequently, Proposition 8.1 induces the action of $\text{Br}(D)$ on V_{Fock} described in Proposition 3.1.

Remark 8.3. Consider the zig-zag algebra B of Dynkin type A_k . In [KS02, §4], a complex of finitely generated, projective B -modules is associated to any ‘admissible’ curve in the k -punctured disk. Equivalently, this is a complex of \mathbf{P} in $\mathbf{Kom}(\mathcal{H}')$ (for example $\mathbf{P}_1\langle -1 \rangle \rightarrow \mathbf{P}_2$). It would be interesting to generalize that result as follows: to any ‘admissible’ n -tuple of curves in the k -punctured disk, associate a complex of elements in \mathcal{H}' where each term is a sum of a product of exactly n \mathbf{P} (for example, the complexes appearing in the statement of Proposition 8.1).

8.2 A conjectural intertwiner

At this point we have two braid group actions; one is via the complexes Σ_i and the other is directly on the 2-category $\mathbf{Kom}(\mathcal{H})$. We conjecture that these two actions are related as follows.

CONJECTURE 8.4. Consider a 2-representation \mathcal{K} of \mathcal{H} and denote by \mathbf{R} an arbitrary 1-morphism in $\mathbf{Kom}(\mathcal{H})$. Then $\sigma_i(\mathbf{R}) \circ \Sigma_i \cong \Sigma_i \circ \mathbf{R}$.

In particular, this means that the braid group action on $\mathbf{Kom}(\mathcal{H})$ from §7 is just conjugation using the complexes Σ_i .

8.3 Vertex operators and braid groups

In this section we suppose that our Dynkin diagram D is of affine type (and still simply laced). Denote by \mathfrak{g} the associated affine Lie algebra. In [CL11a] we defined what it means to have a 2-representation of \mathfrak{g} . Roughly, this consists of a 2-category where the objects are indexed by weights of \mathfrak{g} , 1-morphisms are generated by $E_i^{(r)}$ and $F_i^{(r)}$ where $i \in I, r \in \mathbb{N}$ and there are various 2-morphisms with relations. This definition is analogous to the one from this paper for 2-representations of $\widehat{\mathfrak{h}}$.

Suppose that \mathcal{K} is an integrable 2-representation of $\widehat{\mathfrak{h}}$. In [CL11b] we showed that $\mathbf{Kom}(\mathcal{K})$ can be given the structure of a 2-representation of \mathfrak{g} . This categorifies the Frenkel–Kac–Segal vertex operator construction. Roughly, we did the following.

- We defined $\mathbf{1}_\lambda \mapsto \mathbf{1}_n$ if $\lambda = w \cdot \Lambda_0 - n\delta$ for some Weyl element w and $\mathbf{1}_\lambda \mapsto 0$ otherwise. Here λ is a weight, Λ_0 is the fundamental weight corresponding to the affine node and δ is the imaginary root.
- We mapped E_i, F_i as follows

$$E_i \mathbf{1}_\lambda \mapsto [\dots \rightarrow P_i^{(l)} Q_i^{(1^{k+l})} \langle -l \rangle \rightarrow \dots \rightarrow P_i Q_i^{(1^{k+1})} \langle -1 \rangle \rightarrow Q_i^{(1^k)}] \tag{16}$$

$$\mathbf{1}_\lambda F_i \mapsto [P_i^{(1^k)} \rightarrow P_i^{(1^{k+1})} Q_i \langle 1 \rangle \rightarrow \dots \rightarrow P_i^{(1^{k+l})} Q_i^{(l)} \langle l \rangle \rightarrow \dots] \tag{17}$$

if $k := \langle \lambda, \alpha_i \rangle + 1 \geq 0$ (and to similar complexes if $k < 0$).

We argued that this action extends to give a 2-representation of \mathfrak{g} . This means that there also exist complexes for divided power $E_i^{(r)} \mathbf{1}_\lambda$ and $\mathbf{1}_\lambda F_i^{(r)}$. We gave the following conjectural explicit description of these complexes (proven when $r = 1, 2$). Although we did not identify these complexes explicitly (apart from the cases $r = 1, 2$) we did conjecture that, in general, if $k := \langle \lambda, \alpha_i \rangle + r \geq 0$, then

$$E_i^{(r)} \mathbf{1}_\lambda := \left[\dots \rightarrow \bigoplus_{w(\mu) \leq r, |\mu|=l} P_i^{(\mu^t)} Q_i^{(r^k, \mu)} \langle -l \rangle \rightarrow \dots \rightarrow P_i Q_i^{(r^k, 1)} \langle -1 \rangle \rightarrow Q_i^{(r^k)} \right] \left\langle -\binom{r}{2} \right\rangle \left[\binom{r}{2} \right] \tag{18}$$

for certain explicit differentials. Here the direct sum is over all partitions μ of size $|\mu|$ which fit in a box of width r ($w(\mu)$ denotes the width of μ). We also conjectured similar formulas if $k \leq 0$ and likewise for $\mathbf{1}_\lambda F_i^{(r)}$.

8.3.1 *Associated braid group actions.* Given any 2-representation of \mathfrak{g} , we considered in [CK12] the Rickard complex defined by

$$T_i \mathbf{1}_\lambda := [\dots \rightarrow E_i^{(-\langle \lambda, \alpha_i \rangle + s)} F_i^{(s)} \langle -s \rangle \mathbf{1}_\lambda \rightarrow \dots \rightarrow E_i^{(-\langle \lambda, \alpha_i \rangle + 1)} F_i \langle -1 \rangle \mathbf{1}_\lambda \rightarrow E_i^{(-\langle \lambda, \alpha_i \rangle)} \mathbf{1}_\lambda] \quad (19)$$

if $\langle \lambda, \alpha_i \rangle \leq 0$ (and similarly if $\langle \lambda, \alpha_i \rangle \geq 0$). We then showed [CK12, Theorem 2.10] that these complexes satisfy the braid relations in $\text{Br}(D)$. Note that the domain and range of $T_i \mathbf{1}_\lambda$ are given by

$$T_i \mathbf{1}_\lambda : \lambda \rightarrow s_i \cdot \lambda \quad \text{where } s_i \cdot \lambda = \lambda - \langle \lambda, \alpha_i \rangle \alpha_i.$$

If $\mathbf{1}_\lambda \mapsto \mathbf{1}_n$ under the map from [CL11b], then it is easy to check that $\mathbf{1}_{s_i \cdot \lambda} \mapsto \mathbf{1}_n$. Thus, if we compose the complexes for T_i from (19) with those for E and F (from (16) and (17)) then we obtain complexes $T_i \in \text{Kom}(\mathcal{K})$ with domain and range n .

Example 1. Suppose $\mathbf{1}_\lambda \mapsto \mathbf{1}_n$ under the map from [CL11b] with $\langle \lambda, \alpha_i \rangle = -n$. Then, the complex from (18) with $k = 0$ and $r = n$ gives the following expression for $E_i^{(n)} \mathbf{1}_\lambda$

$$\left[\dots \rightarrow \bigoplus_{w(\mu) \leq n, |\mu|=l} P_i^{(\mu^t)} Q_i^{(\mu)} \langle -l \rangle \mathbf{1}_n \rightarrow \dots \rightarrow P_i Q_i \langle -1 \rangle \mathbf{1}_n \rightarrow \mathbf{1}_n \right] \left\langle -\binom{n}{2} \right\rangle \left[\binom{n}{2} \right]. \quad (20)$$

Note that the terms in (20) are zero if $|\mu| > n$ so the extra condition that $w(\mu) \leq n$ is not necessary. Subsequently, $E_i^{(n)} \mathbf{1}_\lambda$ is the same as our complex $\Sigma_i \mathbf{1}_n$. On the other hand, it is not difficult to check that in this case $F_i^{(s)} \mathbf{1}_\lambda = 0$ for any $s > 0$. Thus, the expression in (19) simplifies to give $T_i \mathbf{1}_\lambda = E_i^{(n)} \mathbf{1}_\lambda$. Thus, $T_i \mathbf{1}_\lambda = \Sigma_i \mathbf{1}_n$ and, using [CK12], we recover the braiding of the $\Sigma_i \mathbf{1}_n$ (Theorem 1.1).

Example 2. Suppose $\langle \lambda, \alpha_i \rangle = 1$ and that, under the map in [CL11b], $\mathbf{1}_\lambda \mapsto \mathbf{1}_2$. Then

$$T_i \mathbf{1}_\lambda = [F_i^{(2)} E_i \langle -1 \rangle \mathbf{1}_\lambda \rightarrow F_i \mathbf{1}_\lambda] : \lambda \rightarrow s_i \cdot \lambda.$$

Using the definitions in [CL11b], one can check that

$$F_i \mathbf{1}_\lambda \mapsto [\mathbf{1}_2 \rightarrow P_i Q_i \langle 1 \rangle \mathbf{1}_2 \rightarrow P_i^{(1^2)} Q_i^{(2)} \langle 2 \rangle \mathbf{1}_2] \\ F_i^{(2)} \mathbf{1}_{\lambda + \alpha_i} \mapsto P_i^{(2)} \langle 1 \rangle [-1] \mathbf{1}_0 \quad \text{and} \quad E_i \mathbf{1}_\lambda \mapsto Q_i^{(1^2)} \mathbf{1}_2.$$

Combining this together gives that

$$P_i^{(1^2)} Q_i^{(2)} [-1] \mathbf{1}_2 \rightarrow [\mathbf{1}_2 \rightarrow P_i Q_i \langle 1 \rangle \mathbf{1}_2 \rightarrow P_i^{(2)} Q_i^{(1^2)} \langle 2 \rangle \mathbf{1}_2].$$

This collapses to give a complex of the form

$$[P_i^{(1^2)} Q_i^{(2)} \mathbf{1}_2 \oplus \mathbf{1}_2 \rightarrow P_i Q_i \langle 1 \rangle \mathbf{1}_2 \rightarrow P_i^{(2)} Q_i^{(1^2)} \langle 2 \rangle \mathbf{1}_2]. \quad (21)$$

Note that this complex is *not* of the form of $\Sigma_i \mathbf{1}_2$.

9. Conclusion

Using [CK12], and assuming the conjectural expressions for $E_i^{(r)}$ and $F_i^{(r)}$ from [CL11b] (such as that in (18)) we recover the main result in this paper. On the other hand, as Example 2 above illustrates, the full categorical action from [CL11b] and the braid group action it induces via [CK12] gives us a larger collection of complexes in $\text{Kom}(\mathcal{H})$ which satisfy the braid relations (equation (21) is an example of one such complex). It would be interesting to explicitly identify all of these complexes in $\text{Kom}(\mathcal{H})$ directly.

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Appendix A. Proof of Proposition 7.2

What one needs to check is that the image of any two equivalent 2-morphisms (i.e. related by some 2-relation) under any $\sigma_i^{\pm 1}$ are identical.

There are many 2-relations so we will not check all of them in this paper. We illustrate by checking one of the most difficult relations, namely

$$\sigma_i(T_{jj}1_j) \circ \sigma_i(1_j T_{jj}) \circ \sigma_i(T_{jj}1_j) = \sigma_i(1_j T_{jj}) \circ \sigma_i(T_{jj}1_j) \circ \sigma_i(1_j T_{jj})$$

whenever $\langle i, j \rangle = -1$.

By direct computation, $\sigma_i(T_{jj}1_j) \circ \sigma_i(1_j T_{jj}) \circ \sigma_i(T_{jj}1_j)$ is the following map of complexes:

$$\begin{array}{ccccccc}
 & & P_i P_i P_j \langle -2 \rangle & & P_i P_j P_j \langle -1 \rangle & & \\
 & & \oplus & & \oplus & & \\
 P_i P_i P_i \langle -3 \rangle & \xrightarrow{a} & P_i P_j P_i \langle -2 \rangle & \xrightarrow{b} & P_j P_i P_j \langle -1 \rangle & \xrightarrow{c} & P_j P_j P_j \\
 \downarrow A & & \oplus & & \oplus & & \downarrow D \\
 & & P_j P_i P_i \langle -2 \rangle & & P_j P_j P_i \langle -1 \rangle & & \\
 & & \downarrow B & & \downarrow C & & \\
 & & P_i P_i P_j \langle -2 \rangle & & P_i P_j P_j \langle -1 \rangle & & \\
 P_i P_i P_i \langle -3 \rangle & \xrightarrow{d} & P_i P_j P_i \langle -2 \rangle & \xrightarrow{e} & P_j P_i P_j \langle -1 \rangle & \xrightarrow{f} & P_j P_j P_j \\
 & & \oplus & & \oplus & & \\
 & & P_j P_i P_i \langle -2 \rangle & & P_j P_j P_i \langle -1 \rangle & &
 \end{array}$$

where

$$a = \begin{pmatrix} 1_i 1_i X_i^j \\ 1_i X_i^j 1_i \\ X_i^j 1_i 1_i \end{pmatrix}, \quad b = \begin{pmatrix} 1_i X_i^j 1_j & 1_i 1_j X_i^j & 0 \\ X_i^j & 0 & 1_j 1_i X_i^j \\ 0 & X_i^j 1_j 1_i & 1_j X_i^j 1_i \end{pmatrix}, \quad c = \begin{pmatrix} X_i^j 1_j 1_j & 1_j X_i^j 1_j & 1_j 1_j X_i^j \end{pmatrix},$$

$$d = \begin{pmatrix} 1_i X_i^j 1_i \\ 1_i 1_i X_i^j \\ X_i^j 1_i 1_i \end{pmatrix}, \quad e = \begin{pmatrix} 1_i 1_j X_i^j & 1_i X_i^j 1_j & 0 \\ X_i^j & 0 & 1_j X_i^j 1_i \\ 0 & X_i^j 1_i 1_j & 1_j 1_i X_i^j \end{pmatrix}, \quad f = \begin{pmatrix} X_i^j 1_j 1_j & 1_j 1_j X_i^j & 1_j X_i^j 1_j \end{pmatrix},$$

$$A = ((T_{ii}1_i) \circ (1_i T_{ii}) \circ (T_{ii}1_i)),$$

$$B = \begin{pmatrix} 0 & (T_{ji}1_i) \circ (1_j T_{ii}) \circ (T_{ij}1_i) & 0 \\ 0 & 0 & (T_{ii}1_j) \circ (1_i T_{ji}) \circ (T_{ji}1_i) \\ (T_{ij}1_i) \circ (1_i T_{ij}) \circ (T_{ii}1_j) & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & (T_{ji}1_j) \circ (1_jT_{ji}) \circ (T_{jj}1_i) \\ (T_{jj}1_i) \circ (1_jT_{ij}) \circ (T_{ij}1_j) & 0 & 0 \\ 0 & (T_{ij}1_j) \circ (1_iT_{jj}) \circ (T_{ji}1_j) & 0 \end{pmatrix},$$

$$D = ((T_{jj}1_j) \circ (1_jT_{jj}) \circ (T_{jj}1_j)).$$

Similarly, $\sigma_i(1_jT_{jj}) \circ \sigma_i(T_{jj}1_j) \circ \sigma_i(1_jT_{jj})$ is a map of complexes:

$$\begin{array}{ccccccc} & & P_iP_iP_j\langle -2 \rangle & & P_iP_jP_j\langle -1 \rangle & & \\ & & \oplus & & \oplus & & \\ P_iP_iP_i\langle -3 \rangle & \xrightarrow{a} & P_iP_jP_i\langle -2 \rangle & \xrightarrow{b} & P_jP_iP_j\langle -1 \rangle & \xrightarrow{c} & P_jP_jP_j \\ \downarrow A' & & \oplus & & \oplus & & \downarrow D' \\ & & P_jP_iP_i\langle -2 \rangle & & P_jP_jP_i\langle -1 \rangle & & \\ & & \downarrow B' & & \downarrow C' & & \\ & & P_iP_iP_j\langle -2 \rangle & & P_iP_jP_j\langle -1 \rangle & & \\ & & \oplus & & \oplus & & \\ P_iP_iP_i\langle -3 \rangle & \xrightarrow{d} & P_iP_jP_i\langle -2 \rangle & \xrightarrow{e} & P_jP_iP_j\langle -1 \rangle & \xrightarrow{f} & P_jP_jP_j \\ & & \oplus & & \oplus & & \\ & & P_jP_iP_i\langle -2 \rangle & & P_jP_jP_i\langle -1 \rangle & & \end{array}$$

where

$$A' = ((1_iT_{ii}) \circ (T_{ii}1_i) \circ (1_iT_{ii})),$$

$$B' = \begin{pmatrix} 0 & (1_iT_{ij}) \circ (T_{ii}1_j) \circ (1_iT_{ji}) & 0 \\ 0 & 0 & (1_iT_{ji}) \circ (T_{ji}1_i) \circ (1_jT_{ii}) \\ (1_jT_{ii}) \circ (T_{ij}1_i) \circ (1_iT_{ij}) & 0 & 0 \end{pmatrix},$$

$$C' = \begin{pmatrix} 0 & 0 & (1_iT_{jj}) \circ (T_{ji}1_j) \circ (1_jT_{ji}) \\ (1_jT_{ij}) \circ (T_{ij}1_j) \circ (1_iT_{jj}) & 0 & 0 \\ 0 & (1_jT_{ji}) \circ (T_{jj}1_i) \circ (1_jT_{ij}) & 0 \end{pmatrix},$$

$$D' = ((1_jT_{jj}) \circ (T_{jj}1_j) \circ (1_jT_{jj})).$$

Equalities of matrices $A = A'$, $B = B'$, $C = C'$, $D = D'$ follow from the three-strand relation in the category.

Appendix B. Proof of Theorem 7.1

To prove Theorem 7.1 one needs to fix isomorphisms $\sigma_i\sigma_i^{-1}M \rightarrow M$, $\sigma_i^{-1}\sigma_iM \rightarrow M$, $\sigma_i\sigma_{i+1}\sigma_iM \rightarrow \sigma_{i+1}\sigma_i\sigma_{i+1}M$ on all generating 1-morphisms of $\text{Kom}(\mathcal{H}')$ and then show that these isomorphisms are natural with respect to all generating 2-morphisms.

B.0.2 Reidemeister 2-relations on 1-morphisms. Recall the homotopy equivalences

$$\nu_{P_i} : \sigma_i(P_i) \xrightarrow{\sim} P_i\langle -2 \rangle[1] \quad \text{and} \quad \zeta_{P_i} : \sigma_i^{-1}(P_i) \xrightarrow{\sim} P_i\langle 2 \rangle[-1].$$

We can use these to define isomorphisms

$$\sigma_i^{-1}\sigma_i(P_i) \xrightarrow{\sigma_i^{-1}(\nu_{P_i})} \sigma_i^{-1}(P_i\langle -2 \rangle[1]) \xrightarrow{\zeta_{P_i}} P_i \quad \text{and} \quad \sigma_i\sigma_i^{-1}(P_i) \xrightarrow{\sigma_i(\zeta_{P_i})} \sigma_i(P_i\langle 2 \rangle[-1]) \xrightarrow{\nu_{P_i}} P_i.$$

On the other hand, if $\langle i, j \rangle = -1$, then we use the following maps

$$\begin{array}{ccccc}
 \sigma_i^{-1}\sigma_i(\mathbb{P}_j) & = & \mathbb{P}_i\langle -1 \rangle & \xrightarrow{(X_i^j \ 1 \ X_i^i)} & \mathbb{P}_j \oplus \mathbb{P}_i\langle -1 \rangle \oplus \mathbb{P}_i\langle 1 \rangle & \xrightarrow{(X_j^i \ 0 \ -\epsilon_{ij})} & \mathbb{P}_i\langle 1 \rangle \\
 \downarrow & & \downarrow & & \downarrow (1 \ -X_i^j \ 0) & & \downarrow \\
 \mathbb{P}_j & = & 0 & \longrightarrow & \mathbb{P}_j & \longrightarrow & 0 \\
 \downarrow & & \downarrow & & \downarrow (1 \ 0 \ \epsilon_{ij}X_j^i) & & \downarrow \\
 \sigma_i^{-1}\sigma_i(\mathbb{P}_j) & = & \mathbb{P}_i\langle -1 \rangle & \xrightarrow{(X_i^j \ 1 \ X_i^i)} & \mathbb{P}_j \oplus \mathbb{P}_i\langle -1 \rangle \oplus \mathbb{P}_i\langle 1 \rangle & \xrightarrow{(X_j^i \ 0 \ -\epsilon_{ij})} & \mathbb{P}_i\langle 1 \rangle
 \end{array}$$

where the rightmost column is in cohomological degree one. The vertical composition is

$$\begin{array}{ccccc}
 \mathbb{P}_i\langle -1 \rangle & \longrightarrow & \mathbb{P}_j \oplus \mathbb{P}_i\langle -1 \rangle \oplus \mathbb{P}_i\langle 1 \rangle & \longrightarrow & \mathbb{P}_i\langle 1 \rangle \\
 \downarrow 0 & \nearrow D_0 & \downarrow \gamma_0 & \nearrow D_1 & \downarrow 0 \\
 \mathbb{P}_i\langle -1 \rangle & \longrightarrow & \mathbb{P}_j \oplus \mathbb{P}_i\langle -1 \rangle \oplus \mathbb{P}_i\langle 1 \rangle & \longrightarrow & \mathbb{P}_i\langle 1 \rangle
 \end{array}$$

where

$$\gamma_0 = \begin{pmatrix} 1 & -X_i^j & 0 \\ 0 & 0 & 0 \\ \epsilon_{ij}X_i^j & -X_i^i & 0 \end{pmatrix}.$$

Here $D_0 = (0 \ -1 \ 0)$ and $D_1 = (0 \ 0 \ \epsilon_{ij})$ give a homotopy between this composition and the identity map.

B.0.3 Reidemeister 3-relations on 1-morphisms.

PROPOSITION B.1. For $\langle i, j \rangle = -1$ there is an isomorphism

$$\gamma_{\mathbb{P}_i} : \sigma_i\sigma_j\sigma_i(\mathbb{P}_i) \rightarrow \sigma_j\sigma_i\sigma_j(\mathbb{P}_i).$$

Proof. One checks that the map of complexes $\beta : \sigma_i\sigma_j(\mathbb{P}_i) \rightarrow \mathbb{P}_j\langle -1 \rangle$ given by

$$\begin{array}{ccccc}
 \mathbb{P}_i\langle -2 \rangle & \xrightarrow{\begin{pmatrix} -\epsilon_{ij} \\ 0 \\ X_i^j \end{pmatrix}} & \mathbb{P}_i\langle -2 \rangle \oplus \mathbb{P}_i\langle 1 \rangle \oplus \mathbb{P}_j\langle -1 \rangle & \xrightarrow{(X_i^i \ 1 \ X_j^i)} & \mathbb{P}_i \\
 \downarrow & & \downarrow (\epsilon_{ij}X_i^j \ 0 \ 1) & & \downarrow \\
 0 & \longrightarrow & \mathbb{P}_j & \longrightarrow & 0
 \end{array}$$

is a homotopy equivalence with inverse map $\bar{\beta}$. Then $\beta \circ \sigma_i\sigma_j(\nu_{\mathbb{P}_i}) : \sigma_i\sigma_j\sigma_i(\mathbb{P}_i) \rightarrow \mathbb{P}_j\langle -3 \rangle[2]$ is a homotopy equivalence.

Similarly $\nu_{\mathbb{P}_j} \circ \sigma_j(\beta_{\mathbb{P}_j}) : \sigma_j\sigma_i\sigma_j(\mathbb{P}_i) \rightarrow \mathbb{P}_j\langle -3 \rangle[-2]$ is a homotopy equivalence. Finally, we define

$$\gamma_{\mathbb{P}_i} = -\sigma_{i+1}(\bar{\beta}_{\mathbb{P}_{i+1}}) \circ \bar{\nu}_{\mathbb{P}_{i+1}} \circ \beta_{\mathbb{P}_{i+1}} \circ \sigma_i\sigma_{i+1}(\nu_{\mathbb{P}_i}). \quad \square$$

PROPOSITION B.2. For $\langle i, j \rangle = -1$ there is an isomorphism

$$\gamma_{\mathbb{P}_j} : \sigma_i\sigma_j\sigma_i(\mathbb{P}_j) \rightarrow \sigma_j\sigma_i\sigma_j(\mathbb{P}_j).$$

Proof. The proof of this is similar to that of Proposition B.1. □

Next, if $\langle i, j \rangle = -1 = \langle j, k \rangle$ and $\langle i, k \rangle = 0$ one needs to write down a homotopy equivalence $\sigma_i \sigma_j \sigma_i(\mathbf{P}_k) \xrightarrow{\sim} \sigma_j \sigma_i \sigma_j(\mathbf{P}_k)$. A direct calculation shows that

$$\begin{aligned} \sigma_i \sigma_j \sigma_i(\mathbf{P}_k) &= [\mathbf{P}_i \langle -2 \rangle \xrightarrow{X_i^j} \mathbf{P}_j \langle -1 \rangle \xrightarrow{X_j^k} \mathbf{P}_k] \\ \sigma_j \sigma_i \sigma_j(\mathbf{P}_k) &= [\mathbf{P}_j \langle -3 \rangle \xrightarrow{A} \mathbf{P}_j \langle -3 \rangle \oplus \mathbf{P}_j \langle -1 \rangle \oplus \mathbf{P}_i \langle -2 \rangle \xrightarrow{B} \mathbf{P}_j \langle -1 \rangle \oplus \mathbf{P}_j \langle -1 \rangle \xrightarrow{C} \mathbf{P}_k] \end{aligned}$$

where

$$A = \begin{pmatrix} 1 & 0 & X_j^i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 0 \\ X_j^j & 1 & X_j^i \end{pmatrix}, \quad C = \begin{pmatrix} X_j^k & X_j^k \end{pmatrix}.$$

It is not difficult to check that the following maps are homotopy equivalences.

$$\begin{array}{ccccc} \sigma_i \sigma_j \sigma_i(\mathbf{P}_k) & = & \mathbf{P}_j \langle -3 \rangle & \xrightarrow{A} & \mathbf{P}_j \langle -3 \rangle \oplus \mathbf{P}_j \langle -1 \rangle \oplus \mathbf{P}_i \langle -2 \rangle & \xrightarrow{B} & \mathbf{P}_j \langle -1 \rangle \oplus \mathbf{P}_j \langle -1 \rangle & \xrightarrow{C} & \mathbf{P}_k \\ \downarrow \phi & & \downarrow & & \downarrow \begin{pmatrix} -X_j^i & 0 & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 1 \end{pmatrix} & & \downarrow 1 \\ \sigma_j \sigma_i \sigma_j(\mathbf{P}_k) & = & 0 & \xrightarrow{\quad} & \mathbf{P}_i \langle -2 \rangle & \xrightarrow{X_i^j} & \mathbf{P}_j \langle -1 \rangle & \xrightarrow{X_j^k} & \mathbf{P}_k \\ \downarrow \bar{\phi} & & \downarrow & & \downarrow \begin{pmatrix} 0 & -X_j^i & 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 \end{pmatrix} & & \downarrow 1 \\ \sigma_i \sigma_j \sigma_i(\mathbf{P}_k) & = & \mathbf{P}_j \langle -3 \rangle & \xrightarrow{A} & \mathbf{P}_j \langle -3 \rangle \oplus \mathbf{P}_j \langle -1 \rangle \oplus \mathbf{P}_i \langle -2 \rangle & \xrightarrow{B} & \mathbf{P}_j \langle -1 \rangle \oplus \mathbf{P}_j \langle -1 \rangle & \xrightarrow{C} & \mathbf{P}_k \end{array}$$

Finally, if i, j, k form a triangle then there is a more complicated homotopy equivalence $\sigma_i \sigma_j \sigma_i(\mathbf{P}_k) \xrightarrow{\sim} \sigma_j \sigma_i \sigma_j(\mathbf{P}_k)$ which we omit (the interested reader can contact the authors for more details).

B.0.4 Reidemeister 2-relations on 2-morphisms. First, assuming $\langle j, k \rangle = -1$, one needs to check that the following diagrams commute

$$\begin{array}{ccc} \sigma_i^{-1} \sigma_i(\mathbf{P}_k \langle -1 \rangle) & \xrightarrow{\sigma_i^{-1} \sigma_i(X_k^j)} & \sigma_i^{-1} \sigma_i(\mathbf{P}_j) \\ \downarrow & & \downarrow \\ \mathbf{P}_k \langle -1 \rangle & \xrightarrow{X_k^j} & \mathbf{P}_j \end{array} \qquad \begin{array}{ccc} \sigma_i \sigma_i^{-1}(\mathbf{P}_k \langle -1 \rangle) & \xrightarrow{\sigma_i \sigma_i^{-1}(X_k^j)} & \sigma_i \sigma_i^{-1}(\mathbf{P}_j) \\ \downarrow & & \downarrow \\ \mathbf{P}_k \langle -1 \rangle & \xrightarrow{X_k^j} & \mathbf{P}_j \end{array}$$

where the vertical maps are those from the previous section.

Next, for any i, j, k , one needs to check the commutativity of the following squares.

$$\begin{array}{ccc} \sigma_i^{-1} \sigma_i(\mathbf{P}_j \mathbf{P}_k) & \xrightarrow{\sigma_i^{-1} \sigma_i(T_{jk})} & \sigma_i^{-1} \sigma_i(\mathbf{P}_j \mathbf{P}_k) \\ \downarrow & & \downarrow \\ \mathbf{P}_j \mathbf{P}_k & \xrightarrow{T_{jk}} & \mathbf{P}_j \mathbf{P}_k \end{array} \qquad \begin{array}{ccc} \sigma_i \sigma_i^{-1}(\mathbf{P}_j \mathbf{P}_k) & \xrightarrow{\sigma_i \sigma_i^{-1}(T_{jk})} & \sigma_i \sigma_i^{-1}(\mathbf{P}_j \mathbf{P}_k) \\ \downarrow & & \downarrow \\ \mathbf{P}_j \mathbf{P}_k & \xrightarrow{T_{jk}} & \mathbf{P}_j \mathbf{P}_k \end{array}$$

B.0.5 Reidemeister 3-relations on 2-morphisms. First, assuming $\langle i, j \rangle = -1$ and $\langle l, k \rangle = -1$, one needs to check that the following diagram commutes.

$$\begin{array}{ccc} \sigma_i \sigma_j \sigma_i(\mathbf{P}_k \langle -1 \rangle) & \xrightarrow{\sigma_i \sigma_{i+1} \sigma_i(X_k^l)} & \sigma_i \sigma_j \sigma_i(\mathbf{P}_l) \\ \downarrow & & \downarrow \\ \sigma_j \sigma_i \sigma_j(\mathbf{P}_k \langle -1 \rangle) & \xrightarrow{\sigma_j \sigma_i \sigma_j(X_k^l)} & \sigma_j \sigma_i \sigma_j(\mathbf{P}_l) \end{array}$$

Next, for any i, j, k, l with $\langle i, j \rangle = -1$, one needs to show that the following diagram commutes.

$$\begin{array}{ccc}
 \sigma_i \sigma_j \sigma_i(\mathbf{P}_l \mathbf{P}_k) & \xrightarrow{\sigma_i \sigma_j \sigma_i(T_{lk})} & \sigma_i \sigma_j \sigma_i(\mathbf{P}_k \mathbf{P}_l) \\
 \downarrow \gamma_{\mathbf{P}_l} \gamma_{\mathbf{P}_k} & & \downarrow \gamma_{\mathbf{P}_k} \gamma_{\mathbf{P}_l} \\
 \sigma_j \sigma_i \sigma_j(\mathbf{P}_l \mathbf{P}_k) & \xrightarrow{\sigma_j \sigma_i \sigma_j(T_{lk})} & \sigma_j \sigma_i \sigma_j(\mathbf{P}_k \mathbf{P}_l)
 \end{array}$$

Checking that all of these diagrams commute breaks up into many cases. Each case, although not difficult, is a bit tedious (the interested reader can contact the authors for more details about these calculations).

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