ON THE CALCULATION OF REFRACTION IN MODEL ATMOSPHERES

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ABSTRACT

Approximations have been removed from the derivation of the coefficients in the binomial expansion for astronomical refraction, $\Delta z_1 = \tan z_1(Y_0 - \frac{1}{2} Y_1 \sec^2 z_1 + \frac{3}{8} Y_2 \sec^4 z_1 - \ldots)$, allowing the calculation of any number of terms to any precision desired. The range of the refraction formula has been extended to greater zenith distances (<90°) by inserting a damping factor into the binomial formula, truncating the expansion at a proper point, and rearranging the terms. Another, computer-manipulated series has been developed for zenith distances at or near the horizon. Further applications include the calculation of photogrammetric and parallactic refractions, as well as range corrections in satellite geodesy.

NOTATION

The notation has been taken from an earlier work (Saastamoinen, 1972-73) on the same subject matter, although important changes and additions have been made to facilitate the derivation of general formulas not presented before. The following symbols appear in the text without explanation:

β	vertical gradient of temperature
e	base of natural logarithms
g	intensity of gravity
i, k	0, 1, 2, ...
j	1, 2, 3, ...
n	refractive index
r	radius vector
R	gas constant of dry air
T	absolute temperature
z	zenith distance
n_1, r_1, etc.	values at the point of observation

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r_a, r_b, etc. values at the lower and upper limits, respectively, of an atmospheric layer with β const.

The same symbol may occasionally be used in different meanings (e also for partial pressure of water vapor) if confusion is unlikely and the notation is familiar from geodetic and meteorological literature .

1. ASTRONOMICAL REFRACTION

1.1 Introduction.

For the evaluation of astronomical refraction

$$\Delta z_1 = \int_0^{\log_e n_1} \tan z \, d\log_e n \tag{1}$$

in a spherically symmetric model atmosphere, the law of refraction

$$y \sin z = \sin z_1 \tag{2}$$

where

$$y = nr/(n_1 r_1)$$

gives

$$\tan z = \tan z_1 [1 + (y^2-1)\sec^2 z_1]^{-\frac{1}{2}}$$

$$= \tan z_1 \left[1 + (-)^k \sum_{k=1}^{\infty} \frac{1.3.5...(2k-1)}{2.4.6...\ (2k)} (y^2-1)^k \sec^{2k} z_1\right]. \tag{3}$$

Binomial series (3) is convergent if $(y^2-1)\sec^2 z_1 < 1$.

Providing that the condition for convergence is satisfied, the contribution to the astronomical refraction of a layer of air between radii vectors r_a and r_b is consequently

$$\Delta z_1(a;b) = \tan z_1 (Y_0 - \frac{1}{2} Y_1 \sec^2 z_1 + \frac{3}{8} Y_2 \sec^4 z_1 - \frac{5}{16} Y_3 \sec^6 z_1 + \ ...) \tag{4}$$

where the coefficients

$$Y_k = \int_{\log_e n_b}^{\log_e n_a} (y^2 - 1)^k d\log_e n \tag{5}$$

are functions of the model atmosphere employed. Total astronomical

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refraction (1) is the sum of the contributions from all the layers between the point of observation and the top of the atmosphere.

Power series (4), or the variant

$$\Delta z_1(a;b) = \tan z_1(Y_0' - \frac{1}{2} Y_1'\tan^2 z_1 + \frac{3}{8} Y_2'\tan^4 z_1 - \frac{5}{16} Y_3'\tan^6 z_1 + \ldots) \quad (4')$$

for which binomial series

$$\tan z = \frac{1}{y} \tan z_1\left[1 + \frac{1}{y^2}(y^2-1)\tan^2 z_1\right]^{-\frac{1}{2}}$$

$$= \tan z_1\left[\frac{1}{y} + (-)^k \sum_{k=1}^{\infty} \frac{1.3.5\ldots(2k-1)}{2.4.6\ldots(2k)} \frac{1}{y^{2k+1}}(y^2-1)^k \tan^{2k} z_1\right] \quad (3')$$

gives the coefficients

$$Y_k' = \int_{\log_e n_b}^{\log_e n_a} (1/y)^{2k+1}(y^2-1)^k d\log_e n, \quad (5')$$

is the common foundation of most formulas proposed for the calculation of astronomical refraction (Teleki, 1974). Essential differences are found, however, in the way of calculation of the coefficients.

1.2 Calculation of coefficients Y_k in terms of atmospheric integrals.

Multiplication of the identity,

$$y^2-1 = \frac{2}{n_1{}^2 r_1}(r - r_1) - \frac{2}{n_1}(n_1-n) + \frac{1}{n_1{}^2 r_1{}^2}(r - r_1)^2$$

$$+ \frac{4}{n_1{}^2 r_1}(n - 1)(r - r_1) + \frac{1}{n_1{}^2}(n_1 - n)^2 + \frac{2}{n_1{}^2 r_1{}^2}(n - 1)(r - r_1)^2$$

$$+ \frac{2}{n_1{}^2 r_1}(n - 1)^2(r - r_1) + \frac{1}{n_1{}^2 r_1{}^2}(n - 1)^2(r - r_1)^2,$$

by

$$d\log_e n = dn/n = dn - (n - 1)dn + (n - 1)^2 dn - (n - 1)^3 dn + \ldots$$

gives

$$(y^2-1)d\log_e n = \frac{2}{n_1{}^2 r_1}(r - r_1)dn - \frac{2}{n_1}(n_1 - n)d\log_e n$$

$$+ \frac{1}{n_1{}^2 r_1{}^2}(r - r_1)^2 dn + \frac{2}{n_1{}^2 r_1}(n - 1)(r - r_1)dn + \frac{1}{n_1{}^2}(n_1 - n)^2 d\log_e n$$

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$$+ \frac{1}{n_1^2 r_1^2}(n-1)(r-r_1)^2 dn. \tag{6}$$

It is easy to verify that equation (6) is also a mathematical identity.

We can now write, in terms of atmospheric integrals given by the definitions

$$P(i,j) = \frac{1}{r_1^j} \int_{n_b}^{n_a} (n-1)^i (r-r_1)^j dn \tag{7}$$

$$P'(i,j) = P(i,j) + \frac{j}{i+1} P(j-1,i+1) \tag{7'}$$

and

$$Q(i) = \int_{\log_e n_b}^{\log_e n_a} (n_1 - n)^i d\log_e n \;, \tag{8}$$

the formulas

$$Y_0 = Q(0) \tag{9}$$

$$n_1^2 Y_1 = 2\,P(0,1) - 2n_1 Q(1) + P'(0,2) + Q(2) + P(1,2) \tag{10}$$

for the calculation of the first two coefficients in series expansion (4).

Similarly, by actual multiplication, is obtained the identity

$$\begin{aligned}
(y^2-1)^2 d\log_e n &= \frac{4}{n_1^4 r_1}(1-n_1^2)(r-r_1)dn + \frac{2}{n_1^4 r_1^2}(3-n_1^2)(r-r_1)^2 dn \\
&+ \frac{4}{n_1^4 r_1}(3-n_1^2)(n-1)(r-r_1)dn + \frac{4}{n_1^2}(n_1-n)^2 d\log_e n + \frac{4}{n_1^4 r_1^3}(r-r_1)^3 dn \\
&+ \frac{2}{n_1^4 r_1^2}(9-n_1^2)(n-1)(r-r_1)^2 dn + \frac{12}{n_1^4 r_1}(n-1)^2(r-r_1)dn \\
&- \frac{4}{n_1^3}(n_1-n)^3 d\log_e n + \frac{1}{n_1^4 r_1^4}(r-r_1)^4 dn + \frac{12}{n_1^4 r_1^3}(n-1)(r-r_1)^3 dn \\
&+ \frac{18}{n_1^4 r_1^2}(n-1)^2(r-r_1)^2 dn + \frac{4}{n_1^4 r_1}(n-1)^3(r-r_1)dn + \frac{1}{n_1^4}(n_1-n)^4 d\log_e n \\
&+ \frac{3}{n_1^4 r_1^4}(n-1)(r-r_1)^4 dn + \frac{12}{n_1^4 r_1^3}(n-1)^2(r-r_1)^3 dn \\
&+ \frac{6}{n_1^4 r_1^2}(n-1)^3(r-r_1)^2 dn + \frac{3}{n_1^4 r_1^4}(n-1)^2(r-r_1)^4 dn \\
&+ \frac{4}{n_1^4 r_1^3}(n-1)^3(r-r_1)^3 dn + \frac{1}{n_1^4 r_1^4}(n-1)^3(r-r_1)^4 dn
\end{aligned} \tag{11}$$

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which gives the formula

$$\begin{aligned} n_1{}^4Y_2 = {} & 4(1-n_1{}^2)P(0,1) + 2(3-n_1{}^2)P'(0,2) + 4n_1{}^2Q(2) \\ & + 4P'(0,3) + 2(9-n_1{}^2)P(1,2) - 4n_1Q(3) + P'(0,4) \\ & + 12P'(1,3) + Q(4) + 3P'(1,4) + 12P(2,3) \\ & + 3P'(2,4) + P(3,4) \end{aligned} \tag{12}$$

for the third coefficient in series expansion (4).

It can be shown that, for any positive integer k, coefficient $n_1{}^{2k}Y_k$ formed in this way will consist of a sum of $1+2k(1+k)$ terms, as follows:

i. The terms containing integrals Q(i) are

$$(-)^k[(2n_1)^kQ(k) - k(2n_1)^{k-1}Q(k+1) + \frac{k\ (k-1)}{2!}(2n_1)^{k-2}Q(k+2)$$
$$- \frac{k(k-1)(k-2)}{3!}(2n_1)^{k-3}Q(k+3) + \ldots(-)^kQ(2k)].$$

The total number of these terms is k+1.

ii. Each coefficient $n_1{}^{2k}Y_k$ contains the odd integrals

$$\begin{aligned} k(k-1)\ldots(k-j+2)(k-j+1)\Big[& \frac{2^j}{(2j-1)!}(2k-1)(2k-3)\ldots(2k-2j+5)(2k-2j+3) \\ & - \frac{2^{j-1}}{1!(2j-3)!}(2k-3)(2k-5)\ldots(2k-2j+5)(2k-2j+3)n_1{}^2 \\ & + \frac{2^{j-2}}{2!(2j-5)!}(2k-5)(2k-7)\ldots(2k-2j+5)(2k-2j+3)n_1{}^4 \\ & - \frac{2^{j-3}}{3!(2j-7)!}(2k-7)(2k-9)\ldots(2k-2j+5)(2k-2j+3)n_1{}^6 \\ & + \ldots(-)^{j-1}\frac{2^1}{(j-1)!1!}n_1{}^{2j-2}\Big](1-n_1{}^2)^{k-2j+1}P(0,2j-1) \end{aligned}$$

and the even integrals

$$\begin{aligned} k(k-1)\ldots(k-j+2)(k-j+1)\Big[& \frac{2^j}{(2j)!}(2k-1)(2k-3)\ldots(2k-2j+3)(2k-2j+1) \\ & - \frac{2^{j-1}}{1!(2j-2)!}(2k-3)(2k-5)\ldots(2k-2j+3)(2k-2j+1)n_1{}^2 \\ & + \frac{2^{j-2}}{2!(2j-4)!}(2k-5)(2k-7)\ldots(2k-2j+3)(2k-2j+1)n_1{}^4 \\ & - \frac{2^{j-3}}{3!(2j-6)!}(2k-7)(2k-9)\ldots(2k-2j+3)(2k-2j+1)n_1{}^6 \\ & + \ldots(-)^j\frac{2^0}{j!0!}n_1{}^{2j}\Big](1-n_1{}^2)^{k-2j}P(0,2j). \end{aligned}$$

There will be k terms of each kind, or a total of 2k terms.

If the multipliers of integrals P(0,2j) and P(0,2j-1) are written out as polynomials, in ascending powers of n_1^2, it will be found that their corresponding terms, taken in the order from left to right, have the ratios (1/2j)(2k-2j+1), (1/2j)(2k-2j-1), ..., (3/2j), and (1/2j), respectively. This will be shown by the symbolic notation

$$P(0,2j) = \frac{1}{2j}(2k-2j+1,2k-2j-1,\ldots,3,1)P(0,2j-1) \qquad (k \geq j)$$

the numbers in parentheses indicating term-by-term multiplication of the polynomial coefficient of P(0,2j-1) in order to form that of P(0,2j). As also holds

$$P(0,2j+1) = \left(\frac{2}{2j+1}\right)(k-j,k-j-1,\ldots,1,0)P(0,2j) \qquad (k > j)$$

all the terms given by the direct formulas can be calculated in succession from the first one, $2k(1-n_1^2)^{k-1}P(0,1)$, the latter developed into a polynomial by the aid of the binomial theorem.

iii. As y is a symmetric function of n and r, numerous relationships exist that aid the calculation of further terms. Using the symbolic notation, previously introduced, we have

for $k \geq j > i$, a total of $\frac{1}{2}k(k+1)$ terms

$$P(2i+1,2j) = \left(\frac{1}{2i+1}\right)(2k-2i-1,2k-2i-3,\ldots,2j-2i+1,2j-2i-1)P(2i,2j)$$

for $k > j > i$, a total of $\frac{1}{2}k(k-1)$ terms

$$P(2i+1,2j+1) = \left(\frac{1}{2i+1}\right)(2k-2i-1,2k-2i-3,\ldots,2j-2i+1)P(2i,2j+1)$$

for $k \geq j > i > 0$, a total of $\frac{1}{2}k(k-1)$ terms

$$P(2i,2j) = \frac{1}{i}(k-i,k-i-1,\ldots,j-i+1,j-i)P(2i-1,2j)$$

and for $k \geq j+1 > i > 0$, a total of $\frac{1}{2}k(k-1)$ terms

$$P(2i,2j+1) = \frac{1}{i}(k-i,k-i-1,\ldots,j-i+2,j-i+1)P(2i-1,2j+1)$$

which altogether add k(2k-1) terms to those previously determined.

iv. All the remaining terms are combinative; they will be included simply by priming the integrals P(i,j), excepting those of the form P(i,i+1), which do not possess a counterpart.

We shall not dwell with all the arguments needed in a rigorous proof of these rules by double induction; let it suffice here to show their general validity in a single instance, say, for integral P(2,3).

Successive application of the symbolic equations, starting from the last one, gives

$$P(2,3) = \frac{1}{3}[(2k-1)^2(k-1)^2,(2k-3)^2(k-2)^2,\ldots,9,0]P(0,1).$$

We know already, from equation (12), that this equality is true if $k = 2$. It remains to be shown that if the equality is true for $k = p$, it is also true for $k = p + 1$.

To form the polynomial multiplier of $P(2,3)$ in the coefficient $n_1^{2p+2}Y_{p+1}$ it is convenient to use a scheme

	$n_1^2(y^2-1) =$
$P(2,2) = \frac{1}{2}[(2p-1)^2(p-1),(2p-3)^2(p-2),\ldots,9,0]P(0,1)$	$(2/r_1)(r-r_1)$
$P(2,1) = [(2p-1)(p-1),(2p-3)(p-2),\ldots,3,0]P(0,1)$	$+(1/r_1^2)(r-r_1)^2$
$P(1,2) = \frac{1}{2}[(2p-1)^2,(2p-3)^2,\ldots,9,1]P(0,1)$	$+(4/r_1)(n-1)(r-r_1)$
$P(1,1) = (2p-1,2p-3,\ldots,3,1)P(0,1)$	$+(2/r_1^2)(n-1)(r-r_1)^2$
$P(0,2) = \frac{1}{2}(2p-1,2p-3,\ldots,3,1)P(0,1)$	$+(2/r_1)(n-1)^2(r-r_1)$
$P(0,1) = (1,1,\ldots,1,1)P(0,1)$	$+(1/r_1^2)(n-1)^2(r-r_1)^2$
$P(0,3) = \frac{1}{3}[(2p-1)(p-1),(2p-3)(p-2),\ldots,3,0]P(0,1)$	$+(n-1)^2$
$P(1,3) = \frac{1}{3}[(2p-1)^2(p-1),(2p-3)^2(p-2),\ldots,9,0]P(0,1)$	$+2(n-1)$
$P(2,3) = \frac{1}{3}[(2p-1)^2(p-1)^2,(2p-3)^2(p-2)^2,\ldots,100,9,0]P(0,1)$	$+(1-n_1^2)$

which displays the identity for y^2-1, slightly modified from the form given previously, together with a set of symbolic equations derived by the rules we assume valid. It is now easy to comprehend that the required polynomial is obtained by multiplying each term of the identity into the equation on the left, and adding up the results. These calculations, shown in a separate table, establish the polynomial in the form

$$P(2,3) = \frac{2}{3}\,p(p+1)[p(2p+1)^2,(p-1)(2p-1)^2,\ldots,50,9](1-n_1^2)^{p-1}$$

which is found equivalent to

$$P(2,3) = \frac{1}{3}[p^2(2p+1)^2,(p-1)^2(2p-1)^2,\ldots,9,0](2p+2)(1-n_1^2)^p$$

if both formulas are expanded using the binomial theorem. Consequently, the original equality is true for $k = p+1$.

1.3 Calculation of atmospheric integrals P(i,j) and Q(i).

The concept of the atmospheric integrals was introduced in an earlier work (Saastamoinen, 1972-73), where formulas for some of the integrals $P(i,j)$ have been derived on the basis of equations

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Table 1-1. Calculation of the multiplier of integral P(2,3) in $n_1^{2p+2} Y_{p+1}$.

$(2p-1)^2(p-1)$,	$(2p-3)^2(p-2)$,....................,9,	0
$(2p-1)(p-1)$,	$(2p-3)(p-2)$,.....................,3,	0
$2(2p-1)^2$,	$2(2p-3)^2$,......................,18,	2
$2(2p-1)$,	$2(2p-3)$,........................,6,	2
$2p-1$,	$2p-3$,...........................,3,	1
1,	1,...............................,1,	1
$\frac{1}{3}(2p-1)(p-1)$,	$\frac{1}{3}(2p-3)(p-2)$,...................,1,	0
$\frac{2}{3}(2p-1)^2(p-1)$,	$\frac{2}{3}(2p-3)^2(p-2)$,..................,6,	0
$\frac{1}{3}(2p-1)^2(p-1)^2$,	$\frac{1}{3}(2p-3)^2(p-2)^2$,.................,3,	0
0,	$\frac{1}{3}(2p-1)^2(p-1)^2(\frac{1}{p-1})$,.....,$\frac{100}{3}(\frac{p-2}{2!})$,	$3(\frac{p-1}{1})$
Sum: $\{\frac{1}{3}p(p+1)(2p+1)^2$,	$\frac{1}{3}(p-1)(p+1)(2p-1)^2$,.......,$\frac{50}{3}(p+1)$,	$3(p+1)\}$ P(0,1)

$$n - 1 = (n_a - 1)\left(\frac{T}{T_a}\right)^{-\frac{g}{R\beta} - 1} \qquad (\beta \neq 0) \qquad (13a)$$

$$n - 1 = (n_a - 1)e^{-\frac{g}{RT}(r - r_a)} \qquad (\beta = 0) \qquad (13b)$$

in an atmosphere consisting of two layers, the troposphere and the stratosphere. We shall no longer restrict the number of layers that may be taken into the atmospheric model, otherwise retaining the various assumptions implicit in equations (13a) and (13b).

In the derivation of a general formula for integrals P(i,j) it is best to take either

$$H = RT/g$$

or

$$h = r - r_1$$

as the independent variable in terms of which all the other quantities are expressed. Setting for brevity

$$m' = -\frac{g}{R\beta} - 1$$

we have then, for $\beta \neq 0$,

$$n - 1 = (n_a - 1)H_a^{-m'}H^{m'} = (n_a - 1)(m' + 1)^{-m'}H_a^{-m'}[h_a + (m'+1)H_a - h]^{m'}$$

$$dn = (n_a - 1)m'H_a^{-m'}H^{m'-1}dH = -(n_a - 1)m'(m'+1)^{-m'}H_a^{-m'}[h_a + (m'+1)H_a - h]^{m'-1}dh$$

$$r - r_1 = h_a + (m'+1)(H_a - H) = h_b - (m'+1)(H - H_b) = h$$

and

$$r_1^j P(i-1,j) = (n_a - 1)^i m' H_a^{-im'} \int_{H_b}^{H_a} [h_a + (m'+1)H_a - (m'+1)H]^j H^{im'-1} dH \qquad (i \neq 0)$$

$$= (n_a - 1)^i m'(m'+1)^{-im'} H_a^{-im'} \int_{h_a}^{h_b} [h_a + (m'+1)H_a - h]^{im'-1} h^j dh$$

The solution of either integral is given by the formula

$$r_1^j P(i-1,j) = \int_{n_b}^{n_a} (n-1)^{i-1}(r - r_1)^j dn \qquad (i \neq 0)$$

$$= \frac{j!}{0!i^j}(im')_j(H_a^j A_i - H_b^j B_i) + \frac{j!}{1!i^{j-1}}(im')_{j-1}(H_a^{j-1}h_a A_i - H_b^{j-1}h_b B_i)$$

$$+ \frac{j!}{2!i^{j-2}}(im')_{j-2}(H_a^{j-2}h_a^2 A_i - H_b^{j-2}h_b^2 B_i)$$

$$+ \ldots + \frac{j}{i}(im')_1(H_a h_a^{j-1}A_i - H_b h_b^{j-1}B_i) + h_a^j A_i - h_b^j B_i \qquad (14)$$

where

$$A_i = \frac{1}{i}(n_a - 1)^i \qquad B_i = \frac{1}{i}(n_b - 1)^i$$

and

$$\frac{im'+i}{im'+1} = (im')_1, \quad \left(\frac{im'+i}{im'+2}\right)(im')_1 = (im')_2, \quad \ldots, \quad \left(\frac{im'+i}{im'+j}\right)(im')_{j-1} = (im')_j .$$

Because

$$\lim_{\beta \to 0} (im')_j = 1,$$

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it is evident that equation (14) is also valid if $\beta = 0$, in which case the factors containing m' are simply left out.

For integrals Q(i), the immediate solution is

$$Q(i) = \int_{n_b}^{n_a} \frac{1}{n}(n_1 - n)^i dn$$

$$= n_1^i \log_e(n_a/n_b) - in_1^{i-1}(n_a - n_b) + \frac{1}{2}\frac{i(i-1)}{2!} n_1^{i-2}(n_a^2 - n_b^2)$$
$$- \frac{1}{3}\frac{i(i-1)(i-2)}{3!} n_1^{i-3}(n_a^3 - n_b^3) + \ldots(-)^i \frac{1}{i}(n_a^i - n_b^i).$$

This expression may be transformed, setting

$$A = n_a - n_b \qquad C = \tfrac{1}{2}(n_a + n_b)$$

and

$$\log_e(n_a/n_b) = (A/C) + \frac{1}{2^2.3}(A/C)^3 + \frac{1}{2^4.5}(A/C)^5 + \ldots,$$

into the following formula suitable for numerical evaluation,

$$Q(i) = (n_1 - C)^i (A/C)$$
$$+ \frac{1}{2^2.3}\left[n_1^2 + (i-2)n_1 C + \frac{1}{2!}(i-1)(i-2)C^2\right](n_1 - C)^{i-2}(A/C)^3$$
$$+ \frac{1}{2^4.5}\left[n_1^4 + (i-4)n_1^3 C + \frac{1}{2!}(i-3)(i-4)n_1^2 C^2 + \frac{1}{3!}(i-2)(i-3)(i-4)n_1 C^3\right.$$
$$\left. + \frac{1}{4!}(i-1)(i-2)(i-3)(i-4)C^4\right](n_1 - C)^{i-4}(A/C)^5$$
$$+ \frac{1}{2^6.7}\left[n_1^6 + (i-6)n_1^5 C + \ldots + \frac{1}{6!}(i-1)(i-2)\ldots(i-6)C^6\right](n_1 - C)^{i-6}(A/C)^7$$
$$+ \ldots, \qquad (15)$$

where i may take any value (i = 0, 1, 2, ...).

1.4 Extension of the range of the refraction formula.

We shall now go back to the binomial expansion for tan z, and replace equation (3) by

$$\tan z = f \tan z_1 [1 + f^2(y^2-1)\sec^2 z_1 - (1-f^2)]^{-\frac{1}{2}}$$
$$= f \tan z_1 \left\{1 + (-)^k \sum_{k=1}^{\infty} \frac{1.3.5\ldots(2k-1)}{2.4.6\ldots(2k)}[f^2(y^2-1)\sec^2 z_1 - (1-f^2)]^k\right\} \qquad (16)$$

where f stands for a positive number less than 1. This series is convergent if $|f^2(y^2-1)\sec^2 z - (1-f^2)| < 1$; consequently, it can be made convergent for any zenith distance $z_1 < 90°$ by taking f sufficiently small.

The idea behind this arrangement is to truncate series (16), after k+1 terms, whereupon it is permissible to restore the original order of terms in ascending powers of y^2-1. The contribution to the astronomical refraction of a layer of air between radii vectors r_a and r_b then becomes

$$\Delta z_1(a;b) = \tan z_1\Big[F_0Y_0 - \frac{1}{2}F_1Y_1\sec^2 z_1 + \frac{3}{8}F_2Y_2\sec^4 z_1 - \frac{5}{16}F_3Y_3\sec^6 z_1 + \ldots(-)^k\frac{1.3\ldots(2k-1)}{2.4\ldots(2k)}F_kY_k\sec^{2k}z_1\Big] + R_{k+1} \qquad (17)$$

where

$$\begin{aligned}
F_0 &= f\{1+\tfrac{1}{2}(1-f^2) + \ldots + \frac{1.3\ldots(2k-1)}{2.4\ldots(2k)}(1-f^2)^k\},\\
F_1 &= f^3\{1+\tfrac{3}{2}(1-f^2) + \ldots + \frac{3.5\ldots(2k-1)}{2.4\ldots(2k-2)}(1-f^2)^{k-1}\},\\
F_2 &= f^5\{1+\tfrac{5}{2}(1-f^2) + \ldots + \frac{5.7\ldots(2k-1)}{2.4\ldots(2k-4)}(1-f^2)^{k-2}\},\\
&\ldots\ldots\ldots, \qquad\qquad (18)\\
F_{k-1} &= f^{2k-1}\{1 + \frac{2k-1}{2}(1-f^2)\},\\
F_k &= f^{2k+1}.
\end{aligned}$$

Formula (17) is valid if

$$f^2 < \frac{2}{(y_b{}^2-1)\sec^2 z_1 + 1}$$

but in order to keep remainder R_{k+1} small, without the necessity of including an excessive number of terms, f should be chosen so that

$$\left|f^2 - \frac{1}{(y^2-1)\sec^2 z_1 + 1}\right| = \min. \qquad (19)$$

considering all the values of y and z_1 involved.

With given numerical values of f and k, equations (18) provide a set of <u>damping factors</u>, F_0, F_1, ..., F_k, by which the coefficients of the first k+1 terms of refraction formula (4) are multiplied.

Upon truncation of series (16), the remaining terms of tan z are

$$(-)^{k+1}\frac{1.3.5\ldots(2k+1)}{2.4.6\ldots(2k+2)} f\tan z_1\Big[1 - \left(\frac{2k+3}{2k+4}\right)V + \frac{(2k+3)(2k+5)}{(2k+4)(2k+6)}V^2 - \frac{(2k+3)(2k+5)(2k+7)}{(2k+4)(2k+6)(2k+8)}V^3 + \ldots\Big]V^{k+1}$$

where

$$V = f^2(y^2-1)\sec^2 z_1 - (1-f^2).$$

This gives the approximate formula

$$R_{k+1} \sim (-)^{k+1} \frac{1.3.5\ldots(2k+1)}{2.4.6\ldots(2k+2)} f \tan z_1 \int_{n_b}^{n_a} \frac{V^{k+1}}{1 + \frac{2k+3}{2k+4} V} dn \qquad (20)$$

with

$$dn = -\left(\frac{m'}{m'+1}\right)\left(\frac{n-1}{H}\right) dr$$

for the evaluation of the magnitude of remainder R_{k+1} by numerical integration. This evaluation is necessary because the (k+1)th term of formula (17), unlike that of equation (4), does not give an indication of the accuracy achieved.

If $\tan z_1$ is numerically large, we may choose to substitute $f \cos z_1$ for f in equation (16), and consider the series

$$\begin{aligned} \tan z &= f \sin z_1 [1 + f^2(y^2 - 1) - (1 - f^2\cos^2 z_1)]^{-\frac{1}{2}} \\ &= f \sin z_1 \{1 + (-)^k \sum_{k=1}^{\infty} \frac{1.3.5\ldots(2k-1)}{2.4.6\ldots(2k)} [f^2(y^2-1) - (1 - f^2\cos^2 z_1)]^k\} \end{aligned} \qquad (16')$$

where f now stands for a positive number greater than 1. The contribution to the astronomical refraction of a layer of air between radii vectors r_a and r_b then becomes

$$\Delta z_1(a;b) = \sin z_1 \left[F_0 Y_0 - \frac{1}{2} F_1 Y_1 + \frac{3}{8} F_2 Y_2 - \frac{5}{16} F_3 Y_3 + \ldots (-)^k \frac{1.3.5\ldots(2k-1)}{2.4.6\ldots(2k)} F_k Y_k\right] + R_{k+1} \qquad (17')$$

where the damping factors

$$\begin{aligned} F_0 &= f\{1 + \tfrac{1}{2}(1-f^2\cos^2 z_1) + \ldots + \frac{1.3\ldots(2k-1)}{2.4\ldots(2k)}(1 - f^2\cos^2 z_1)^k\} \\ F_1 &= f^3\{1 + \tfrac{3}{2}(1-f^2\cos^2 z_1) + \ldots + \frac{3.5\ldots(2k-1)}{2.4\ldots(2k-2)}(1 - f^2\cos^2 z_1)^{k-1}\} \\ F_2 &= f^5\{1 + \tfrac{5}{2}(1-f^2\cos^2 z_1) + \ldots + \frac{5.7\ldots(2k-1)}{2.4\ldots(2k-4)}(1 - f^2\cos^2 z_1)^{k-2}\} \\ &\ldots\ldots\ldots\ldots \\ F_{k-1} &= f^{2k-1}\{1 + \frac{2k-1}{2}(1 - f^2\cos^2 z_1)\} \\ F_k &= f^{2k+1} \end{aligned} \qquad (18')$$

may be written out as polynomials of $\cos^2 z_1$. Series (16') is convergent if

$$f^2 < \frac{2}{y_b{}^2 - \sin^2 z_1}$$

but again, f should be chosen so that

$$|f^2 - \frac{1}{y^2 - \sin^2 z_1}| = \text{min.} \tag{19'}$$

for all the values of y and z_1 involved. Remainder R_{k+1} is evaluated by numerical integration using the formulas

$$W = f^2(y^2 - 1) - (1 - f^2\cos^2 z_1)$$

$$dn = -\left(\frac{m'}{m'+1}\right)\left(\frac{n-1}{H}\right)dr$$

and

$$R_{k+1} \sim (-)^{k+1} \frac{1.3.5\ldots(2k+1)}{2.4.6\ldots(2k+2)} f \sin z_1 \int_{n_b}^{n_a} \frac{W^{k+1}}{1 + \frac{2k+3}{2k+4} W} dn. \tag{20'}$$

If $z_1 = 90°$, equation (17) can not be applied unless the point of observation is moved along the light ray to the base (r_a) of a higher layer, where

$$\tan z_a = \{[n_a r_a/(n_1 r_1 \sin z_1)]^2 - 1\}^{-\frac{1}{2}}$$

has a suitable value. The refraction component for the nearly horizontal section of the light ray may be calculated using the series expansions in 1.5.

1.5 Calculation of astronomical refraction at or near the horizon.

We shall now consider the integral

$$\Delta z_1(a;b) = \sin z_a \int_{\log_e n_b}^{\log_e n_a} (\eta^2 - \sin^2 z_a)^{-\frac{1}{2}} d\log_e n \tag{21}$$

where

$$\eta = nr/(n_a r_a)$$

in the calculation of astronomical refraction for any zenith distance $0° \leq z_a \leq 90°$, especially for $z_a = 90°$. In the solution given below, the integrand will be expressed in the form of a convergent series, and integrated term by term.

Let

$$x = (r - r_a)/H_a$$

be chosen as the independent variable. We have then, from equation (13a),

$$(n - 1)/(n_a - 1) = (1 - \frac{x}{m'+1})^{m'}$$

$$= 1 - m_1 x + m_1 m_2 x^2 - m_1 m_2 m_3 x^3 + \ldots$$

and

$$n/n_a = 1 - a_0 m_1 x + a_0 m_1 m_2 x^2 - a_0 m_1 m_2 m_3 x^3 + \ldots \quad (22)$$

with

$$a_0 = (n_a - 1)/n_a; \quad m_j = \frac{1}{j} - \frac{1}{m'+1} \quad (\text{or } m_j = \frac{1}{j}, \text{ if } \beta = 0).$$

The derivative of series (22)

$$\frac{dn}{dx}/n_a = - a_0 m_1 + 2a_0 m_1 m_2 x - 3a_0 m_1 m_2 m_3 x^2 + \ldots$$

divided by the series itself gives

$$d\log_e n = - a_0 m_1 [1 - (2m_2 - a_0 m_1)x + (3m_2 m_3 - 3a_0 m_1 m_2 + a_0{}^2 m_1{}^2)x^2$$

$$- (4m_2 m_3 m_4 - 4a_0 m_1 m_2 m_3 - 2a_0 m_1 m_2{}^2 + 4a_0{}^2 m_1{}^2 m_2 - a_0{}^3 m_1{}^3)x^3 + \ldots]dx. \quad (23)$$

Multiplication of series (22) by the binom

$$r/r_a = 1 + (H_a/r_a)x = 1 + b_0 x$$

further gives

$$\eta = 1 + (b_0 - a_0 m_1)x + a_0 m_1 (m_2 - b_0)x^2 - a_0 m_1 m_2 (m_3 - b_0)x^3 + \ldots$$

whence

$$\eta^2 - \sin^2 z_a = \cos^2 z_a + 2(b_0 - a_0 m_1)x + 2(a_0 m_1 m_2 - 2a_0 m_1 b_0$$

$$+ \frac{1}{2} a_0{}^2 m_1{}^2 + \frac{1}{2} b_0{}^2)x^2 - 2a_0 m_1 (m_2 m_3 - 2b_0 m_2 + a_0 m_1 m_2 - a_0 m_1 b_0 + b_0{}^2)x^3 + \ldots$$

$$= \cos^2 z_a + c_1 x + c_2 x^2 - c_3 x^3 + \ldots \quad (24)$$

The remaining part of the calculation consists of the extraction of the inverse square root of series (24), multiplication by differential (23), and integration of the product according to equation (21).

If $z_a = 90°$, we have $\sin^2 z_a = 1$, $\cos^2 z_a = 0$

$$\eta^2 - 1 = c_1 x(1 + \frac{c_2}{c_1} x - \frac{c_3}{c_1} x^2 + \ldots) = c_1 x(1 + X)$$

$$(\eta^2 - 1)^{-\frac{1}{2}} = \frac{1}{\sqrt{c_1}} x^{-\frac{1}{2}}(1 - \frac{1}{2} X + \frac{3}{8} X^2 - \frac{5}{16} X^3 + \ldots) \qquad (X < 1)$$

The integrand is obtained in the form

$$(n^2 - 1)^{-\frac{1}{2}} d\log_e n = - \frac{a_0 m_1}{\sqrt{c_1}} x^{-\frac{1}{2}}(1 - d_1 x + d_2 x^2 - \ldots)dx$$

with

$$d_1 = 2m_2 - a_0 m_1 + \frac{1}{2}(c_2/c_1)$$

$$d_2 = 3m_2(m_3 - a_0 m_1) + \frac{1}{2c_1}(2c_2 m_2 + c_3 - a_0 c_2 m_1) + \frac{3}{8}(c_2/c_1)^2 + a_0{}^2 m_1{}^2$$

....

which gives

$$\Delta z_1(a;b) = \frac{2a_0 m_1}{\sqrt{c_1}} \sqrt{x_b}(1 - \frac{1}{3} d_1 x_b + \frac{1}{5} d_2 x_b{}^2 - \ldots) \quad (25)$$

as the final result.

If $\cos^2 z_a$ is numerically small, the extraction of the inverse square root succeeds similarly if we first find (by iteration) a number, x_0, such that

$$\cos^2 z_a - c_1 x_0 + c_2 x_0{}^2 + c_3 x_0{}^3 + \ldots = 0$$

and substitute a new variable $w = x + x_0$ that eliminates the constant term in equation (24). This procedure, which is equivalent to extending the light ray to its lowest point where the tangent line is horizontal, also finds application in the calculation of refraction if $z_a > 90°$.

(to be continued)

REFERENCES

Saastamoinen, J.: 1972-73, "Contributions to the theory of atmospheric refraction", Bulletin Géodésique, Nos. 105-107

Teleki, G.: (edited by) "The present state and future of the astronomical refraction investigations". Proceedings of the Study Group on Astronomical Refraction of the International Astronomical Union Commission 8, Belgrade, 1974.

APPENDIX A

Numerical applications

A few illustrative samples are given on the calculation of astronomical refraction in a spherically symmetric model (Atmospheric Model No. 2, (Saastamoinen, 1972-73)) specified as follows:

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$r_1 = 6380$ km $T_1 = 285.08$ K $n_1 = 1.000280868$

Troposphere 0 - 10.4 km: $\beta = -6.45$ K.km^{-1} $R/g = (2.8704/98)$ km.K^{-1}

Stratosphere 10.4 - ∞ km: $\beta = 0$ $R/g = (2.8704/98)$ km.K^{-1}

The diminution of the refractive index with height is given by equations (13a) and (13b); for the purpose of calculation, the specified values are assumed to be exact.

1. Coefficients Y_k.

Layer:	0 - 10.4 km	10.4 - 24 km	24 - ∞ km	Binomial multiplier
Y_0	39".614630	16".134229	2".176190	1
10^2Y_1	4.768940	6.821834	1.956960	-1/2
10^4Y_2	0.834121	3.067383	1.847449	3/8
10^6Y_3	0.170238	1.466399	1.854364	-5/16
10^8Y_4	0.037758	0.742198	2.007800	35/128
$10^{10}Y_5$	0.008818	0.394937	2.379461	-63/256
$10^{12}Y_6$	0.002132	0.219156	3.124625	231/1024
$10^{14}Y_7$	0.000529	0.125831	4.581027	-429/2048
$10^{16}Y_8$	0.000133	0.074246	7.511767	6435/32768
$10^{18}Y_9$	0.000034	0.044772	13.731743	-12155/65536

2. Examples of formulas for astronomical refraction.

$s = 10^{-2}\sec^2 z_1$

Zenith distances $0^\circ \le z_1 \le 80^\circ$. $(f = 1)$

$$\Delta z_1 = \tan z_1(57''.92505 - 6''.77387s + 2''.15586s^2 - 1''.09094s^3 + 0''.76228s^4 - 0''.68493s^5 + 0''.75479s^6 - 0''.98607s^7 + 1''.48977s^8 - 2''.55514s^9).$$

max $R_{10} \sim 0''.0003$

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Zenith distances $0° \le z_1 \le 82°$. (f = 0.9)

$$\Delta z_1 = \tan z_1(57''\!.92505 - 6''\!.77386s + 2''\!.15580s^2 - 1''\!.09052s^3 + 0''\!.75966s^4 - 0''\!.67073s^5 + 0''\!.68851s^6 - 0''\!.72595s^7 + 0''\!.64970s^8 - 0''\!.34516s^9).$$

max $R_{10} \sim 0''\!.0001$

Zenith distances $82° \le z_1 \le 84°$. (f = 0.9, 0 - 10.4 km; f = 0.75, 10.4 - ∞ km)

$$\Delta z_1 = \tan z_1(57''\!.92398 - 6''\!.76680s + 2''\!.13081s^2 - 1''\!.02570s^3 + 0''\!.61974s^4 - 0''\!.40724s^5 + 0''\!.25657s^6 - 0''\!.13688s^7 + 0''\!.05285s^8 - 0''\!.01080s^9).$$

max $R_{10} \sim 0''\!.0001$

Zenith distances $84° \le z_1 \le 86°$. (f = 0.9, 0 - 10.4 km; f = 0.75, 10.4 - 24 km; f = 0.56, 24 - ∞ km)

$$\Delta z_1 = \tan z_1(57''\!.909480 - 6''\!.696487s + 1''\!.962573s^2 - 0''\!.758515s^3 + 0''\!.3019568s^4 - 0''\!.1082585s^5 + 0''\!.03197350s^6 - 0''\!.007131299s^7 + 0''\!.001056695s^8 - 0''\!.000077801s^9).$$

max $R_{10} \sim 0''\!.001$

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3. Contributions of the atmospheric integrals to the coefficients

Integral	k = 0	k = 1	k = 2	k = 3	k = 4
Q(0)	57″.92504 93				
P(0,1)		7″.57784 80	-0″.63824 36	0″.04479 67	-0″.00293 46
Q(1)		-0.81327 43			
P'(0,2)		0.00921 64	2.76257 89	-0.38785 11	0.03811 30
Q(2)		0.00007 61	0.02283 64		
P'(0,3)			0.00823 12	1.36993 73	-0.26934 02
P(1,2)		0.00000 04	0.00045 03	0.05619 56	-0.01105 00
Q(3)			-0.00000 48	-0.00080 15	
P'(0,4)			0.00000 84	0.00835 36	0.97252 94
P'(1,3)			0.00000 06	0.00028 54	0.02490 88
Q(4)				0.00000 03	0.00003 15
P'(0,5)				0.00002 10	0.00979 73
P'(1,4)				0.00000 06	0.00016 61
P(2,3)					0.00001 09
Q(5)					
P'(0.6)					0.00004 42
P'(1,5)					0.00000 06
P'(2,4)					
Q(6)					
P(0.7)					0.00000 01
P'(1,6)					
P'(2,5)					
P(3,4)					
P(0,8)					
P(1,7)					
P(2,6)					
P'(3,5)					
P(0,9)					
P(1,8)					
P(2,7)					
P(3,6)					
P(4,5)					
P(0,10)					
P(1,9)					
P(2,8)					
P(0,11)					
P(1,10)					
P(0,12)					
P(0,13)					
Totals	57″.92504 93	6″.77386 66	2″.15585 74	1″.09093 79	0″.76227 71

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in the formula for astronomical refraction.

k = 5	k = 6	k = 7	k = 8	k = 9	Integral
					Q(0)
0".00018 54	-0".00001 14	0".00000 07			P(0,1)
					Q(1)
-0.00321 01	0.00024 78	-0.00001 81	0".00000 13	-0".00000 01	P'(0,2)
					Q(2)
0.03403 25	-0.00350 36	0.00031 97	-0.00002 69	0.00000 21	P'(0,3)
0.00139 63	-0.00014 38	0.00001 31	-0.00000 11	0.00000 01	P(1,2)
					Q(3)
-0.24594 10	0.03798 70	-0.00462 21	0.00048 67	-0.00004 65	P'(0,4)
-0.00630 09	0.00097 33	-0.00011 84	0.00001 25	-0.00000 12	P'(1,3)
					Q(4)
0.87915 29	-0.27188 64	0.04963 91	-0.00696 97	0.00083 17	P'(0,5)
0.01146 23	-0.00354 63	0.00064 76	-0.00009 09	0.00001 09	P'(1,4)
0.00069 95	-0.00021 64	0.00003 95	-0.00000 55	0.00000 07	P(2,3)
-0.00000 13					Q(5)
0.01324 24	0.96729 38	-0.35378 42	0.07454 60	-0.01186 38	P'(0,6)
0.00011 41	0.00657 48	-0.00240 61	0.00050 71	-0.00008 07	P'(1,5)
0.00000 64	0.00033 15	-0.00012 13	0.00002 56	-0.00000 41	P'(2,4)
	0.00000 01				Q(6)
0.00009 29	0.02039 33	1.25605 98	-0.53052 56	0.12672 79	P(0,7)
0.00000 06	0.00008 78	0.00436 42	-0.00184 46	0.00044 07	P'(1,6)
	0.00000 34	0.00015 02	-0.00006 35	0.00001 52	P'(2,5)
	0.00000 02	0.00001 06	-0.00000 45	0.00000 11	P(3,4)
0.00000 04	0.00020 43	0.03533 45	1.88095 44	-0.90128 51	P(0,8)
	0.00000 06	0.00007 53	0.00329 85	-0.00158 19	P(1,7)
		0.00000 20	0.00008 03	-0.00003 85	P(2,6)
		0.00000 01	0.00000 52	-0.00000 25	P'(3,5)
	0.00000 12	0.00047 80	0.06810 53	3.19123 05	P(0,9)
		0.00000 06	0.00007 17	0.00280 77	P(1,8)
			0.00000 14	0.00004 74	P(2,7)
			0.00000 01	0.00000 23	P(3,6)
				0.00000 01	P(4,5)
		0.00000 40	0.00119 47	0.14461 76	P(0,10)
			0.00000 08	0.00007 51	P(1,9)
				0.00000 10	P(2,8)
			0.00001 32	0.00319 04	P(0,11)
				0.00000 10	P(1,10)
			0.00000 01	0.00004 47	P(0,12)
				0.00000 04	P(0,13)
0".68493 24	0".75479 12	0".98606 88	1".48977 26	2".55514 42	Totals

4. Sample calculation of R_{k+1}.

($k = 9$; $z_1 = 85°$)

h, km	$-10^6 \frac{dn}{dr}$	$10^2(y^2 - 1)$	V	$-0''.4154(\frac{dn}{dr})f\ V^{10}/(1 + \frac{21}{22}V)$	
0	27.28	0.0000	-0.1900	$0''.764 \times 10^{-6}$	
2.6	22.34	0.0686	-0.1168	0.004	
5.2	18.07	0.1397	-0.0410	0.000	f = 0.9
7.8	14.39	0.2128	0.0369	0.000	
10.4	11.28	0.2878	0.1168	0.002	
10.4	13.90	0.2878	-0.2244	1.786	
13.8	8.163	0.3871	-0.1508	0.018	
17.2	4.793	0.4896	-0.0749	0.000	f = 0.75
20.6	2.814	0.5939	0.0023	0.000	
24	1.652	0.6993	0.0804	0.000	
24	1.6523	0.6993	-0.3977	61.31	
36	0.2523	1.075	-0.242	0.05	
48	0.0385	1.453	-0.086	0.00	f = 0.56
60	0.0059	1.833	0.070	0.00	
72	0.0009	2.212	0.227	0.00	

Integrals; 0 – 10.4 km: $0''.5$; 10.4 – 24 km: $2''$; 24 – 72 km: $195''$ $\} \times 10^{-6}$

$R_{10} \sim 0''.0002$

5. Astronomical refraction in the troposphere at $z_1 = 90°$.

$$x = 0.11976\,14110\,h \qquad \text{(h in km)}$$

$$\Delta z_1(r_1;r) = 2020''.53687\sqrt{x}\,(1 - 0.21827\,814\,x + 0.03226\,0119\,x^2$$

$$- 0.00286\,4275\,x^3 + 0.00021\,7379\,x^4 - 0.00002\,3471\,x^5 + 0.00000\,2731\,x^6$$

$$- 0.00000\,0321\,x^7 + 0.00000\,0040\,x^8 - 0.00000\,0005\,x^9).$$

(Total $1743''.330$).

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APPENDIX B

Coefficients Y_k

1. Polynomial multipliers of integrals P(i,j) in $n_1^{2k} Y_k$.

$$k(k-1)\ldots(k-p+1)(k-p)(P_n(i,j)(1-n_1^2)^{k-i-j}P(i,j)$$

where

$$p = \tfrac{1}{2}(j-2) \text{ if } j \text{ is even, or}$$

$$p = \tfrac{1}{2}(j-1) \text{ if } j \text{ is odd.}$$

$$P_n(0,1) = 2$$

$$P_n(0,2) = (2k-1) - n_1^2$$

$$\frac{3}{2} P_n(0,3) = (2k-1) - 3n_1^2$$

$$P_n(1,2) = (2k-1)^2 - (8k-6)n_1^2 + n_1^4$$

$$6\, P_n(0,4) = (2k-1)(2k-3) - 6(2k-3)n_1^2 + 3n_1^4$$

$$\frac{3}{2} P_n(1,3) = (2k-1)^2 - 2(8k-7)n_1^2 + 9n_1^4$$

$$15\, P_n(0,5) = (2k-1)(2k-3) - 5(4k-6)n_1^2 + 15n_1^4$$

$$6\, P_n(1,4) = (2k-1)^2(2k-3) - (2k-3)(26k-25)n_1^2 + 3(22k-35)n_1^4 - 9n_1^6$$

$$\frac{3}{2} P_n(2,3) = (k-1)(2k-1)^2 - (28k^2-58k+31)n_1^2 + (41k-55)n_1^4 - 9n_1^6$$

$$6\, P_n(0,6) = \frac{1}{15}(2k-1)(2k-3)(2k-5) - (2k-3)(2k-5)n_1^2 + 3(2k-5)n_1^4 - n_1^6$$

$$15\, P_n(1,5) = (2k-1)^2(2k-3) - (2k-3)(38k-39)n_1^2 + 5(34k-57)n_1^4 - 75n_1^6$$

$$2\, P_r(2,4) = \frac{1}{3}(k-1)(2k-1)^2(2k-3) - (2k-3)(14k^2-31k+18)n_1^2$$
$$+ (74k^2-229k+180)n_1^4 - (47k-82)n_1^6 + 3n_1^8$$

$$3\, P_n(0,7) = \frac{1}{105}(2k-1)(2k-3)(2k-5) - \frac{1}{5}(2k-3)(2k-5)n_1^2 + (2k-5)n_1^4 - n_1^6$$

$$3\, P_n(1,6) = \frac{1}{30}(2k-1)^2(2k-3)(2k-5) - \frac{2}{15}(2k-3)(2k-5)(13k-14)n_1^2$$
$$+ (6k-15)(4k-7)n_1^4 - (22k-56)n_1^6 + 25n_1^8$$

$$5\, P_n(2,5) = \frac{1}{3}(k-1)(2k-1)^2(2k-3) - \frac{1}{3}(2k-3)(58k^2-135k+83)n_1^2$$
$$+ (158k^2-521k+441)n_1^4 - 5(39k-77)n_1^6 + 50n_1^8$$

$$3\, P_n(3,4) = \frac{1}{6}(k-1)(2k-1)^2(2k-3)^2 - \frac{1}{2}(2k-3)^2(20k^2-48k+31)n_1{}^2 + (172k^3 -852k^2+1433k-819)n_1{}^4-(232k^2-866k+819)n_1{}^6 + \frac{3}{2}(49k-93)n_1{}^8 - \frac{3}{2}n_1{}^{10}$$

$$6\, P_n(0,8) = \frac{1}{420}(2k-1)(2k-3)(2k-5)(2k-7) - \frac{1}{15}(2k-3)(2k-5)(2k-7)n_1{}^2 + \frac{1}{2}(2k-5)(2k-7)n_1{}^4 - (2k-7)n_1{}^6 + \frac{1}{4}n_1{}^8$$

$$105\, P_n(1,7) = \frac{1}{3}(2k-1)^2(2k-3)(2k-5) - \frac{4}{3}(2k-3)(2k-5)(17k-19)n_1{}^2 + 14(2k-5)(16k-29)n_1{}^4 - 20(35k-91)n_1{}^6 + 245n_1{}^8$$

$$90\, P_n(2,6) = (k-1)(2k-1)^2(2k-3)(2k-5) - 2(2k-3)(2k-5)(38k^2-92k+59)n_1{}^2 + 10(2k-5)(88k^2-305k+273)n_1{}^4 - 60(59k^2-276k+322)n_1{}^6 + 15(137k-361)n_1{}^8 - 150n_1{}^{10}$$

$$15\, P_n(3,5) = \frac{1}{3}(k-1)(2k-1)^2(2k-3)^2 - \frac{2}{3}(2k-3)^2(40k^2-102k+71)n_1{}^2 + 2(332k^3-1740k^2+3121k-1917)n_1{}^4 - 4(374k^2-1543k+1638)n_1{}^6 + 25(43k-99)n_1{}^8 - 150n_1{}^{10}$$

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DISCUSSION

J. Saastamoinen: replied to a question from Garfinkel, that in the refraction calculations we need atmosphere models with at least 5-6 layers. Two layers models are not enough.

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