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Geometric Study of Minkowski Differences of Plane Convex Bodies

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Abstract. In the Euclidean plane $\mathbb{R}^2$, we define the Minkowski difference $\mathcal{K}-\mathcal{L}$ of two arbitrary convex bodies $\mathcal{K}$, $\mathcal{L}$ as a rectifiable closed curve $\mathcal{H}_h \subset \mathbb{R}^2$ that is determined by the difference $h = h_{\mathcal{K}} - h_{\mathcal{L}}$ of their support functions. This curve $\mathcal{H}_h$ is called the hedgehog with support function h. More generally, the object of hedgehog theory is to study the Brunn–Minkowski theory in the vector space of Minkowski differences of arbitrary convex bodies of Euclidean space $\mathbb{R}^{n+1}$, defined as (possibly singular and self-intersecting) hypersurfaces of $\mathbb{R}^{n+1}$. Hedgehog theory is useful for: (i) studying convex bodies by splitting them into a sum in order to reveal their structure; (ii) converting analytical problems into geometrical ones by considering certain real functions as support functions. The purpose of this paper is to give a detailed study of plane hedgehogs, which constitute the basis of the theory. In particular: (i) we study their length measures and solve the extension of the Christoffel–Minkowski problem to plane hedgehogs; (ii) we characterize support functions of plane convex bodies among support functions of plane hedgehogs and support functions of plane hedgehogs among continuous functions; (iii) we study the mixed area of hedgehogs in $\mathbb{R}^2$ and give an extension of the classical Minkowski inequality (and thus of the isoperimetric inequality) to hedgehogs.

1 Introduction

The set $\mathcal{K}^{n+1}$ of convex bodies of $(n+1)$-Euclidean vector space $\mathbb{R}^{n+1}$ is usually equipped with Minkowski addition and multiplication by non-negative real numbers which are respectively defined by:

$$\forall (K,L) \in (\mathcal{K}^{n+1})^2,\ K+L = \{u+v \mid u \in K, v \in L\};$$
$$\forall \lambda \in \mathbb{R}_+, \forall K \in \mathcal{K}^{n+1},\ \lambda.K = \{\lambda u \mid u \in K\}.$$

Of course, $(\mathcal{K}^{n+1}, +, .)$ does not constitute a vector space since we cannot subtract convex bodies in $\mathcal{K}^{n+1}$. Now, in the same way as we construct the group of integers from the set of natural numbers, we can construct the real vector space $(\mathcal{H}^{n+1}, +, .)$ of formal differences of convex bodies of $\mathbb{R}^{n+1}$ from $(\mathcal{K}^{n+1}, +, .)$. Then the so-called theory of hedgehogs consists in considering $\mathcal{K}^{n+1}$ as a convex cone of this vector space $(\mathcal{H}^{n+1}, +, .)$. More precisely, it consists in: (i) considering each formal difference of convex bodies of $\mathbb{R}^{n+1}$ as a (possibly singular and self-intersecting) hypersurface of $\mathbb{R}^{n+1}$ called a *hedgehog*; (ii) extending the mixed volume $V\colon (\mathcal{K}^{n+1})^{n+1} \to \mathbb{R}$ to a symmetric $(n+1)$-linear form on $\mathcal{H}^{n+1}$; (iii) considering the Brunn–Minkowski theory in $\mathcal{H}^{n+1}$. For $n \leq 2$, this idea goes back to a paper by H. Geppert [3] who introduced hedgehogs under the German names *stützbare Bereiche* $(n = 1)$ and *stützbare Flächen* $(n = 2)$, in an attempt to extend certain parts of the Brunn–Minkowski theory. Later and independently, R. Langevin, G. Levitt and H. Rosenberg [7] gave a

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study of hedgehogs with a C^2 support function in which hedgehogs are seen as *envelopes parametrized by their Gauss map*. The author has been working at developing this hedgehog theory since 1985 [8–22].

The relevance of this theory can be illustrated by the following two principles: (1) to study convex bodies by splitting them judiciously (that is, according to the problem under consideration) into a sum of hedgehogs in order to reveal their structure; (2) to convert analytical problems into geometrical ones by considering certain real functions on the unit sphere $\mathbb{S}^n$ of $\mathbb{R}^{n+1}$ as support functions of hedgehogs (or of "multi-hedgehogs" [7]). The first principle permitted us to invalidate an old conjectured characterization of the 2-sphere [15] and the second one to give a geometrical proof of the Sturm–Hurwitz theorem [21]. The reader will find a short introduction to the theory in [17]. For an elementary survey of hedgehogs with a C^2 support function, we refer the reader to [19].

In hedgehog theory, plane hedgehogs (*i.e.,* Minkowski differences of convex bodies of $\mathbb{R}^2$) play an important role for at least three reasons: (1) the theory is of course easier in dimension 2 than in higher dimensions, so that it is often convenient to consider the case of plane hedgehogs first; (2) general hedgehogs (*i.e.,* Minkowski differences of arbitrary convex bodies of $\mathbb{R}^{n+1}$) are defined by induction on the dimension, replacing *support sets* by *support hedgehogs* (see below), so that it is often necessary to consider first the case of plane hedgehogs; (3) a classical type of proof in the theory consists in proceeding by induction on the dimension by means of orthogonal projections, see for instance [8, 10, 13, 15, 16].

The purpose of this paper is to give a detailed study of hedgehogs of $\mathbb{R}^2$, defined geometrically as Minkowski differences of arbitrary convex bodies of $\mathbb{R}^2$.

In Section 2, we shall define geometrically general hedgehogs of $\mathbb{R}^{n+1}$ as Minkowski differences of arbitrary convex bodies of $\mathbb{R}^{n+1}$. This definition makes clear that a good understanding of plane hedgehogs is a prerequisite to a study of general hedgehogs of $\mathbb{R}^{n+1}$. However, Section 2 may be omitted in a first reading.

In Section 3, we shall see that plane hedgehogs can always be seen as rectifiable closed curves having exactly one co-oriented support line in each direction. Every $h \in C^2(\mathbb{S}^1; \mathbb{R})$ defines a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ that can be seen as the envelope of the family of co-oriented lines with equation $\langle x, u\rangle = h(u)$, $(u \in \mathbb{S}^1)$. Note that it is still possible to associate such an envelope $\mathcal{H}_h$ to every $h \in C^1(\mathbb{S}^1; \mathbb{R})$ but it generally cannot be interpreted as a difference of two plane convex bodies when h is only of class C^1. Such "hedgehogs" can even be fractal curves, that is, nowhere differentiable curves of infinite length, as we shall recall at the end of the section.

In Section 4, we shall study length measures and areas of hedgehogs of $\mathbb{R}^2$. The area measure of order 1 of a convex body $\mathcal{K} \subset \mathbb{R}^2$ (that is, the length measure of its boundary $\partial\mathcal{K}$) is defined as a (positive) Borel measure $S_1(\mathcal{K}, \,.\,)$ on $\mathbb{S}^1$, see [24, §4.3]. Recall that S_1 is Minkowski linear, that is $S_1(\lambda\mathcal{K}+\mu\mathcal{L}, \,.\,) = \lambda S_1(\mathcal{K}, \,.\,)+\mu S_1(\mathcal{L}, \,.\,)$, for all $(\mathcal{K}, \mathcal{L}) \in (\mathcal{K}^2)^2$ and $(\lambda, \mu) \in \mathbb{R}_+^2$. By considering $\mathcal{K}^2$ as a convex cone of $\mathcal{H}^2$, this notion of length measure can be extended by linearity to hedgehogs of $\mathbb{R}^2$; the length measure of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ is then a (possibly signed) Borel measure l_h on $\mathbb{S}^1$. In the same way, the area of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ will be defined as an algebraic area in order to extend the mixed area $A\colon \mathcal{K}^2 \times \mathcal{K}^2 \to \mathbb{R}$, $(\mathcal{K}, \mathcal{L}) \longmapsto A(\mathcal{K}, \mathcal{L})$ to a symmetric bilinear form $a\colon \mathcal{H}^2 \times \mathcal{H}^2 \to \mathbb{R}$. Of course, (algebraic) length measures and (algebraic) areas will be interpreted and studied from a geometrical point of view.

On the way, we shall prove that plane hedgehogs are determined up to translations by their length measure (which solves the Christoffel–Minkowski problem for plane hedgehogs). Moreover, we shall characterize support functions of plane convex bodies among support functions of plane hedgehogs and support functions of plane hedgehogs among continuous functions.

In Section 5, we shall give an extension of the classical Minkowski inequality (and thus of the isoperimetric inequality) to hedgehogs.

In Section 6, we shall notice that hedgehogs of $\mathbb{R}^2$ do not only constitute a real vector space $(\mathcal{H}^2, +, .)$ but also a commutative and associative $\mathbb{R}$-algebra $(\mathcal{H}^2, +, ., *)$, where $*$ is given by the convolution product of support functions. This convolution product was introduced and studied by H. Görtler in [4, 5].

2 Genesis of Hedgehogs as Differences of Arbitrary Convex Bodies

Let us see how we can define geometrically general hedgehogs of $\mathbb{R}^{n+1}$ as differences of arbitrary convex bodies. To this aim, let us recall some basic facts about convex bodies of $\mathbb{R}^{n+1}$. (1) Every convex body $\mathcal{K} \subset \mathbb{R}^{n+1}$ is determined by its support function $h_{\mathcal{K}}\colon \mathbb{R}^{n+1} \to \mathbb{R}, u \mapsto \max\{\langle x, u\rangle \mid x \in \mathcal{K}\}$ (which is convex and thus continuous on $\mathbb{R}^{n+1}$); note that $h_{\mathcal{K}}(u)$ can be interpreted as the signed distance of the support hyperplane with outer normal vector u from the origin ($h_{\mathcal{K}}(u) < 0$ if and only if u points into the open halfspace containing the origin). (2) For $u \in \mathbb{S}^n$, the support set of $\mathcal{K}$ with normal vector u is defined as the intersection of $\mathcal{K}$ with its support hyperplane with normal vector u, that is by $\mathcal{K}_u = \{x \in \mathcal{K} \mid \langle x, u\rangle = h_{\mathcal{K}}(u)\}$. Note that $\mathcal{K}_u = \{h_{\mathcal{K}}(u)u\} + K_u$, where K_u is the convex body of the n-dimensional subspace of $\mathbb{R}^{n+1}$ that is orthogonal to u, say $u^{\perp}$, with support function $h'_{\mathcal{K}}(u; v) = \lim_{t\downarrow 0}[h_{\mathcal{K}}(u + tv) - h_{\mathcal{K}}(u)]/t$. (3) The boundary of $\mathcal{K}$ is constituted of the union of all its support sets $\mathcal{K}_u$, where $u \in \mathbb{S}^n$. For more details, we refer the reader to the book by R. Schneider [24].

Our definition of general hedgehogs is based on the three following remarks. (1) In $\mathbb{R}$, every convex body $\mathcal{K}$ is determined by its support function as the segment $[-h_{\mathcal{K}}(-1), h_{\mathcal{K}}(1)]$, where $-h_{\mathcal{K}}(-1) \leq h_{\mathcal{K}}(1)$, so that the difference $\mathcal{K} - \mathcal{L}$ of two convex bodies $\mathcal{K}, \mathcal{L}$ can be defined as an oriented segment of $\mathbb{R}$: $\mathcal{K} - \mathcal{L} := [-(h_{\mathcal{K}} - h_{\mathcal{L}})(-1), (h_{\mathcal{K}} - h_{\mathcal{L}})(1)]$. (2) If $\mathcal{K}$ and $\mathcal{L}$ are two convex bodies of $\mathbb{R}^{n+1}$, then for all $u \in \mathbb{S}^n$, their support sets $\mathcal{K}_u$ and $\mathcal{L}_u$ can be identified with convex bodies K_u and L_u of the n-dimensional Euclidean vector space $u^{\perp} \simeq \mathbb{R}^n$. (3) Addition of two convex bodies $\mathcal{K}, \mathcal{L} \subset \mathbb{R}^{n+1}$ corresponds to that of their support sets with same normal vector: $(\mathcal{K}+\mathcal{L})_u = \mathcal{K}_u + \mathcal{L}_u$ for all $u \in \mathbb{S}^n$; therefore, the difference $\mathcal{K} - \mathcal{L}$ of two convex bodies $\mathcal{K}, \mathcal{L} \subset \mathbb{R}^{n+1}$ must be defined in such a way that $(\mathcal{K} - \mathcal{L})_u = \mathcal{K}_u - \mathcal{L}_u$ for all $u \in \mathbb{S}^n$.

A natural way of defining geometrically general hedgehogs as differences of arbitrary convex bodies is therefore to proceed by induction on the dimension by extending the notion of *support set with normal vector u* to a notion of *support hedgehog with normal vector u*.

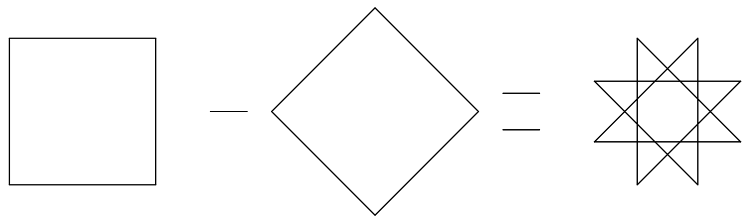

Figure 1

Notations and Definitions

Let $\mathcal{K}_{n+1}$ be the set of support functions of convex bodies of $\mathbb{R}^{n+1}$ and let $\mathcal{H}_{n+1}$ be the subspace spanned by $\mathcal{K}_{n+1}$ in the real vector space of continuous real functions on $\mathbb{R}^{n+1}$. (Note that $\mathcal{H}_{n+1}$ can be identified with the real vector space of formal differences of convex bodies of $\mathbb{R}^{n+1}$.) To any $h \in \mathcal{H}_{n+1}$ we associate a geometric realization by induction on $n+1 \in \mathbb{N}$. For any $h \in \mathcal{H}_1$, we define the hedgehog with support function h as the oriented segment $\mathcal{H}_h = [-h(-1), h(1)]$. Given $n \geq 1$, we assume that the notion of hedgehog with support function $h \in \mathcal{H}_k$ has already been defined for $k = n$. Then for any $h \in \mathcal{H}_{n+1}$, we associate to each $u \in \mathbb{S}^n$ the hedgehog H_h^u of $u^\perp$ with support function $h'(u; v) = \lim_{t \downarrow 0}[h(u+tv) - h(u)]/t$. We denote by $\mathcal{H}_h$ and call the *hedgehog with support function* h, the datum of all the support hedgehogs $\mathcal{H}_h^u = \{h(u)u\} + H_h^u$, where $u \in \mathbb{S}^n$.

3 Construction of Plane Hedgehogs From Their Support Function

As we saw in Section 2, (1) the set $\mathcal{K}_2$ of support functions of convex bodies of $\mathbb{R}^2$ spans a subspace $\mathcal{H}_2$ in the vector space of continuous real functions on $\mathbb{R}^2$: $\mathcal{H}_2 = \{h_{\mathcal{K}} - h_{\mathcal{L}} \mid (h_{\mathcal{K}}, h_{\mathcal{L}}) \in \mathcal{K}_2 \times \mathcal{K}_2\}$. (2) Each $h = h_{\mathcal{K}} - h_{\mathcal{L}} \in \mathcal{H}_2$ is the *support function* of a hedgehog $\mathcal{H}_h$, which can be seen as the geometrical realization of the formal difference $\mathcal{K} - \mathcal{L}$. (3) This hedgehog $\mathcal{H}_h$ is obtained by associating to each $u \in \mathbb{S}^1$, the oriented segment $\sigma_h(u) = \{h(u)u\} + [-h'(u; -u^\top), h'(u; u^\top)]$, where $u^\top \in \mathbb{S}^1$ is the unit vector such that $(u, u^\top)$ is a direct orthonormal frame of $\mathbb{R}^2$, which we assume equipped with the standard orientation. Let us illustrate this definition of plane hedgehogs as Minkowski differences of plane convex bodies by a very simple example.

Example 3.1 Let $\mathcal{K}$ and $\mathcal{L}$ be the convex bodies of $\mathbb{R}^2$ with support function $h_{\mathcal{K}}(x) = |\langle x, e_1\rangle| + |\langle x, e_2\rangle|$ and $h_{\mathcal{L}}(x) = |\langle x, e_3\rangle| + |\langle x, e_4\rangle|$, where $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^2$, (e_1, e_2) the canonical basis of $\mathbb{R}^2$ and $e_3, e_4 \in \mathbb{R}^2$ the vectors defined by $e_3 = \frac{1}{\sqrt{2}}(e_1 + e_2)$ and $e_4 = \frac{1}{\sqrt{2}}(e_1 - e_2)$. These convex bodies are two squares whose formal difference $\mathcal{K} - \mathcal{L}$ can be realized geometrically as the hedgehog with support function $h = h_{\mathcal{K}} - h_{\mathcal{L}}$ as represented in Figure 1.

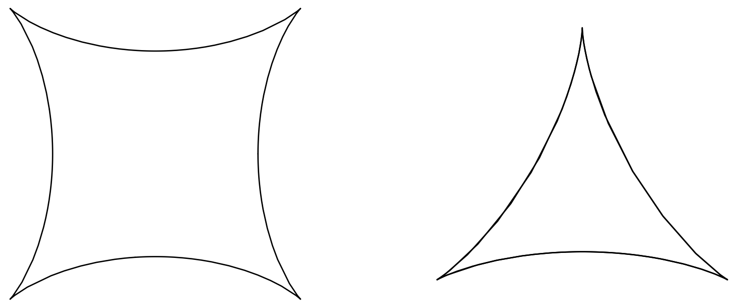

Figure 2

In this section, we shall reformulate this construction of plane hedgehogs and we shall observe that they can be seen as oriented rectifiable closed curves of $\mathbb{R}^2$. But before that, let us recall some basic facts on the particular case of hedgehogs whose support function is of class C^2 on the unit circle $\mathbb{S}^1$, see for instance [13].

Let $h\colon \mathbb{R}^2 \to \mathbb{R}$ be a function whose restriction to $\mathbb{S}^1$ is of class C^2. Then,

(i) The function h defines a plane hedgehog $\mathcal{H}_h$, that is $h \in \mathcal{H}_2$.

(ii) This hedgehog $\mathcal{H}_h$ can be seen as the envelope of a family of co-oriented lines having exactly one co-oriented support line with a given unit normal vector (see Figure 2(a), which shows the case where $h(x_1, x_2) = x_1^2 - x_2^2$). More precisely, the hedgehog $\mathcal{H}_h$ is the envelope of the family of lines with equation $\langle x, u\rangle = h(u)$, where $u \in \mathbb{S}^1$.

(iii) This envelope $\mathcal{H}_h$ admits a natural parametrization, namely the map $x_h\colon \mathbb{S}^1 \to \mathbb{R}^2$, $u(\theta) = (\cos\theta, \sin\theta) \mapsto x_h(\theta) = p(\theta)\, u(\theta) + p'(\theta)\, u'(\theta)$, where $p(\theta) = h(u(\theta))$. Indeed, $x_h(\theta)$ is the unique solution of the system

$$\langle x, u(\theta)\rangle = p(\theta),$$
$$\langle x, u'(\theta)\rangle = p'(\theta),$$

for all $\theta \in J = [0, 2\pi[$. Note that x_h can be interpreted as the inverse of the Gauss map in this sense that at each regular point $x_h(u)$ of $\mathcal{H}_h$, u is a normal vector to $\mathcal{H}_h$.

(iv) The algebraic area of $\mathcal{H}_h$ can be defined as the integral

$$a(h) = \int_{\mathbb{R}^2 - \mathcal{H}_h} i_h(x)\, d\mathcal{L}(x),$$

where $i_h(x)$ is the winding number of $\mathcal{H}_h$ around x and $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}^2$. The winding number $i_h(x)$ can also be seen as the algebraic intersection number of almost every oriented half-line with origin x with $\mathcal{H}_h$ equipped with its transverse orientation (this number is independent of the oriented half-line for an open dense set of directions).

Let us recall that a convex body $\mathcal{K} \subset \mathbb{R}^{n+1}$ is said to be of class C^2_+ if its boundary $\partial\mathcal{K}$ is a C^2-hypersurface of $\mathbb{R}^{n+1}$ with positive Gauss curvature. Every hedgehog $\mathcal{H}_h \subset \mathbb{R}^{n+1}$ with support function $h \in C^2(\mathbb{S}^n; \mathbb{R})$ can be seen as a difference of two convex bodies of class C^2_+ since $h + r$ is the support function of a convex body of class C^2_+ for every sufficiently large positive real number r.

Example 3.2 If $p(\theta) = \sin 3\theta$ (see Figure 2(b)) the algebraic area of $\mathcal{H}_h$ is equal to -4π, *i.e.,* to -2 area(D), where D is the domain delimited by $\mathcal{H}_h$. The minus sign comes from the fact that D is concave at the regular points of its boundary and the factor 2 from the fact that the parametrization describes the curve twice.

Let us recall how the boundary of a plane convex body $\mathcal{K} \subset \mathbb{R}^2$ is determined by its support function $h = h_{\mathcal{K}} : \mathbb{R}^2 \to \mathbb{R}$. Its boundary, say $\partial\mathcal{K}$, is constituted of the union of all its support sets $\mathcal{K}_u = \{x \in \mathcal{K} \mid \langle x, u\rangle = h_{\mathcal{K}}(u)\}$, where $u \in \mathbb{S}^1$:

$$\partial\mathcal{K} = \bigcup_{u \in \mathbb{S}^1} \mathcal{K}_u.$$

For every $u \in \mathbb{S}^1$, the convexity of $h \colon \mathbb{R}^2 \to \mathbb{R}$ implies the existence of

$$h'(u; v) = \lim_{t \downarrow 0} \frac{h(u + tv) - h(u)}{t},$$

for all $v \in \mathbb{R}^2$ (*cf.* [24, Remark 1.5.3]) and $h'(u; \cdot)$ is the support function of the support set with normal vector u [24, Theorem 1.7.2], so that

$$\mathcal{K}_u = \{x \in \mathbb{R}^2 \mid \forall v \in \mathbb{R}^2, \langle x, v\rangle \leq h'(u; v)\}.$$

Now, let us see the support function h as a 2π-periodic function p on $\mathbb{R}$: $p(\theta) = h(u(\theta))$, where $u(\theta) = (\cos\theta, \sin\theta)$. The left derivative p'_l and the right derivative p'_r of p are everywhere defined on $\mathbb{R}$ and they are respectively given by $p'_l(\theta) = -h'(u(\theta); -u'(\theta))$ and $p'_r(\theta) = h'(u(\theta); u'(\theta))$. The calculation gives

$$h'(u(\theta); \alpha\, u(\theta) + \beta\, u'(\theta)) = \alpha\, p(\theta) + \varepsilon\, \beta\, h'(u(\theta); \varepsilon\, u'(\theta)),$$

where $\varepsilon = \operatorname{sgn}(\beta)$. Thus $\forall \theta \in J = [0, 2\pi[$,

$$\begin{aligned} \mathcal{K}_{u(\theta)} &= \{x \in \mathbb{R}^2 \mid \forall v \in \mathbb{R}^2, \langle x, v\rangle \leq h'(u(\theta); v)\} \\ &= \{x \in \mathbb{R}^2 \mid \exists\, t \in [p'_l(\theta), p'_r(\theta)], x = p(\theta)u(\theta) + t\, u'(\theta)\}. \end{aligned}$$

Therefore, the boundary of $\mathcal{K}$ is constituted of the union of all the segments $\sigma_h(\theta) = [x_h^-(\theta), x_h^+(\theta)]$, $(\theta \in J = [0, 2\pi[)$, where $x_h^-(\theta) = p(\theta)u(\theta) + p'_l(\theta)u'(\theta)$ and $x_h^+(\theta) = p(\theta)u(\theta) + p'_r(\theta)u'(\theta)$:

$$\partial\mathcal{K} = \bigcup_{\theta \in J} \sigma_h(\theta).$$

Recall that this boundary $\partial\mathcal{K}$ is a rectifiable simple closed curve. Let us begin by proving that for any $h \in \mathcal{H}_2$, the segments $\sigma_h(\theta) = [x_h^-(\theta), x_h^+(\theta)]$, $(\theta \in J)$, are well defined and still constitute a rectifiable (but not necessarily simple) closed curve. The proof is based on the following proposition.

Proposition 3.3 *For every $h \in \mathcal{H}_2$, the following four properties are satisfied:*

(i) *the function $p = h \circ u$, where $u(\theta) = (\cos\theta, \sin\theta)$, is Lipschitzian on $\mathbb{R}$;*

(ii) *the function p admits a left derivative p_l' (resp., a right derivative p_r') that is continuous from the left (resp., from the right) on $\mathbb{R}$, and we have*

$$\forall\theta \in \mathbb{R},\ p_l'(\theta) = \lim_{\alpha\to\theta^-} p_r'(\alpha) \quad \textit{and} \quad p_r'(\theta) = \lim_{\alpha\to\theta^+} p_l'(\alpha);$$

(iii) *the family $((p_r' - p_l')(\theta))_{\theta\in J=[0,2\pi[}$ is absolutely summable;*

(iv) *the functions p_l' and p_r' are of bounded variation on $I = [0, 2\pi]$.*

Sketch of the Proof It is sufficient to check the result for $h \in \mathcal{K}_2$. Let us continue with the function $h = h_{\mathcal{K}}$ and with the notations we introduced above.

(i) Property (i) follows from the convexity of h, *cf.* [24, Theorem 1.5.1].

(ii) As $h_\theta(t) = h(u(\theta) + t\,u'(\theta))$ is convex on $\mathbb{R}$, h_θ admits a left derivative $(h_\theta)_l'$ that is continuous from the left and a right derivative $(h_\theta)_r'$ that is continuous from the right and such that $(h_\theta)_l' \le (h_\theta)_r'$, *cf.* [24, Theorem 1.5.2]. Property (ii) follows by expressing $(h_\theta)_l'$ and $(h_\theta)_r'$ in terms of the functions p, p_l' and p_r'.

(iii) Property (iii) results from the fact that ∂K is a rectifiable simple closed curve that is constituted of the union of all the segments $\sigma_h(\theta) = [x_h^-(\theta), x_h^+(\theta)]$, $(\theta \in J = [0, 2\pi[)$, the relative interiors of which are pairwise disjoint.

(iv) As for every increasing sequence $(\theta_i)_{i\ge 0}$ of I, the segments $\sigma_h(\theta_i)$ are placed in the increasing order of subscripts on the anticlockwise oriented curve ∂K, it also follows that x_h^- and x_h^+ are of bounded variation on I. Using (i), we then deduce that the maps $p_l'u'$ and $p_r'u'$ are also of bounded variation on I. Noting that the functions p_l' and p_r' are necessarily bounded, we at last deduce Property (iv). ∎

Given any function h in $\mathcal{H}_2$, let us consider the union of all the segments $\sigma_h(\theta) = [x_h^-(\theta), x_h^+(\theta)]$, $(\theta \in J = [0, 2\pi[)$. These segments, which are well defined from property (ii), make up the image of the map

$$x_h\colon D_{x_h} \to \mathbb{R}^2,\ (\theta, t) \mapsto p(\theta)u(\theta) + tu'(\theta),$$

where $u(\theta) = (\cos\theta, \sin\theta)$ and $p(\theta) = h(\cos\theta, \sin\theta)$ for all $\theta \in \mathbb{R}$ and where $D_{x_h} = \{(\theta, t) \in J \times \mathbb{R} \mid (t - p_l'(\theta))(t - p_r'(\theta)) \le 0\} \cup \{(2\pi, p_l'(2\pi))\}$. Note that the points $A = (0, p_l'(0))$ and $B = (2\pi, p_l'(2\pi))$ satisfy $x_h(A) = x_h(B)$.

Let us show how $x_h(D_{x_h})$ can be seen as a closed curve of $\mathbb{R}^2$. To this aim, let us equip the set D_{x_h} with the metric defined by

$$d((\theta_1, t_1), (\theta_2, t_2)) = \begin{cases} |t_1 - t_2| & \text{if } \theta_1 = \theta_2, \\ \theta_j - \theta_i + |p_r'(\theta_i) - t_i| + s(\theta_i, \theta_j) + |t_j - p_l'(\theta_j)| & \text{if } \theta_i < \theta_j, \end{cases}$$

where $s(\theta_i, \theta_j) = \sum_{\theta_i<\alpha<\theta_j} |(p_r' - p_l')(\alpha)|$, *cf.* property (iii).

Let us observe that this metric is such that

(i) the map $d_h\colon D_{x_h} \to \mathbb{R}, M \mapsto d(A, M)$ is an isometry from D_{x_h} onto $I_h = [0, F_h]$, where $F_h = d_h(B) = 2\pi + \sum_{\theta\in J} |(p_r' - p_l')(\theta)|$ (there is no particular difficulty in proving this point);

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(ii) The map x_h is continuous on D_{x_h} (reduce this point to the continuity of $D_{x_h} \to \mathbb{R}, (\theta, t) \mapsto t$ and make use of property (ii)).

This allows us to define the map

$$\gamma_h = x_h \circ d_h^{-1} : I_h \to \mathbb{R}^2, \lambda \mapsto x_h(\theta(\lambda), t(\lambda)),$$

where $(\theta(\lambda), t(\lambda)) = d_h^{-1}(\lambda)$, and to assert that it is continuous and such that $\gamma_h(I_h) = x_h(D_{x_h})$. The curve of $\mathbb{R}^2$ that it defines is closed since $x_h(A) = x_h(B)$ and we can state the following.

Theorem 3.4 *For every $h \in \mathcal{H}_2$, the map $\gamma_h \colon I_h \to \mathbb{R}^2$ defines a closed curve of $\mathbb{R}^2$ whose geometric realization $\gamma_h(I_h)$ is the union of the segments $\sigma_h(\theta) = [x_h^-(\theta), x_h^+(\theta)]$, ($\theta \in J = [0, 2\pi[$).*

Definition 3.5 For every $h \in \mathcal{H}_2$, the closed curve of $\mathbb{R}^2$ that is defined by $\gamma_h \colon I_h \to \mathbb{R}^2$ is denoted by $\mathcal{H}_h$ and called hedgehog with support function h. Regular parts of $\mathcal{H}_h$ are assumed to be equipped with the transverse orientation for which the unit normal at $\gamma_h(\lambda) = x_h(\theta(\lambda), t(\lambda))$ is $u(\theta(\lambda))$. For all $\theta \in J$, the oriented segment $\sigma_h(\theta) = [x_h^-(\theta), x_h^+(\theta)]$ is called support hedgehog of $\mathcal{H}_h$ in direction $u(\theta)$.

Theorem 3.6 *For every $h \in \mathcal{H}_2$, the hedgehog $\mathcal{H}_h$ is a rectifiable curve of $\mathbb{R}^2$. Let us denote its length by $L(h)$. This length satisfies the inequality*

$$L(h) \geq \int_0^{F_h} \|\gamma_h'(\lambda)\| \, d\lambda,$$

where $\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^2$.

Proof Given any partition $\sigma = (\lambda_0, \dots, \lambda_n)$ of $I_h = [0, F_h]$, let $L(\sigma, h)$ denote the sum

$$\sum_{i=0}^{n-1} \|\gamma_h(\lambda_{i+1}) - \gamma_h(\lambda_i)\| = \sum_{i=0}^{n-1} \|x_h(\theta(\lambda_{i+1}), t(\lambda_{i+1})) - x_h(\theta(\lambda_i), t(\lambda_i))\|.$$

Recall that the length of $\mathcal{H}_h$ can be defined by

$$L(h) = \sup_{\sigma} L(\sigma, h),$$

where the supremum is taken over all partitions of I_h. By definition, the curve $\mathcal{H}_h$ is rectifiable if and only if it has a finite length (which means analytically that the components of $\gamma_h \colon I_h \to \mathbb{R}^2$ are functions of bounded variation on I_h). In this case, the derivative γ_h' exists almost everywhere on I_h and the inequality

$$L(h) \geq \int_0^{F_h} \|\gamma_h'(\lambda)\| \, d\lambda$$

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holds. Consequently, it suffices to prove the existence of some real constant C such that $L(\sigma, h) \leq C$ for every partition σ of I_h. Put $u(\theta) = (\cos\theta, \sin\theta)$ and $p(\theta) = h(u(\theta))$ for all $\theta \in \mathbb{R}$. Writing $p(\alpha)u(\alpha) - p(\beta)u(\beta)$ under the form $p(\alpha)(u(\alpha) - u(\beta)) + (p(\alpha) - p(\beta))u(\beta)$ and noting that $(\theta(\lambda_i))_{i=0}^{n}$ is an increasing sequence of I, we get at once

$$\text{(A)} \qquad \sum_{i=0}^{n-1} \|p(\theta(\lambda_{i+1}))u(\theta(\lambda_{i+1})) - p(\theta(\lambda_i))u(\theta(\lambda_i))\| \leq 2\pi(p_h + q_h),$$

where $p_h = \sup_{\theta\in I} |p(\theta)|$ and where q_h is the best Lipschitz constant of $p\colon \mathbb{R} \to \mathbb{R}$. Similarly, we get

$$\text{(B)} \qquad \sum_{i=0}^{n-1} \|t(\lambda_{i+1})u'(\theta(\lambda_{i+1})) - t(\lambda_i)u'(\theta(\lambda_i))\| \leq 2\pi t_h + \sum_{i=0}^{n-1} |t(\lambda_{i+1}) - t(\lambda_i)|,$$

where $t_h = \sup(\{|t| \mid \exists\theta \in I, (\theta, t) \in D_{x_h}\})$. Besides, using properties (iii) and (iv) we get

$$s_h := \sup_{\sigma} \sum_{i=0}^{n-1} |t(\lambda_{i+1}) - t(\lambda_i)| < +\infty,$$

where the supremum is taken over all partitions of I_h. As $L(\sigma, h)$ is less than or equal to the sum of left-hand sides of inequalities (A) or (B), we now conclude that we can take $C = 2\pi(p_h + q_h + t_h) + s_h$. ■

Remark 3.7 For every $h \in \mathcal{H}_2$, let $X_h\colon [0, L(h)] \to \mathcal{H}_h \subset \mathbb{R}^2$, $s \mapsto X_h(s) = (X_h^1(s), X_h^2(s))$ be the parametrization by arclength of $\mathcal{H}_h$ (for which the length of every subarc $X_h\colon [0, L] \to \mathcal{H}_h$ $(0 \leq L \leq L(h))$ is equal to L).This parametrization $X_h = (X_h^1, X_h^2)$ is such that X_h^1 and X_h^2 are absolutely continuous (and thus almost everywhere differentiable) on $[0, L(h)]$, and there exists an increasing map $s \mapsto \theta_s$ of $[0, L(h)]$ into J such that, for almost every $s \in [0, L(h)]$, $X_h(s) \in \sigma_h(\theta_s)$ and $X_h'(s) = \varepsilon_h(s)u'(\theta_s)$, where $\varepsilon_h(s) \in \{-1, 1\}$.

Note that the length of $\mathcal{H}_h$ can be interpreted in terms of the 1-dimensional (outer) Hausdorff measure Λ_1 in $\mathbb{R}^2$:

$$L(h) = \int_{\mathbb{R}^2} n_h(x)\, d\Lambda_1(x)$$

where $n_h(x)$ is the number of $\lambda \in [0, F_h]$ such that $\gamma_h(\lambda) = x$, see [23, pp. 125–126].

Remark 3.8 In definitions and results following Proposition 3.3, we can replace $\mathcal{H}_2$ by the linear subspace consisting of all functions of $C(\mathbb{S}^1; \mathbb{R})$ that satisfy properties (i)–(iv) . But we shall see later that this subspace is nothing but $\mathcal{H}_2$.

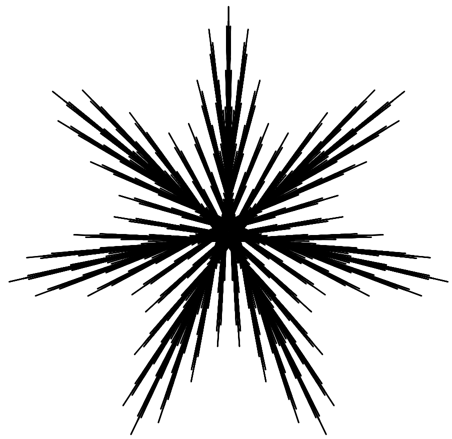

Figure 3

Remark 3.9 If $h\colon \mathbb{R}^2 \to \mathbb{R}$ is a function whose restriction to $\mathbb{S}^1$ is of class C^1, then we can still define $\mathcal{H}_h$ as the envelope of the family of lines with equation $\langle x, u(\theta)\rangle = p(\theta)$, where $p(\theta) = h(u(\theta))$. Moreover, this envelope $\mathcal{H}_h$ can still be parametrized by the map $x_h\colon \mathbb{S}^1 \to \mathbb{R}^2$, $u(\theta) \mapsto p(\theta)u(\theta)+p'(\theta)\, u'(\theta)$. Indeed, $x_h(\theta) = p(\theta)u(\theta)+p'(\theta)u'(\theta)$ is still the unique solution of the system

$$\langle x, u(\theta)\rangle = p(\theta), \langle x, u'(\theta)\rangle = p'(\theta),$$

for all $\theta \in J$. But in general, this envelope $\mathcal{H}_h$ does not represent the difference of two plane convex bodies. In fact, such a hedgehog can even be a fractal curve.

Theorem 3.10 ([14]) *If $p(\theta) = h(\cos\theta, \sin\theta)$ is a Möbius function of the form*

$$p(\theta) = \sum_{n=1}^{+\infty} \frac{\sin(\beta^n\theta)}{\alpha^n}.$$

where β is an odd natural number and α is a real number such that $\alpha > \beta$ and $\beta^2 > \alpha(1+\frac{3\pi}{2})$, then the envelope $\mathcal{H}_h$ is a continuous but nowhere differentiable curve whose length is infinite.

Figure 3 represents the fractal envelope $\mathcal{H}_h$ for $\alpha = 8$ and $\beta = 7$.

Proposition 3.11 *For every $h \in \mathcal{H}_2$, Proposition 3.3 ensures that $p = h \circ u$ admits a left derivative p_l' and a right derivative p_r' on $\mathbb{R}$. These left and right derivatives of p admit a common derivative at almost every $\theta \in \mathbb{R}$. We shall simply denote it by $p''(\theta)$.*

Proof From property (iv), p_l' and p_r' are of bounded variation on I and thus almost everywhere differentiable on $\mathbb{R}$. Now, property (i) ensures that p is Lipschitzian and thus almost everywhere differentiable on $\mathbb{R}$. Therefore, p_l' and p_r' coincide almost everywhere on $\mathbb{R}$, so that their derivatives must also coincide almost everywhere on $\mathbb{R}$. ∎

Proposition 3.12 *For every $h \in \mathcal{K}_2$, the function $p(\theta) = h(u(\theta))$ satisfies the two following properties:*

(v) $p_l'(\theta) \le p_r'(\theta)$ *for all* $\theta \in \mathbb{R}$;
(vi) $(p + p'')(\theta) \ge 0$ *for almost every* $\theta \in \mathbb{R}$.

This proposition is an immediate consequence of the following characterization, which is due to M. Kallay [6]: $\forall h \in C(\mathbb{S}^1;\mathbb{R})$,

$$(h \in \mathcal{K}_2) \Leftrightarrow \left(\forall \theta \in I, \forall \alpha \in \left[0, \frac{\pi}{2}\right], p(\theta+\alpha) + p(\theta-\alpha) \ge 2p(\theta)\cos\alpha\right),$$

where $p(\theta) = h(u(\theta))$. We just have to observe that

$$p''(\theta) = \lim_{\alpha \to 0} \frac{p(\theta+\alpha) + p(\theta-\alpha) - 2p(\theta)}{\alpha^2}.$$

As we shall see in Section 4 (*cf.* Theorem 4.12), properties (v) and (vi) do not characterize support functions of convex bodies in $\mathcal{H}_2$.

4 Length and Area Measures of Plane Hedgehogs

In Section 3, we saw that every formal difference of two convex bodies of $\mathbb{R}^2$ can be seen as a (transversely oriented) rectifiable curve, which we called a hedgehog. In this section, we introduce and study the notions of length measure and mixed area for hedgehogs. Whereas the length measure $L(\mathcal{C}, \cdot\,)$ of a convex curve $\mathcal{C} \subset \mathbb{R}^2$ is defined as a (positive) Borel measure on $\mathbb{S}^1$, the length measure of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ will be defined as a (possibly signed) Borel measure l_h on $\mathbb{S}^1$ in order that the map $\mathcal{H}_h \mapsto l_h$ be linear. This algebraic length measure will of course be interpreted and studied from a geometrical point of view. In the same way, the area of a hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ will be defined as an algebraic area in order to extend the mixed area $A\colon \mathcal{K}^2 \times \mathcal{K}^2 \to \mathbb{R}$, $(\mathcal{K}, \mathcal{L}) \mapsto A(\mathcal{K}, \mathcal{L})$ to a symmetric bilinear form $a\colon \mathcal{H}^2 \times \mathcal{H}^2 \to \mathbb{R}$. The area of $\mathcal{H}_h$ will be interpreted as the integral over $\mathbb{R}^2 - \mathcal{H}_h$ of the winding number $i_h(x)$ of $\mathcal{H}_h$ with respect to $x \in \mathbb{R}^2 - \mathcal{H}_h$. We shall see in Section 5 that this bilinear form satisfies a partial extension of the Minkowski inequality $A(\mathcal{K}, \mathcal{L}) \ge A(\mathcal{K})\,.\,A(\mathcal{L})$ which leads to a natural extension of the isoperimetric inequality to plane hedgehogs. On the way, we shall solve the Christoffel–Minkowski problem for plane hedgehogs by giving a necessary and sufficient condition for a (possibly signed) Borel measure on $\mathbb{S}^1$ to be the length measure of a hedgehog. Moreover, we shall characterize support functions of plane convex bodies among support functions of plane hedgehogs and support functions of plane hedgehogs among continuous functions.

Let us begin by recalling some basic facts concerning the area measure of order 1 of plane convex bodies, that is the length measure of plane convex curves. We shall use notations of Section 3 and $\mathcal{B}(\mathbb{S}^1)$ will denote the σ-algebra of Borel subsets of $\mathbb{S}^1$.

The area measure of order 1 of a convex body $\mathcal{K} \subset \mathbb{R}^2$ (that is, the length measure of its boundary $\partial\mathcal{K}$) is the (positive) Borel measure $S_1(\mathcal{K}, \,.\,)$ defined as follows:

(i) if $\mathcal{K}$ is contained in a line, then

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1), S_1(\mathcal{K}, \Omega) := \sum_{u \in \Omega} \text{Length}[\sigma_{h_K}(u)];$$

(ii) if $\mathcal{K}$ is not contained in a line, then

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1), S_1(\mathcal{K}, \Omega) := \Lambda_1 \Big[\bigcup_{u \in \Omega} \sigma_{h_K}(u) \Big],$$

where Λ_1 denotes the 1-dimensional (outer) Hausdorff measure in $\mathbb{R}^2$. This area measure of order 1 determines $\mathcal{K}$ up to a translation. More precisely (see [24, Theorems 4.3.1; 4.3.3]), we have the following existence and uniqueness result for plane convex bodies with prescribed area measure of order 1.

Theorem 4.1 *Let m be a (positive) Borel measure on $\mathbb{S}^1$. If m satisfies*

$$\text{(C)} \qquad \int_{\mathbb{S}^1} u \, dm(u) = 0_{\mathbb{R}^2},$$

then m is the length measure of a unique (up to translations) convex body of $\mathbb{R}^2$.

Recall that integral condition (C) is necessary from the translation invariance of the area of $\mathcal{K}$. Let us recall the following formula for the perimeter of a plane convex body.

Theorem 4.2 (Barbier 1860 [1]) *Let $\mathcal{K}$ be a convex body of $\mathbb{R}^2$. The perimeter of $\mathcal{K}$, that is the length $L(\partial\mathcal{K}) := S_1(\mathcal{K}, \mathbb{S}^1)$ of its boundary $\partial\mathcal{K}$, is given by*

$$L(\partial\mathcal{K}) = \int_0^{2\pi} p(\theta) \, d\theta,$$

where $p(\theta) = h_{\mathcal{K}}(\cos\theta, \sin\theta)$.

Remark 4.3 As is well known, if the restriction of $h = h_{\mathcal{K}}$ to $\mathbb{S}^1$ is of class C^2, then

(i) $\partial\mathcal{K}$ can be parametrized by

$$x_h \colon \mathbb{S}^1 \to \partial\mathcal{K} \subset \mathbb{R}^2, \ u(\theta) = (\cos\theta, \sin\theta) \mapsto x_h(\theta) = p(\theta)\, u(\theta) + p'(\theta) u'(\theta),$$

where $p(\theta) = h(u(\theta))$;

(ii) x_h is of class C^1 on $\mathbb{S}^1$ and we have $\forall \theta \in J, \ x_h'(\theta) = (p + p'')(\theta)\, u'(\theta)$;

(iii) $R_h \colon \mathbb{S}^1 \to \mathbb{R}, u(\theta) \mapsto R_h(\theta) := (p + p'')(\theta)$ is non-negative and $R_h(\theta)$ can be interpreted as the (principal) radius of curvature of $\mathcal{H}_h$ at $x_h(\theta)$. Therefore, in this case the length measure of $\partial\mathcal{K}$ is given by

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1), \ S_1(\mathcal{K}, \Omega) = \int_\Omega R_h \, d\sigma,$$

where σ is the circular Lebesgue measure on $\mathbb{S}^1$.

Algebraic Length Measure

Definition 4.4 Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^2$. For every $\Omega \in \mathcal{B}(\mathbb{S}^1)$, we put

$$l_h(\Omega) := S_1(\mathcal{K}, \Omega) - S_1(\mathcal{L}, \Omega),$$

where $\mathcal{K}$ and $\mathcal{L}$ are two convex bodies of $\mathbb{R}^2$ such that $h = h_{\mathcal{K}} - h_{\mathcal{L}}$, that is, two convex bodies of which $\mathcal{H}_h$ is the difference. As $h_{\mathcal{K}}$ and $S_1(\mathcal{K}, \cdot)$ depend linearly on $\mathcal{K} \in \mathcal{K}^2 \subset \mathcal{H}^2$, this definition does not depend on the choice of $(\mathcal{K}, \mathcal{L}) \in \mathcal{K}^2 \times \mathcal{K}^2$. This signed measure l_h is called the algebraic length measure of $\mathcal{H}_h$. Naturally, if h is the support function of a convex body $\mathcal{K} \in \mathcal{K}^2$, then $l_h(\cdot) = S_1(\mathcal{K}, \cdot)$.

Definition 4.5 Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^2$. The algebraic length of $\mathcal{H}_h$ is defined by $l(h) := l_h(\mathbb{S}^1)$. By the definition of l_h, it follows from Theorem 4.2 that $l(h)$ is given by

$$l(h) = \int_0^{2\pi} p(\theta)\, d\theta,$$

where $p(\theta) = h(\cos\theta, \sin\theta)$.

We shall see later in the section how l_h and $l(h)$ can be interpreted from a geometrical point of view. The following generalization of Theorem 4.1 solves the Christoffel–Minkowski problem for plane hedgehogs.

Theorem 4.6 *Let m be a (possibly signed) Borel measure on $\mathbb{S}^1$. If m satisfies*

$$\int_{\mathbb{S}^1} u\, dm(u) = 0_{\mathbb{R}^2},$$

then m is the (algebraic) length measure of a unique (up to translations) hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$.

Proof *Existence:* Let $m = m^+ - m^-$ be the Jordan decomposition of m:

$$m^+ = \frac{1}{2}(|m| + m) \quad \text{and} \quad m^- = \frac{1}{2}(|m| - m),$$

where $|m|$ is the total variation measure of m. From the assumption, there exists some $(a, b) \in \mathbb{R}^2$, such that:

$$\int_{\mathbb{S}^1} u\, dm^+(u) = \int_{\mathbb{S}^1} u\, dm^-(u) = (a, b).$$

Let m_f be the Borel measure given by $\forall\Omega \in \mathcal{B}(\mathbb{S}^1)$ $m_f(\Omega) = \int_\Omega f(u)\, d\sigma(u)$, where σ is the circular Lebesgue measure and $f(u(\theta)) := c - \frac{1}{\pi}(a\cos\theta + b\sin\theta)$, c being some constant larger than $\frac{1}{\pi}\sqrt{a^2 + b^2}$. The Borel measures defined by $\mu = m^+ + m_f$ and $\nu = m^- + m_f$ are positive and such that

$$m = \mu - \nu \quad \text{and} \quad \int_{\mathbb{S}^1} u\, d\mu(u) = \int_{\mathbb{S}^1} u\, d\nu(u) = 0_{\mathbb{R}^2}.$$

Thus, from Theorem 4.1, there exists $(\mathcal{K},\mathcal{L}) \in (\mathcal{K}^2)^2$, such that $\mu = S_1(\mathcal{K},\,\cdot\,)$ and $\nu = S_1(\mathcal{L},\,\cdot\,)$, so that

$$m = \mu - \nu = S_1(\mathcal{K},\,\cdot\,) - S_1(\mathcal{L},\,\cdot\,) = l_h,$$

where $h = h_{\mathcal{K}} - h_{\mathcal{L}}$.

Uniqueness up to translations: Assume that $\widetilde{h} = h_{\widetilde{\mathcal{K}}} - h_{\widetilde{\mathcal{L}}} \in \mathcal{H}_2$ is such that $l_h = l_{\widetilde{h}}$, where $(\widetilde{\mathcal{K}},\widetilde{\mathcal{L}}) \in (\mathcal{K}^2)^2$. We then have

$$S_1(\mathcal{K},\,\cdot\,) - S_1(\mathcal{L},\,\cdot\,) = S_1(\widetilde{\mathcal{K}},\,\cdot\,) - S_1(\widetilde{\mathcal{L}},\,\cdot\,).$$

As S_1 is Minkowski linear, we thus have

$$S_1(\widetilde{\mathcal{K}} + \mathcal{L},\,\cdot\,) = S_1(\widetilde{\mathcal{K}},\,\cdot\,) + S_1(\mathcal{L},\,\cdot\,) = S_1(\mathcal{K},\,\cdot\,) + S_1(\widetilde{\mathcal{L}},\,\cdot\,) = S_1(\mathcal{K}+\widetilde{\mathcal{L}},\,\cdot\,).$$

From Theorem 4.1, it follows that $\widetilde{\mathcal{K}}+\mathcal{L}$ and $\mathcal{K}+\widetilde{\mathcal{L}}$ are translates of each other, so that $h_{\widetilde{\mathcal{K}}+\mathcal{L}} - h_{\mathcal{K}+\widetilde{\mathcal{L}}}$ is a linear form ϕ on $\mathbb{R}^2$. As $h_{\mathcal{K}+\widetilde{\mathcal{L}}} = h_{\mathcal{K}} + h_{\widetilde{\mathcal{L}}}$ and $h_{\widetilde{\mathcal{K}}+\mathcal{L}} = h_{\widetilde{\mathcal{K}}} + h_{\mathcal{L}}$, it follows that $\widetilde{h} = h + \phi$, which completes the proof ■

Curvature Function

Definition 4.7 Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^2$ and let σ denote the circular Lebesgue measure on $\mathbb{S}^1$. From the Lebesgue decomposition theorem, there is a unique pair of mutually singular measures l_h^a and l_h^s such that

$$l_h = l_h^a + l_h^s,$$

where l_h^a is absolutely continuous with respect to σ and l_h^s mutually singular with σ. Furthermore, from the Radon–Nykodym theorem, there is a unique $R_h \in L^1(\sigma)$ such that

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ l_h^a(\Omega) = \int_\Omega R_h\, d\sigma.$$

$R_h \in L^1(\sigma)$ is called the curvature function of $\mathcal{H}_h$. For the sake of simplicity, we shall often consider R_h as a function of $\theta \in \mathbb{R}$ by putting $R_h(\theta) = R_h(u(\theta))$.

Remark 4.8 The curvature function of $\mathcal{H}_h$ is such that

$$R_h(\theta) = \lim_{\alpha \downarrow 0} \frac{l_h^a([\theta, \theta + \alpha])}{\alpha},$$

for almost every $\theta \in \mathbb{R}$. If the restriction of h to $\mathbb{S}^1$ is a function of class C^2 then $R_h(\theta) = (p + p'')(\theta)$, where $p(\theta) = h(u(\theta))$, for every $\theta \in \mathbb{R}$ and $R_h(\theta)$ can be interpreted as the (principal) radius of curvature of $\mathcal{H}_h$ at $x_h(\theta)$ (see Remark 3.9 for notations).

Absolute Length Measure

Definition 4.9 For every hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$, let L_h denote the total variation of l_h, that is the (positive) Borel measure $|l_h|$ defined by:

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ L_h(\Omega) = \sup_{(\Omega_i)_{i\in\Im} \in P(\Omega)} \sum_{i\in\Im} |l_h(\Omega_i)|,$$

where $P(\Omega)$ denotes the set of all partitions $(\Omega_i)_{i\in\Im}$ of Ω. This Borel measure L_h is called the absolute length measure of $\mathcal{H}_h$. Naturally, if $h \in \mathcal{K}_2$, then L_h is the length measure $S_1(\mathcal{K}, \cdot)$, where $\mathcal{K}$ is the convex body with support function h.

Remark 4.10 Let $\mathcal{H}_h$ be a hedgehog of $\mathbb{R}^2$ and let σ denote the circular Lebesgue measure on $\mathbb{S}^1$. From the Lebesgue decomposition theorem, there is a unique pair of mutually singular measures L_h^a and L_h^s such that

$$L_h = L_h^a + L_h^s,$$

where L_h^a is absolutely continuous with respect to σ and L_h^s mutually singular with σ. It is an easy exercise to check that these measures L_h^a and L_h^s are respectively the total variations $|l_h^a|$ and $|l_h^s|$ of l_h^a and l_h^s, so that we have in particular,

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ L_h^a(\Omega) = \int_\Omega |R_h|\, d\sigma.$$

Remark 4.11 As it will be checked at the end of the section, for every $h \in \mathcal{H}_2$, $L_h(\mathbb{S}^1)$ is the (absolute) length $L(h)$ of the rectifiable curve of $\mathcal{H}_h$.

Length Function

For every $h \in \mathcal{H}_2$, let us define a *length function* $\mathcal{L}_h \colon I = [0, 2\pi] \to \mathbb{R}$ as follows: for every $\theta \in I$, let $\mathcal{L}_h(\theta)$ denote the length of the rectifiable curve

$$\gamma_h \colon [0, \lambda^-(\theta)] \to \mathbb{R}^2, \lambda \longmapsto x_h(\theta(\lambda), t(\lambda)) = p(\theta(\lambda))u(\theta(\lambda)) + t(\lambda)u'(\theta(\lambda)),$$

where $\lambda^-(\theta) = d_h(\theta, p_l'(\theta))$, with notations of Section 3. In other words, $\mathcal{L}_h(\theta)$ denotes the length of the subarc of $\mathcal{H}_h$ beginning at $x_h^-(0)$ and ending at $x_h^-(\theta)$. In the case where $h = h_{\mathcal{K}} \in \mathcal{K}_2$, we thus have $\mathcal{L}_h(\theta) = L_h(u([0,\theta[))$, for all $\theta \in I$. This length function $\mathcal{L}_h$ is obviously increasing and thus of bounded variation. Therefore, $\mathcal{L}_h$ admits a decomposition of the form

$$\mathcal{L}_h = j_h + r_h + s_h,$$

where j_h is a jump function (with a derivative equal to 0 except for an at most countable set of jump discontinuities), r_h an absolutely continuous function and s_h a continuous singular function (with a derivative equal to 0 almost everywhere). This decomposition is unique provided these three functions are required to be equal to 0 at $\theta = 0$.

Let us consider the case where $h = h_{\mathcal{K}} \in \mathcal{K}_2$. In this case, the length measure $L_h \colon \mathcal{B}(\mathbb{S}^1) \to \mathbb{R}$ of $\partial\mathcal{K}$ is (inherited from) the Lebesgue–Stieltjes measure, say μ_h, associated with the increasing and left continuous function $\mathcal{L}_h$:

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ L_h(\Omega) = \mu_h(\Omega_J),$$

where $\Omega_J = \{\theta \in J | u(\theta) \in \Omega\}$. Moreover, the unique decomposition of L_h into discrete, absolutely continuous and continuous singular parts (with respect to the circular Lebesgue measure σ on $\mathbb{S}^1$) then corresponds to the decomposition $\mathcal{L}_h = j_h + r_h + s_h$ (in which j_h, r_h and s_h are required to be equal to 0 at $\theta = 0$). Let us notice that in this case

(i) the jump function j_h is given by

$$j_h(\theta) = \sum_{0 \leq \alpha < \theta} (p_r'(\alpha) - p_l'(\alpha))$$

(remember that $p_r' - p_l' \geq 0$ from Proposition 3.12(v)), so that the discrete part of L_h is given by

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ J_h(\Omega) := \sum_{\alpha \in \Omega_J} (p_r'(\alpha) - p_l'(\alpha));$$

(ii) the absolutely continuous part of L_h is the measure L_h^a, given by

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ L_h^a(\Omega) = \int_\Omega R_h \, d\sigma$$

(note that the curvature function R_h is σ-almost everywhere ≥ 0 from Remark 4.8);

(iii) the continuous singular part of L_h is a positive measure on $\mathcal{B}(\mathbb{S}^1)$, so that the continuous singular function s_h is increasing on I.

Characterization of Support Functions of Plane Convex Bodies Among All Support Functions of Plane Hedgehogs

Let us prove the following characterization of support functions of plane convex bodies among support functions of plane hedgehogs.

Theorem 4.12 *Let $h \in \mathcal{H}_2$. We have $h \in \mathcal{K}_2$ if and only if the three following conditions are satisfied:*

(i) $p_r' - p_l' \geq 0$ *on I;*

(ii) $R_h \geq 0$ *σ-almost everywhere on $\mathbb{S}^1$;*

(iii) *the continuous singular part of the length function $\mathcal{L}_h$ is increasing on I.*

Proof It follows from the previous study that these three conditions are necessary. Let us check that they are also sufficient. Let us assume that these three conditions are satisfied and denote by L_{s_h} the (measure inherited from the) Lebesgue–Stieltjes measure associated with the continuous singular part of $\mathcal{L}_h$. Then, the Borel measure

$$l_h\colon \mathcal{B}(\mathbb{S}^1) \to \mathbb{R},\ \Omega \mapsto \int_\Omega R_h\, d\sigma + \sum_{\alpha\in\Omega_J} (p_r'(\alpha) - p_l'(\alpha)) + L_{s_h}(\Omega),$$

where $\Omega_J = \{\theta \in J \mid u(\theta) \in \Omega\}$, is positive. Moreover, as it is of the form $L_f - L_g$, where $(f, g) \in (\mathcal{K}_2)^2$, it satisfies

$$\int_{\mathbb{S}^1} u\, dm(u) = 0_{\mathbb{R}^2}.$$

Therefore, Theorem 4.1 ensures that there exists some $k \in \mathcal{K}_2$ such that $L_k = l_h$ and Theorem 4.6 that $\mathcal{H}_h$ and $\mathcal{H}_k$ must be translates, which completes the proof. ■

Remark 4.13 It follows from the previous study that the perimeter of a convex body $\mathcal{K} \subset \mathbb{R}^2$ is given by

$$L(\partial\mathcal{K}) = \int_0^{2\pi} (p + p'')(\theta)\, d\theta + \sum_{\theta\in J}(p_r'(\theta) - p_l'(\theta)) + L_{s_h}(J),$$

where $h = h_{\mathcal{K}}$ and $p(\theta) = h(u(\theta))$, L_{s_h} denoting the Lebesgue–Stieltjes measure associated with the continuous singular part of $\mathcal{L}_h$. Let us give an example where

$$L(\partial\mathcal{K}) > \int_0^{2\pi} (p + p'')(\theta)\, d\theta + \sum_{\theta\in J}(p_r'(\theta) - p_l'(\theta)).$$

To this aim, let us consider the odd function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies

$$f(t) = \begin{cases} s(t) & \text{if } 0 \le t \le 1, \\ 1 & \text{if } t \ge 1, \end{cases}$$

where s is the Cantor–Lebesgue function on $[0, 1]$. Now define $h\colon \mathbb{R}^2 \to \mathbb{R}$ by $\forall(x, y) \in \mathbb{R}^2$,

$$h(x, y) = \begin{cases} |x| & \text{if } y = 0, \\ |y|(1 + \int_1^{x/y} f(t)\, dt) & \text{if } y \neq 0. \end{cases}$$

It is then easy to check that h is the support function of a centered convex body $\mathcal{K} \subset \mathbb{R}^2$ for which the required inequality is satisfied.

Vector Length Measure

For every $h \in \mathcal{H}_2$, let us define a vector length measure $\overrightarrow{l_h} : \mathcal{B}(\mathbb{S}^1) \to \mathbb{R}^2$ whose components are (inherited from) the signed Lebesgue–Stieltjes measures associated with the components of the map $x_h^- : I \to \mathbb{R}^2$ (which are left continuous functions of bounded variation). As noted in Remark 3.7, the arclength parametrization $X_h : [0, L(h)] \to \mathcal{H}_h \subset \mathbb{R}^2$ is almost everywhere differentiable on $[0, L(h)]$ and there exists an increasing map $s \mapsto \theta_s$ of $[0, L(h)]$ into J such that, for almost every $s \in [0, L(h)]$, $X_h(s) \in \sigma_h(\theta_s)$ and $X_h'(s) = \varepsilon_h(s)u'(\theta_s)$, where $\varepsilon_h(s) \in \{-1, 1\}$. It is easy to check that

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ \overrightarrow{l_h}(\Omega) = \int_{\Omega_h} X_h'(s)\, ds = \int_{\Omega_h} \varepsilon_h(s) u'(\theta_s)\, ds,$$

where $\Omega_h = \{s \in [0, L(h)] | u(\theta_s) \in \Omega\}$.

Remark 4.14 If the restriction of h to $\mathbb{S}^1$ is of class C^2, then we have of course

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ \overrightarrow{l_h}(\Omega) = \int_{\Omega_J} x_h'(\theta)\, d\theta = \int_{\Omega_J} R_h(\theta) u'(\theta) d\theta,$$

where $\Omega_J = \{\theta \in J | u(\theta) \in \Omega\}$.

Proposition 4.15 *For every $h \in \mathcal{H}_2$, we have $d\overrightarrow{l_h}(u) = u^\top\, dl_h(u)$, where $u^\top \in \mathbb{S}^1$ is the unit vector such that $(u, u^\top)$ is a direct orthonormal frame of $\mathbb{R}^2$, which we assume equipped with the standard orientation.*

Proof By linearity, it suffices to prove it for $h \in \mathcal{K}_2$. Let l be the Lebesgue measure on $[0, L(h)]$ and let $u_h : [0, L(h)] \to \mathbb{S}^1$ be the measurable map defined by $u_h(s) = u(\theta_s)$. For $h \in \mathcal{K}_2$, l_h is nothing but the image measure of l by u_h, so that:

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1), \int_\Omega u^\top\, dl_h(u) = \int_{\Omega_h} u'(\theta_s)\, ds = \overrightarrow{l_h}(\Omega),$$

where $\Omega_h = \{s \in [0, L(h)] | u(\theta_s) \in \Omega\}$. ∎

Geometrical Interpretation of l_h and L_h for $h \in \mathcal{H}_2$

The following result ensures that the absolute length measure (resp., the algebraic length measure) of $\mathcal{H}_h$ can indeed be interpreted as the length measure of $\mathcal{H}_h$ (resp., the length measure of $\mathcal{H}_h$ counted with the sign of $\varepsilon_h(s) = \langle X_h'(s), u'(\theta_s)\rangle$).

Theorem 4.16 *Let $h \in \mathcal{H}_2$. The algebraic length measure of $\mathcal{H}_h$ is given by*

$$\forall \Omega \in \mathcal{B}(\mathbb{S}^1),\ l_h(\Omega) = \int_{\Omega_h} \varepsilon_h(s)\, ds,$$

where $\Omega_h = \{s \in [0, L(h)] \mid u(\theta_s) \in \Omega\}$ and $\varepsilon_h(s) = \operatorname{sgn}(\langle X_h'(s), u'(\theta_s)\rangle)$.

Proof From Proposition 4.15, we have indeed $\forall\Omega \in \mathcal{B}(\mathbb{S}^1)$,

$$\begin{aligned} l_h(\Omega) &= \int_\Omega \langle u^\top \, d\overrightarrow{l_h}(u)\rangle \\ &= \int_{\Omega_h} \langle u'(\theta_s), X_h'(s)\rangle \, ds \\ &= \int_{\Omega_h} \varepsilon_h(s)\, ds, \end{aligned}$$

where $\langle \cdot , \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^2$. ■

Algebraic Area

For every $h \in \mathcal{H}_2$, let us define the algebraic area of $\mathcal{H}_h$ as the integral

$$a(h) = \int_{\mathbb{R}^2 - \mathcal{H}_h} i_h(x)\, d\mathcal{L}(x),$$

where $i_h(x)$ is the winding number of $\mathcal{H}_h$ around $x \in \mathbb{R}^2 - \mathcal{H}_h$ and $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}^2$.

Theorem 4.17 *For every $h \in \mathcal{H}_2$, we have*

$$a(h) = \frac{1}{2} \int_{\mathbb{S}^1} h(u)\, dl_h(u).$$

The quadratic form $a\colon \mathcal{H}_2 \to \mathbb{R}$, $h \longmapsto a(h)$ satisfies: $\forall h \in \mathcal{H}_2$,

$$a(h) = \frac{1}{2} \int_0^{2\pi} \left(p^2 - (p')^2\right)(\theta)\, d\theta,$$

where $p(\theta) = h(u(\theta))$.

Proof Let us define the body of $\mathcal{H}_h$ as the set

$$\mathcal{K}_h = \mathcal{H}_h \cup \{x \in \mathbb{R}^2 - \mathcal{H}_h \mid i_h(x) \neq 0\}.$$

Given $m \in \mathbb{R}^2$, let us consider $\mathcal{K}_h$ as a part of the image of $\Delta_h = [0,1] \times [0, L(h)]$ under the map

$$X_h^m\colon \Delta_h \longrightarrow \mathbb{R}^2, \qquad (r,s) \longmapsto m + r(X_h(s) - m).$$

We shall say that a half-line $L \subset \mathbb{R}^2$ with origin m is transverse to $\mathcal{H}_h$ if, for every $s \in [0, L(h)]$ such that $X_h(s) \in L$, the vector $X_h'(s)$ exists and is transverse to L. Now, almost every $x \in X_h^m(\Delta_h)$ belongs to such a half-line and in this case $i_h(x)$ is given by

$$i_h(x) = \sum_{(r,s)\in E_h(x)} i_h(r,s),$$

where $E_h(x) = (X_h^m)^{-1}(x)$ and $i_h(r,s) = \operatorname{sgn}[\langle X_h(s) - m, u(\theta_s)\rangle \varepsilon_h(s)]$, that is $i_h(r,s) = \operatorname{sgn}[p_m(\theta_s)\varepsilon_h(s)]$ where $p_m(\theta) = p(\theta) - \langle m, u(\theta)\rangle$. We thus have

$$\begin{aligned}
a(h) &= \iint_{\mathbb{R}^2 - \mathcal{H}_h} i_h(x_1, x_2)\, dx_1 dx_2 \\
&= \iint_{\Delta_h} i_h(r,s) \Big| \det\Big[\frac{\partial X_h^m}{\partial r}(r,s), \frac{\partial X_h^m}{\partial s}(r,s) \Big] \Big|\, drds \\
&= \iint_{\Delta_h} \det[X_h(s) - m, rX_h'(s)]\, drds \\
&= \iint_{\Delta_h} r p_m(\theta_s)\varepsilon_h(s)\, drds \\
&= \int_0^1 r\, dr \int_0^{L(h)} p_m(\theta_s)\varepsilon_h(s)\, ds \\
&= \frac{1}{2} \int_{\mathbb{S}^1} (h(u) - \langle m, u\rangle)\, dl_h(u) \\
&= \frac{1}{2} \int_{\mathbb{S}^1} h(u)\, dl_h(u),
\end{aligned}$$

since $\int_{\mathbb{S}^1} u\, dl_h(u) = 0_{\mathbb{R}^2}$. Therefore, the map $a\colon \mathcal{H}_2 \to \mathbb{R}$ is a quadratic form.

As the relation

$$a(h) = \frac{1}{2} \int_0^{2\pi} (p^2 - (p')^2)(\theta)\, d\theta$$

is well known for $h \in \mathcal{K}_2$ (see [25, p. 188]), we can thus claim that it remains true for every $h \in \mathcal{H}_2$. ∎

Corollary 4.18 *Let $a\colon (\mathcal{H}_2)^2 \to \mathbb{R}$, $(h,k) \longmapsto a(h,k)$ be the symmetric bilinear form obtained by polarizing $a\colon \mathcal{H}_2 \to \mathbb{R}$. For every $(h,k) \in (\mathcal{H}_2)^2$, $a(h,k)$ may be interpreted as an algebraic mixed area of $\mathcal{H}_h$ and $\mathcal{H}_k$ and can be given by*

$$a(h,k) = \frac{1}{2} \int_{\mathbb{S}^1} (h(u)\, dk_h(u) + k(u)\, dh_h(u)) = \frac{1}{2} \int_0^{2\pi} (p\,q - p'q')(\theta)\, d\theta,$$

where $p(\theta) = h(u(\theta))$ and $q(\theta) = k(u(\theta))$.

Remark 4.19 Another way to prove the relation

$$a(h) = \frac{1}{2} \int_0^{2\pi} (p^2 - (p')^2)(\theta)\, d\theta$$

is to consider $a(h)$ as the difference of two areas. Indeed, it is easy to check that

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(i) the integral

$$\int_0^{2\pi} \frac{p(\theta)^2}{2}\, d\theta$$

can be seen as the area of the pedal curve of $\mathcal{H}_h$ with respect to the origin. Recall that this pedal curve, say $P(\mathcal{H}_h)$, is defined as follows: to each $x_h(\theta,t) \in \mathcal{H}_h$, we assign the foot $P(x_h(\theta,t)) = p(\theta)u(\theta)$ of the perpendicular from the origin to the support line of $\mathcal{H}_h$ at $x_h(\theta,t)$. Naturally, the area $a[P(\mathcal{H}_h)]$ of the pedal curve $P(\mathcal{H}_h)$ is defined by

$$a[P(\mathcal{H}_h)] := \int_{\mathbb{R}^2 - P(\mathcal{H}_h)} i_{P(\mathcal{H}_h)}(x)\, d\mathcal{L}(x),$$

where $i_{P(\mathcal{H}_h)}(x)$ is the winding number of $P(\mathcal{H}_h)$ around $x \in \mathbb{R}^2 - P(\mathcal{H}_h)$ and $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}^2$;

(ii) the integral

$$\int_0^{2\pi} \frac{p'(\theta)^2}{2}\, d\theta$$

can be seen as the area of the image of $\Sigma_h = \{(\theta,t) \in D_h \times \mathbb{R} \mid t(t - p'(\theta)) < 0\}$, where $D_h = \{\theta \in J \mid p'(\theta) \text{ exists}\}$, under the map

$$T\colon \Sigma_h \to \mathbb{R}^2, \qquad (\theta,t) \longmapsto p(\theta)u(\theta) + tu'(\theta).$$

This area $a[T(\mathcal{H}_h)]$ is of course defined by:

$$a[T(\mathcal{H}_h)] := \int_{T(\Sigma_h)} t_h(x)\, d\mathcal{L}(x),$$

where $t_h(x) = \mathrm{Card}(\{(\theta,t) \in \Sigma_h \mid T(\theta,t) = x\})$ and can be given by

$$a[T(\mathcal{H}_h)] = \iint_{\Sigma_h} \Big| \det\Big[\frac{\partial T}{\partial \theta}(\theta,t), \frac{\partial T}{\partial t}(\theta,t)\Big] \Big|\, d\theta dt.$$

To prove the second relation of Theorem 4.17, it then suffices to observe that for $\mathcal{L}$-almost every $x \in \mathbb{R}^2 - (\mathcal{H}_h \cup P(\mathcal{H}_h))$, we have $i_h(x) = i_{P(\mathcal{H}_h)}(x) - t_h(x)$.

Characterization of Support Functions of Plane Hedgehogs Among Continuous Functions

Theorem 4.20 *Functions of $\mathcal{H}_2$ are exactly functions of $C(\mathbb{S}^1;\mathbb{R})$ that satisfy properties* (i)–(iv) *of Proposition* 3.3.

Proof We already know that functions of $\mathcal{H}_2$ satisfy these four conditions. It remains to check that if $h \in C(\mathbb{S}^1;\mathbb{R})$ satisfies properties (i)–(iv), then $h \in \mathcal{H}_2$. Let $h \in C(\mathbb{S}^1;\mathbb{R})$ be such a function. As noticed in Remark 3.8, it defines a closed rectifiable curve $\mathcal{H}_h \subset \mathbb{R}^2$. Let $l_h\colon \mathcal{B}(\mathbb{S}^1) \to \mathbb{R}$ be the signed Borel measure inherited

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from the Lebesgue–Stieltjes measure associated with the following left continuous function of bounded variation:

$$\mathcal{L}_h^{\varepsilon}\colon J = [0, 2\pi[\to \mathbb{R}, \quad \theta \mapsto \int_0^{\mathcal{L}_h(\theta)} \varepsilon_h(s)\, ds,$$

where the length function $\mathcal{L}_h(\theta)$ of $\mathcal{H}_h$ is defined as in the case where $h \in \mathcal{H}_2$ and where $\varepsilon_h(s) = \operatorname{sgn}(\langle X_h'(s), u'(\theta_s)\rangle)$ (see Remarks 3.7 and 3.8), that is, $\forall \Omega \in \mathcal{B}(\mathbb{S}^1)$,

$$l_h(\Omega) := \mu_{\mathcal{L}_h^{\varepsilon}}(\Omega_J),$$

where $\mu_{\mathcal{L}_h^{\varepsilon}}$ is the Lebesgue–Stieltjes measure associated to $\mathcal{L}_h^{\varepsilon}$ on J and where $\Omega_J = \{\theta \in J \mid u(\theta) \in \Omega\}$. This signed Borel measure $l_h\colon \mathcal{B}(\mathbb{S}^1) \to \mathbb{R}$ satisfies

$$\int_{\mathbb{S}^1} u\, dl_h(u) = 0_{\mathbb{R}^2}.$$

Indeed, we have

$$\begin{aligned}\int_{\mathbb{S}^1} u^\top\, dl_h(u) &= \int_0^{L(h)} \varepsilon_h(s) u'(\theta_s)\, ds \\ &= \int_0^{L(h)} X_h'(s)\, ds \\ &= X_h(L(h)) - X_h(0),\end{aligned}$$

since components of X_h are absolutely continuous on $[0, L(h)]$, see Remark 3.8. Therefore, l_h is the algebraic length measure of a unique (up to translations) hedgehog of $\mathcal{H}_f \subset \mathbb{R}^2$, where $f \in \mathcal{H}_2$. Now, if $\overrightarrow{l_h}\colon \mathcal{B}(\mathbb{S}^1) \to \mathbb{R}^2$ denotes the vector measure whose components are (inherited from) the signed Lebesgue–Stieltjes measures associated with the components of $x_h^-\colon I \to \mathbb{R}^2$ (which are left continuous functions of bounded variation), then $\forall \theta \in I$,

$$\begin{aligned}x_h^-(\theta) - x_h^-(0) &= \overrightarrow{l_h}(\Omega_\theta) \\ &= \int_0^{\mathcal{L}_h(\theta)} X_h'(s)\, ds = \int_0^{\mathcal{L}_h(\theta)} \varepsilon_h(s) u'(\theta_s)\, ds \\ &= \int_{\Omega_\theta} u^\top\, dl_h(u) = \int_{\Omega_\theta} u^\top\, dl_f(u) \\ &= \overrightarrow{l_f}(\Omega_\theta) = x_f^-(\theta) - x_f^-(0),\end{aligned}$$

where $\Omega_\theta = u([0, \theta[)$. So, for all $\theta \in I$, we have $x_h^-(\theta) = x_f^-(\theta) + x$, where $x = (x_h^- - x_f^-)(0)$, and thus $h(\theta) = f(\theta) + \langle x, u(\theta)\rangle$, so that $h \in \mathcal{H}_2$. ∎

The following result can be seen as a particular case of Theorem 4.6:

Proposition 4.21 *If $\rho\colon \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function that is summable on $I = [0, 2\pi]$ and such that $\int_0^{2\pi} \rho(\theta)u(\theta)d\theta = 0$, then there exists a plane hedgehog $\mathcal{H}_h \subset \mathbb{R}^2$ whose curvature function satisfies $R_h(\theta) = \rho(\theta)$ for almost every $\theta \in I$.*

Proof Our proof is a mere adaptation of [6, Theorem 4]. Using the characterization of functions in $\mathcal{H}_2$, it consists in proving that

$$p(\theta) = \int_0^{\theta} \rho(\alpha)\sin(\theta - \alpha)\,d\alpha,$$

is a 2π-periodic Lipschitzian function on $\mathbb{R}$ which admits an absolutely continuous derivative on I that satisfies $(p + p'')(\theta) = \rho(\theta)$ for almost every $\theta \in I$. A first calculation shows that p admits an absolutely continuous derivative on I, namely

$$p'(\theta) = \int_0^{\theta} \rho(\alpha)\cos(\theta - \alpha)\,d\alpha,$$

and a second one that $(p + p'')(\theta) = \rho(\theta)$ for almost every $\theta \in I$. The integral condition $\int_0^{2\pi} \rho(\theta)u(\theta)\,d\theta = 0$ ensures that p is 2π-periodic on $\mathbb{R}$. ■

Of course, Proposition 4.21 gives only an existence result: there is no uniqueness. For instance, if $\mathcal{H}_h \subset \mathbb{R}^2$ is any polygonal hedgehog, then $p(\theta) = h(u(\theta))$ satisfies $(p + p'')(\theta) = 0$ for almost every $\theta \in I$.

5 Geometric Inequalities for General Plane Hedgehogs

The following theorem gives an extension to hedgehogs of the classical Minkowski inequality (and thus of the isoperimetric inequality) for plane convex bodies:

Theorem 5.1

(i) *If $h \in \mathcal{H}_2$ is such that $l(h) = 0$, then $a(h) \leq 0$.*

(ii) *If $(f, g) \in (\mathcal{H}_2)^2$ is such that $a(g) > 0$, then $a(f, g)^2 \geq a(f) \cdot a(g)$. In particular, $\forall h \in \mathcal{H}_2$, $4\pi a(h) \leq l(h)^2$.*

Proof (i) If the restriction of h to $\mathbb{S}^1$ is a function of class C^2, this is only Wirtinger's inequality. If $h = h_{\mathcal{K}} - h_{\mathcal{L}}$, where $(\mathcal{K}, \mathcal{L}) \in \mathcal{K}^2 \times \mathcal{K}^2$, then we can proceed as follows. We know there exist sequences $(\mathcal{K}_n)$ and $(\mathcal{L}_n)$ of plane convex bodies of class C^2_+ that converge respectively towards $\mathcal{K}$ and $\mathcal{L}$ with respect to the Hausdorff metric on $\mathcal{K}^2$ [2]. Recall that convergence of plane convex bodies with respect to the Hausdorff metric is equivalent to uniform convergence on $\mathbb{S}^1$ of the corresponding support functions [24]. Let (h_n) be the sequence defined by $\forall n \in \mathbb{N}$, $h_n = f_n - \frac{1}{2\pi} l(f_n)$, where $f_n = h_{\mathcal{K}_n} - h_{\mathcal{L}_n}$. Using the assumption $l(h) = 0$, we check at once that (h_n) converges uniformly towards h on $\mathbb{S}^1$. Now, $\forall n \in \mathbb{N}$, $l(h_n) = 0$ and thus $a(h_n) \leq 0$ from Wirtinger's inequality. Using the bilinearity of the algebraic mixed area and the continuity of quermassintegrals on $\mathcal{K}^2$ (see [24]), we deduce that $a(h) = \lim_{n \to +\infty} a(h_n) \leq 0$.

(ii) Since $a(g) > 0$, we have $l(g) \neq 0$. Let τ be the trinomial defined on $\mathbb{R}$ by

$$\tau(t) := a(f + tg) = a(f) + 2ta(f, g) + t^2 a(g).$$

If $a(f, g)^2 < a(f)\, a(g)$, then $\tau(t)$ has no real root and the assumption $a(g) > 0$ implies that $\tau(t) > 0$ for all $t \in \mathbb{R}$. But this is impossible since there exists $\lambda \in \mathbb{R}$ such that $l(f + \lambda g) = l(f) + \lambda l(g) = 0$, which implies $\tau(\lambda) \leq 0$ from (i). ■

6 Convolution

Differences of convex bodies of $\mathbb{R}^2$ do not only constitute a real vector space $(\mathcal{H}^2, +, \cdot)$ but also a commutative and associative $\mathbb{R}$-algebra. Indeed, as noticed by H. Görtler [4, 5], we can define the convolution product of two plane hedgehogs $\mathcal{H}_f$ and $\mathcal{H}_g$ of $\mathbb{R}^2$ as the plane hedgehog whose support function is given by

$$(f * g)(u(\theta)) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta - \alpha) g(\alpha)\, d\alpha,$$

for all $\theta \in I$; and we can check at once that $(\mathcal{H}^2, +, \cdot, *)$ is then a commutative and associative algebra. H. Görtler also noticed that the convolution product of two plane convex bodies is still a plane convex body. The interest of convolution of hedgehogs is that properties of one factor are often transmitted to the product. Of course, we think immediately of regularity properties but we can also mention the following: to be centered (centrally symmetric with center at the origin), to be projective (*i.e.,* to have an antisymmetric support function), to be of constant width. Note that we can also define a convolution product on the real vector space of hedgehogs of $\mathbb{R}^{n+1}$ for $n \geq 2$, but this product is then non commutative.

References

[1] E. Barbier, *Note sur le probleme de l'aiguille et le jeu joint couvert.* J. Math. Pures Apple. **5**(1860), 273–286.

[2] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper.* Springer, Berlin. Reprint: Chelsea Publishing, New York, 1948.

[3] H. Geppert, *Über den Brunn-Minkowskischen Satz.* Math. Z. **42**(1937), 238–254.

[4] H. Görtler, *Erzeugung stützbarer Bereiche. I.* Deutsche Math. **2**(1937), 454–456.

[5] ———, *Erzeugung stützbarer Bereiche. II.* Deutsche Math. **3**(1937), 189–200.

[6] M. Kallay, *Reconstruction of a plane convex body from the curvature of its boundary.* Israel J. Math. **17**(1974), 149–161.

[7] R. Langevin, G. Levitt, and H. Rosenberg, *Hérissons et multihérissons (enveloppes paramétrées par leur application de Gauss).* In: Singularities, Banach Center Publ. 20, PWN, Warsaw, 1988, pp. 245–253.

[8] Y. Martinez-Maure, *Feuilletages des surfaces et hérissons dans* $\mathbb{R}^3$. Thèse de doctorat de 3ème cycle, Université Paris 7, 1985.

[9] ———, *A note on the tennis ball theorem.* Amer. Math. Monthly **103**(1996), 338–340.

[10] ———, *Sur les hérissons projectifs (enveloppes paramétrées par leur application de Gauss).* Bull. Sci. Math. **121**(1997), no. 8, 585–601.

[11] ———, *Hedgehogs of constant width and equichordal points.* Ann. Polon. Math. **67**(1997), no. 3, 285–288.

[12] ———, *De nouvelles inégalités géométriques pour les hérissons.* Arch. Math. (Basel) **72**(1999), 444–453.

[13] ———, *Indice d'un hérisson: étude et applications.* Publ. Mat. **44**(2000), 237–255.

[14] ———, *A fractal projective hedgehog.* Demonstratio Math. **34**(2001), no. 1, 59–63.

[15] ———, *Contre-exemple à une caractérisation conjecturée de la sphère.* C. R. Acad. Sci. Paris, Sér. I Math. **332**(2001), 41–44.

[16] ———, *Hedgehogs and zonoids.* Adv. Math. **158**(2001), 1–17.

[17] ———, *La théorie des hérissons (différences de corps convexes) et ses applications.* Habilitation, Univ. Paris 7, 2001.

[18] ———, *Sommets et normales concourantes des courbes convexes de largeur constante et singularités des hérissons.* Arch. Math. **79**(2002) 489–498.

[19] ———, *Voyage dans l'univers des hérissons.* In: Ateliers Mathematica, Paris, Vuibert, 2003, 445–470.

[20] ———, *Théorie des hérissons et polytopes.* C. R. Acad. Sci. Paris Sér. I Math. **336**(2003), no. 3, 241–244.

[21] ———, *Les multihérissons et le théorème de Sturm–Hurwitz.* Arch. Math. (Basel) **80**(2003), 79–86.

[22] ———, *A Brunn-Minkowski theory for minimal surfaces.* Illinois J. Math **48**(2004), no. 2, 589–607.

[23] H. Ohtsuka, *Dirichlet problems of Riemann surfaces and conformal mappings.* Nagoya Math. J. **3**(1951), 91-137.

[24] R. Schneider, *Convex Bodies: The Brunn-Minkowski Theory.* Encyclopedia of Mathematics and Its Applications 44, Cambridge University Press, Cambridge, 1993.

[25] F. A. Valentine, *Convex Sets.* McGraw-Hill, New York, 1964.

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